[8] W. B. Jurkat and H. -E. Richert, *An improvement of Selberg's sieve method I*, Acta Arith. 11 (1965), pp. 217–240.
[9] J. W. Porter, *Some numerical results in the Selberg sieve method*, ibid. 20 (1972), pp. 417–421.
[10] H. -E. Richert, *Selberg's sieve with weights*, Mathematika 16 (1969), pp. 1–22.
[11] A. Selberg, *Sieve methods*, Proceedings of the Stony Brook Number Theory Conference, 1969, American Mathematical Society.

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An asymptotic formula for the property $(n, f(n)) = 1$ for a class of multiplicative functions

by

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1. Introduction. A number of authors have investigated the problem of estimating the sum

$$\sum_{\substack{1\leqslant n\leqslant x\\ (n,f(n))=1}} 1 \tag{1}$$

for certain types of integer-valued arithmetic functions f. If the arithmetic properties of n and $f(n)$ are more or less unrelated, probabilistic arguments lead one to expect that the sum in (1) is asymptotic to $6x/\pi^2$, and some results in this direction have been obtained, in particular by Watson [15], Erdös and Lorentz [2] and Hall [6], [7], [8]. There is marked contrast between Hall's result for certain strongly additive functions, later improved in some cases by Faĭnleĭb [4], and the result to be derived in this paper for a related class of multiplicative functions. In [6] and [8] Hall considered the strongly additive function given by

$$f(n) = \sum_{p\mid n} p, \tag{2}$$

and in [7] he investigated a class of functions that includes functions of the type

$$f(n) = \sum_{p\mid n} g(p), \tag{3}$$

where g is a polynomial with integer coefficients satisfying some further conditions. Taking $g(x) = x$ in (3) gives (2), and in both cases the sum (1) is asymptotic to $6x/\pi^2$; in fact a very much more precise result was obtained for (2), using a combination of elementary and analytical argu-

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ments, by Hall in [8] and for (3), using a different approach, by Faĭnleĭb in [4].

Multiplicative functions f present a rather different problem, for here the arithmetic properties of $f(n)$ are dependent on those of n. In 1948, Erdös [1] considered Euler's φ-function and he proved that

$$\sum_{\substack{n\leqslant x\\(n,\varphi(n))=1}} 1 \sim \frac{e^{-\gamma}x}{\log\log\log x}; \tag{4}$$

this result is of special interest since the property $(n, \varphi(n)) = 1$ is a necessary and sufficient condition for there to be only one group of order n. The purpose of this paper is to obtain the corresponding result for polynomial-like, multiplicative functions; if there exist polynomials $W_j(x)$ $(j = 1, 2, \ldots)$ with integer coefficients such that for all primes p

$$f(p^j) = W_j(p) \quad (j = 1, 2, \ldots), \tag{5}$$

then f is said to be *polynomial-like*. For example, the function φ and the divisor functions

$$\tau(n) = \sum_{d|n} 1, \quad \sigma_\nu(n) = \sum_{d|n} d^\nu \quad (\nu \text{ a positive integer}) \tag{6}$$

satisfy this condition, whilst the function in (3) is a strongly additive polynomial-like function.

Our objective is to obtain an asymptotic formula for the sum (1) when f is a multiplicative polynomial-like function. We shall assume that the polynomial W_1 of (5) satisfies two further conditions: that the degree k of W_1 is positive, and that $W_1(0) \neq 0$. It is easy to see that, if we drop this latter condition, the sum (1) is rather small. For if $W_1(0) = 0$, $(p, f(p)) = p$ for all primes p, and hence using the multiplicative property of f, we have that $(n, f(n)) = 1$ implies that n is squarefull (that is, $p^2|n$ whenever $p|n$); hence

$$\sum_{\substack{n\leqslant x\\(n,f(n))=1}} 1 \leqslant \sum_{\substack{n\leqslant x\\n \text{ squarefull}}} 1 = O(x^{1/2})$$

by a result of Erdös and Szekeres [3] (see (25) below). In Theorem 3 in § 5, we consider the case when $W_1(x)$ is a non-zero constant.

We now state the main result of this paper:

THEOREM 1. *Let f be a polynomial-like multiplicative function such that W_1 has positive degree and $W_1(0) \neq 0$. Then there exists a positive constant C such that, as $x\to\infty$,*

$$\sum_{\substack{1\leqslant n\leqslant x\\(n,f(n))=1}} 1 \sim \frac{Cx}{(\log\log\log x)^\lambda},$$

where λ $(0 < \lambda \leqslant 1$, λ rational$)$ is the constant of Theorem 2 below applied to the polynomial W_1.

The structure of C will emerge from the proof (see (34)), and for certain subclasses of polynomial-like multiplicative functions we shall obtain a simple expression for it in terms of the positive constant C_1 of (9) below. When $f = \varphi$, we see from (4) that $C = e^{-\gamma}$, $\lambda = 1$, and this is also true, as we shall see in Corollaries 2 and 5 in § 5, for all functions in a certain subclass, containing φ and σ_ν for odd ν, of the functions considered here.

The proof of Theorem 1 is based on Erdös's proof in [1] of (4), but in our rather general situation it is somewhat more complicated, although remaining elementary in character. We divide it into several parts. The preliminary Lemmas that we need are contained in § 3; in § 4, we prove Theorem 1, and in § 5 we investigate some special cases, some of which have been referred to above, and in particular we consider the divisor functions.

In § 2, we shall prove Theorem 2 (stated below); the proof of this theorem in the general case is due to Professor A. Schinzel, and the author is very grateful to him for supplying a proof of this result (when previously (7) below had been assumed as an additional condition on W_1 in Theorem 1) and for permission to include his proof in this paper. Let W be a non-constant polynomial with integer coefficients, and denote by $\varrho(p)$ the number of solutions of the congruence

$$W(x) \equiv 0 \pmod p,$$

where p is prime; let $\mathscr{S}$ denote the set of all primes p for which $\varrho(p) > 0$. Then we have

THEOREM 2 (due to A. Schinzel). *For any non-constant polynomial W with integer coefficients, there exist constants λ $(0 < \lambda \leqslant 1)$ and D such that*

$$\sum_{\substack{p\leqslant x\\p\in\mathscr{S}}} \frac{1}{p} = \lambda\log\log x + D + O\big((\log x)^{-1}\big). \tag{7}$$

Let n be the number of elements in the Galois Group $\mathscr{G}$ of the splitting field of W over the field Q of rational numbers, and suppose that exactly t of these elements leave at least one zero of W fixed; then

$$\lambda = t/n. \tag{8}$$

Some special cases of (7) are well known; for example, when $\mathscr{S}$ is the set of all primes, $\lambda = 1$, and when $\mathscr{S}$ is the set of all primes congruent to $h \pmod k$, where $(h, k) = 1$, $\lambda = 1/\varphi(k)$. The proof in the general case is by an algebraical argument based on the proof in [11] of a Theorem of Frobenius. A less precise result than (7) may be deduced from two papers

[12], [13] by Schulze in which it is shown that the natural density of $\mathscr{S}$ in the set of all primes is λ, where λ is given by (8).

With the notation of Theorem 2, it very readily follows that

COROLLARY. *There exists a positive constant C_1 such that*

(9) $$\prod_{\substack{p\leqslant x\\ p\in\mathscr{S}}}(1-p^{-1}) = C_1\big(1+O((\log x)^{-1})\big)(\log x)^{-\lambda}.$$

Finally, I wish to thank Professor W. Narkiewicz for first drawing my attention to the problem discussed in this paper, and for a number of helpful and encouraging discussions whilst I was working in Wrocław.

2. Schinzel's proof of Theorem 2. We use the notation introduced in § 1 and some well known lemmas in conjunction with the notation and lemmas employed in Chapter 16 of [11] in the proof of Theorem 16.5 (due to Frobenius).

Let $\boldsymbol{K}$ denote a number field of degree n and $\mathscr{G}$ the Galois Group of $\boldsymbol{K}$ over $\boldsymbol{Q}$. For any polynomial F over $\boldsymbol{Q}$, let $\varrho_F(p)$ (or simply $\varrho(p)$ when there is no risk of ambiguity) denote the number of integer solutions of

(10) $$F(x) \equiv 0 \pmod{p},$$

where p is a prime; when required, we may take F to have integer coefficients without altering $\varrho(p)$ except for a finite number of primes p.

LEMMA 1. *If $\mathfrak{p}$ runs over the prime ideals of $\boldsymbol{K}$, then*

$$\sum_{N\mathfrak{p}\leqslant x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}} = \log x+O(1).$$

LEMMA 2. *If F is irreducible over $\boldsymbol{Q}$, then*

$$\sum_{p\leqslant x}\varrho_F(p)\frac{\log p}{p} = \log x+O(1).$$

These lemmas are well known. Lemma 1 can be deduced by partial summation from the Prime Ideal Theorem, as given for example in Sätze 190, 191 of [9], or can be deduced directly and straightforwardly from the result

$$\sum_{N\mathfrak{a}\leqslant x}1 = gx+O(x^{1-1/n}) \quad (g\neq 0)$$

(p. 712 of [16]). Lemma 2 can be deduced from Lemma 1 with $\boldsymbol{K} = \boldsymbol{Q}(\theta)$, where θ is a zero of F; for $N\mathfrak{p} = p$ for $\varrho(p)$ prime ideals $\mathfrak{p}$, and otherwise $N\mathfrak{p} = p^f$ for at most n prime ideals $\mathfrak{p}$, where $2\leqslant f\leqslant n$.

LEMMA 3. *Let $\boldsymbol{K}$ be normal over $\boldsymbol{Q}$, and z be an integer of $\boldsymbol{K}$ whose different in $\boldsymbol{Q}(z)$ over $\boldsymbol{Q}$ is prime to p. Let $\mathscr{H}$ be the Galois group of $\boldsymbol{K}$ over $\boldsymbol{Q}(z)$, and*

$$\mathscr{G} = \mathscr{H}+\mathscr{H}T_2+\ldots \quad (T_v\in\mathscr{G} \text{ for } v = 2, \ldots).$$

Let p belong to the class of $S\in\mathscr{G}$, and let $F(x) = 0$ be the irreducible equation for z over $\boldsymbol{Q}$. Then $\varrho_F(p)$ equals the number of integers v such that

$$T_vST_v^{-1}\in\mathscr{H}.$$

This is Theorem 16.4 of [11] for the case $\boldsymbol{F} = \boldsymbol{Q}$.

LEMMA 4. *In the notation of Lemma 3, let $S\in\mathscr{G}$ be of order m and let N be the normalizer of $\{S^i\}$. Then*

$$(N:\{S^i\}) = \frac{n}{m}\frac{\varphi(m)}{t},$$

where t is the number of elements in the division of S.

See Lemma 16.5.1 of [11]; we recall that the division of S is the set of all elements of the form

$$G^{-1}S^jG \quad \text{with} \quad G\in\mathscr{G} \quad \text{and} \quad (j, m) = 1.$$

LEMMA 5. *In the notation of Lemma 4, let E be the set of primes that belong to the division of S. Then*

(11) $$\sum_{\substack{p\leqslant x\\ p\in E}}\frac{\log p}{p} = \frac{t}{n}\log x+O(1).$$

Proof (based on the proof of Theorem 16.5 of [11]). We use induction on the order of S. If S is the identity element, then (11) follows from Lemma 1 since p has exactly n prime divisors of degree one in $\boldsymbol{K}$. Suppose that S has order $m>1$. Consider the subfield $\boldsymbol{L}$ of $\boldsymbol{K}$ fixed by $\{S^i\}$ and let $\boldsymbol{L} = \boldsymbol{Q}(z)$ (z integral). Let $F(x) = 0$ be the irreducible equation of z over $\boldsymbol{Q}$, and define $\varrho_F(p) = \varrho(p)$ as in the sentence containing (10); assume that $(z-z^{(i)}, p) = 1$, which excludes only a finite number of primes p. By Lemma 3, $\varrho(p) = 0$ for any prime p not belonging to the division of S^d for some d. Suppose now that p belongs to the division of S^d, where $d\mid m$. By Lemma 3, $\varrho(p)$ equals the number of integers v for which

$$T_vS^dT_v^{-1}\in\{S^i\}, \quad \text{or} \quad S^d\in T_v^{-1}\{S^i\}T_v.$$

If N_d is the normalizer of $\{S^{id}\}$, we have therefore by Lemma 4,

(12) $$\varrho(p) = (N_d:\{S^i\}) = (N_d:\{S^{id}\})/(\{S^i\}:\{S^{id}\}) = \frac{n}{m}\frac{\varphi(m/d)}{t_d}$$

where t_d is the number of elements in the division of S^d.

Let E_d denote the set of primes belonging to the division of S^d. We now use Lemma 2, (12) and our induction hypothesis that the result of the lemma is true if S has order less than m, and we have (since $t_1 = t$, $E_1 = E$)

$$\log x + O(1) = \sum_{p \le x} \varrho(p)\frac{\log p}{p} = \sum_{d|m} \frac{n}{m}\frac{\varphi(m/d)}{t_d} \sum_{\substack{p\le x\\ p\in E_d}} \frac{\log p}{p}$$

$$= \sum_{\substack{d|m\\ d>1}} \frac{\varphi(m/d)}{m}\big(\log x + O(1)\big) + \frac{n}{m}\frac{\varphi(m)}{t} \sum_{\substack{p\le x\\ p\in E}} \frac{\log p}{p}$$

$$= \left(1 - \frac{\varphi(m)}{m}\right)\big(\log x + O(1)\big) + \frac{n}{m}\frac{\varphi(m)}{t} \sum_{\substack{p\le x\\ p\in E}} \frac{\log p}{p}$$

whence

$$\sum_{\substack{p\le x\\ p\in E}} \frac{\log p}{p} = \frac{t}{n}\log x + O(1).$$

This completes the proof of the inductive step, and hence of the lemma.

LEMMA 6. *Let F be a monic polynomial without multiple roots over Q (but F need not be irreducible). Let $\mathscr{G}$ be the Galois group of F over Q. Let p be a prime belonging to the class of $S \in \mathscr{G}$ but not dividing the discriminant of F. If S consists of g cycles of length $l_1, \ldots, l_g$, then*

$$F(x) \equiv f_1(x) \ldots f_g(x) \pmod p$$

where f_i is irreducible $(\mathrm{mod}\, p)$ *and of degree l_i $(i = 1, \ldots, g)$.*

See Theorem 16.6 of [11].

Proof of Theorem 2. Our aim is to find an asymptotic formula for the sum

$$\sum_{\substack{p\le x\\ \varrho(p)>0}} p^{-1},$$

where $\varrho(p)$ is the number of integer solutions of the congruence

$$W(x) \equiv 0 \pmod p,$$

where W is an arbitrary non-constant polynomial with integer coefficients. Our sum will only alter by a constant if we replace W by a suitable polynomial F satisfying the conditions of Lemma 6. If p does not divide the discriminant of F, we have by Lemma 6 that $\varrho_F(p) > 0$ if and only if p belongs to a class whose elements have a cycle of length one. Since all classes in a division decompose into cycles in the same way, the last condition can be stated by saying that p belongs to a division whose elements have a cycle of length one. If t is the number of elements in the union of all such divisions and n is the order of $\mathscr{G}$, then by Lemma 5,

$$\sum_{\substack{p\le x\\ \varrho_W(p)>0}} \frac{\log p}{p} = \frac{t}{n}\log x + O(1).$$

On applying partial summation, we deduce that

$$\sum_{\substack{p\le x\\ \varrho_W(p)>0}} p^{-1} = \frac{t}{n}\log\log x + D + O\big((\log x)^{-1}\big),$$

where D is a constant, and the theorem follows.

3. Preliminary lemmas. As in the proof of Theorem 2, let $\mathscr{S}$ be the set of primes p for which $\varrho(p) > 0$, where $\varrho(p)$ is the number of solutions of

(13) $$W_1(x) \equiv 0 \pmod p,$$

and let $\mathscr{S}_0$ be the finite set of primes p such that $p\,|\,W_1(0)$ but (13) has no non-zero solution; thus (13) has a non-zero solution if and only if $p \in \mathscr{S}\setminus\mathscr{S}_0$, and $\mathscr{S}_0$ may of course be the empty set $\emptyset$.

Throughout p, q, r (with or without suffices) denote primes, and $C, C_1, C_2, \ldots, D, D_1, D_2, \ldots$ denote absolute constants (> 0 except perhaps D) unless the parameters on which they depend are explicitly indicated, for example by $C_2(m)$ in Lemma 12.

Let $1 > \varepsilon > 0$, $\delta > 0$ be arbitrary, and define

(14) $$y = (\log\log x)^{1-\varepsilon}, \quad z = (\log\log x)^{1+\varepsilon}, \quad u = (\log z)^{2\lambda+\delta}, \quad X = x^{1/(\log\log x)^2},$$

where λ is the constant of Theorems 1 and 2. Assume that x is sufficiently large for $p < y$ when $p\,|\,W_1(0)$.

LEMMA 7. *Let $T(x)$ be the set of all prime powers q^j such that $q^j \le u$ but $|W_j(q)| > y$, and suppose that for an infinite number of x, $T(x) \ne \emptyset$. Then to each $\eta > 0$, there exists x_0 such that for all $x > x_0$ for which $T(x) \ne \emptyset$, every $q^j \in T(x)$ satisfies*

$$q^j > \eta^{-2}.$$

Proof. Suppose that the lemma is false. Then there exists $\eta_0 > 0$ such that there are arbitrarily large x for which $T(x) \ne \emptyset$ but some $q^j \in T(x)$ satisfies $q^j \le \eta_0^{-2}$. Then q, j and therefore $W_j(q)$ are bounded, so

(15) $$|W_j(q)| \le D_1(\eta_0),$$

where $D_1(\eta_0)$ is a constant depending on η_0 but independent of x. Choose an x for which $y > D_1(\eta_0)$ and the conditions in the second sentence of the proof are satisfied; then on using (15) we obtain a contradiction, for we have for some $q^j \in T(x)$, $|W_j(q)| \leqslant D_1(\eta_0) < y$ but by definition of $T(x)$, $|W_j(q)| > y$.

COROLLARY. *Let $T^*(x)$ denote the set of all positive integers $n \leqslant u$ for which $f(n)$ is non-zero and has a prime divisor exceeding y, and suppose that for an infinite number of x, $T^*(x) \neq \emptyset$. Then to each $\eta > 0$, there exists x_0 such that for all $x > x_0$ for which $T^*(x) \neq \emptyset$, every $n \in T^*(x)$ satisfies*

$$n > \eta^{-2}.$$

Proof. If for some prime $p > y$, $p \mid f(n) \neq 0$, then there exists a prime power q^j such that $q^j \| n$ and $p \mid f(q^j) = W_j(q) \neq 0$. It follows that to each $n \in T^*(x)$, there corresponds at least one $q^j \| n$ with $q^j \in T(x)$, for

$$q^j \leqslant n \leqslant u \quad \text{and} \quad |f(n)| \geqslant |W_j(q)| \geqslant p > y.$$

Thus the result of the lemma is applicable and the corollary follows.

LEMMA 8. *If $p < (\log\log x)^{1-\varepsilon/2}$ and $(p, d) = 1$, then*

$$\sum_{\substack{q<X \\ q\equiv d \pmod p}} \frac{1}{q} > D_2 \frac{\log\log x}{p} > (\log\log x)^{\varepsilon/4},$$

where X is given by (14).

Since the relation

$$\sum_{\substack{p\leqslant v \\ p\equiv h \pmod k}} 1 = \pi(v; k, h) = \big(1+o(1)\big)\frac{v}{\varphi(k)\log v}$$

holds uniformly for $k < \log v$ whenever $(k, h) = 1$, this lemma follows in the same way as Lemma 1 of [1].

LEMMA 9. *If $p \in \mathscr{S} \setminus \mathscr{S}_0$ and $p < (\log\log x)^{1-\varepsilon/2}$, then*

$$\sum_{\substack{n\leqslant x \\ p\nmid f(n)}}' 1 = o\big(x/(\log\log x)^2\big),$$

where $\sum'$ signifies that the sum is over squarefree n satisfying the given conditions.

Proof. Let n denote a squarefree integer, so that

$$f(n) = \prod_{q\mid n} W_1(q).$$

Since $p \in \mathscr{S} \setminus \mathscr{S}_0$, the congruence (13) has a least positive solution d (so $0 < d < p$). If $p \nmid f(n)$, $(p, W_1(q)) = 1$ for all $q \mid n$ and hence in particular $q \not\equiv d \pmod p$. Thus

$$\sum_{\substack{n\leqslant x \\ p\nmid f(n)}}' 1 \leqslant \sum_{\substack{n\leqslant x \\ q\not\equiv d \pmod p \forall q\mid n}}' 1 \leqslant \sum_{\substack{n\leqslant x \\ q\nmid n \text{ if } q\equiv d \pmod p,\, q<X}} 1$$

where n is not restricted to squarefree integers in the final sum and where X is defined in (14). We can apply Brun's sieve to the sum on the right to obtain that it does not exceed

$$D_3 x \prod_{\substack{q<X \\ q\equiv d \pmod p}} (1-q^{-1}) < D_4 x \exp\big(-(\log\log x)^{\varepsilon/4}\big)$$

by Lemma 8, and the result follows. (For Brun's sieve, see pp. 71–78 of [10] or, for a more general form, Theorem 4 of [5].)

LEMMA 10. *For any prime p with $(p, d) = 1$,*

$$\sum_{\substack{p<q\leqslant x \\ q\equiv d \pmod p}} \frac{1}{q} < D_5 \frac{\log p + \log\log x}{p}.$$

This is proved in the same way as Lemma 2 of [1], on using the Brun–Titchmarsh inequality in the form quoted there (see Theorem 2 of [14]). Erdös considered the case $d = 1$.

Recall that $\varrho(q)$ denotes the number of roots (if any) of the congruence (13) (with q instead of p); when $\varrho(q) > 0$, let the roots be d_i $(i = 1, \ldots, \varrho(q))$. It is well known that $\varrho(q) \leqslant \max(q, k)$, where k is the degree of W_1.

LEMMA 11. *Let N_1 denote the number of squarefree integers $n \leqslant x$ such that $(n, f(n)) > 1$ and the least prime divisor in $\mathscr{S}$ of n (if there is one) exceeds z. Then*

$$N_1 = O\big(x/(\log\log x)^{\varepsilon}\big).$$

Proof. First we examine the implication of the condition $(n, f(n)) > 1$ for n contributing to N_1. Since n is squarefree and $(n, W_1(0)) = 1$, on using that $p \mid W_1(0)$ implies that $p \in \mathscr{S}$ and $p < y$, we conclude that there exist distinct primes q, r dividing n such that $(q, W_1(r)) > 1$, whence $\varrho(q) > 0$ (so $q \in \mathscr{S}$) and $r \equiv d_i \pmod q$ for some i, $1 \leqslant i \leqslant \varrho(q)$, where $d_i \neq 0$. Moreover since $q \in \mathscr{S}$ and $q \mid n$, $q > z$. Also $W_1(r) \neq 0$ for $r \mid n$, for otherwise $r \mid W_1(0)$ contradicting $(n, W_1(0)) = 1$, and it is easily seen that

$$|W_1(r)| \ll r^k \quad (\forall r \geqslant 2),$$

where as usual $\ll$ means the same as the O-notation. Hence since $q \mid W_1(r)$,

$$q \leqslant |W_1(r)| \ll r^k.$$

and so

$$r \geqslant q^{\varkappa} \quad \text{where} \quad \varkappa = k^{-1}.$$

Using these facts, we see that the number N_1 of the lemma does not exceed the number of multiples not exceeding x of qr, where q and r satisfy the conditions derived above, whence, since each $d_i \neq 0$,

$$N_1 \leqslant \sum_{\substack{z<q\leqslant x\\ q\in\mathscr{S}}} \sum_{i=1}^{\varrho(q)} \sum_{\substack{q^{\varkappa}\leqslant r\leqslant x\\ r\equiv d_i \pmod q}} \frac{x}{qr}$$

$$= O\left(x \sum_{\substack{z<q\leqslant x\\ q\in\mathscr{S}}} \frac{1}{q}\varrho(q)\left\{\frac{\log q+\log\log x}{q}+q^{-\varkappa}\right\}\right)$$

by Lemma 10, where we include the term $q^{-\varkappa}$ to account for an r satisfying $q^{\varkappa} \leqslant r < q$. Since $\varrho(q) \leqslant k$ and since by well known arguments

$$\sum_{z<q\leqslant x} \frac{\log q}{q^2} = O\left(\frac{\log z}{z}\right), \qquad \sum_{z<q\leqslant x} q^{-2} = O(z^{-1}), \qquad \sum_{z<q\leqslant x} q^{-1-\varkappa} = O(z^{-\varkappa}),$$

we have

$$N_1 = O\left(x\left(\frac{\log z}{z}+\frac{\log\log x}{z}+z^{-\varkappa}\right)\right) = O\left(x/(\log\log x)^{\varepsilon}\right).$$

Definition. Let $B_v(m;x)$ denote the number of squarefree positive integers $n \leqslant x$ such that $(n, m) = 1$, where m is independent of n, and such that the least prime divisor in $\mathscr{S}$ of n exceeds v.

Lemma 12. *Let m be an integer with $p \leqslant v$ for all $p \mid m$. Then*

$$B_v(m;x) = C_2(m)\left(1+O\left((\log v)^{-1}\right)\right)\frac{x}{(\log v)^{\lambda}} + O\left(\frac{x}{v}+4^v\right),$$

where the O-constants are independent of m, and where

$$C_2(m) = C_1 \prod_{p\notin\mathscr{S}} (1-p^{-2}) \prod_{\substack{p\notin\mathscr{S}\\ p\mid m}} (1+p^{-1})^{-1}. \tag{16}$$

Here C_1 and λ are the constants of (9), and $C_2(m)$ satisfies

$$0 < C_2(m) \leqslant C_1 \prod_{p\notin\mathscr{S}} (1-p^{-2}) = C_3. \tag{17}$$

(In our applications, v will be small compared with x, but $v\to\infty$ as $x\to\infty$, and m may be large.)

Proof. Define $B_v^*(m;x)$ similarly to $B_v(m;x)$ except that the word "squarefree" is omitted. Then any n contributing to $B_v^*(m;x)$ but not to $B_v(m;x)$ is divisible by a square of a prime p. The number of integers $n \leqslant x$ that are divisible by p^2 with $p > v$ does not exceed

$$\sum_{p>v} \frac{x}{p^2} = O\left(\frac{x}{v}\right).$$

We calculate the number N_2 of positive integers $\leqslant x$ that are not divisible by a square of a prime $p \leqslant v$, by a prime $p \mid m$ (so $p \leqslant v$), or by a prime $p \in \mathscr{S}$ with $p \leqslant v$; from the above remark, it follows that

$$B_v(m;x) = N_2 + O\left(\frac{x}{v}\right). \tag{18}$$

We use the ideas of the sieve of Eratosthenes. $B_v^*(m;t)$ is the number of positive integers $\leqslant t$ that are not divisible by a prime dividing m or $\in\mathscr{S}$ and $\leqslant v$. Hence

$$B_v^*(m;t) = [t] - \sum_{\substack{p\leqslant v\\ p\mid m \text{ or } p\in\mathscr{S}}} \left[\frac{t}{p}\right] + \sum_{\substack{p_1<p_2\leqslant v\\ p_i\mid m \text{ or } \in\mathscr{S}\ (i=1,2)}} \left[\frac{t}{p_1p_2}\right] - \ldots, \tag{19}$$

the sum clearly being finite. Moreover by similar reasoning,

$$N_2 = B_v^*(m;x) - \sum_{\substack{q\leqslant v\\ q\notin\mathscr{S}\\ q\nmid m}} B_v^*\left(m;\frac{x}{q^2}\right) + \sum_{\substack{q_1<q_2\leqslant v\\ \left.\begin{smallmatrix} q_i\notin\mathscr{S}\\ q_i\nmid m\end{smallmatrix}\right\}(i=1,2)}} B_v^*\left(m;\frac{x}{q_1^2q_2^2}\right) - \ldots \tag{20}$$

On substituting (19) with $t = x,\ x/q^2,\ x/q_1^2q_2^2, \ldots$ in turn in (20), we obtain an expression, the number of whose terms does not exceed

$$\left(1+\binom{\pi(v)}{1}+\binom{\pi(v)}{2}+\ldots+\binom{\pi(v)}{\pi(v)}\right)^2 = 2^{2\pi(v)} \leqslant 4^v.$$

This number is the error incurred by replacing $[t]$ by t throughout in (20) after the substitution of (19). Hence using (18) we have

(21) $B_v(m;x)$

$$= \left\{x - \sum_{\substack{p\leqslant v\\ p\mid m \text{ or } p\in\mathscr{S}}} \frac{x}{p} + \sum_{\substack{p_1<p_2\leqslant v\\ p_i\mid m \text{ or } \in\mathscr{S}\ (i=1,2)}} \frac{x}{p_1p_2} - \ldots\right\} -$$

$$- \sum_{\substack{q\leqslant v\\ q\notin\mathscr{S}\\ q\nmid m}} \left\{\frac{x}{q^2} - \sum_{\substack{p\leqslant v\\ p\mid m \text{ or } p\in\mathscr{S}}} \frac{x}{q^2p} + \sum_{\substack{p_1<p_2\leqslant v\\ p_i\mid m \text{ or } \in\mathscr{S}\ (i=1,2)}} \frac{x}{q^2p_1p_2} - \ldots\right\} +$$

$$+ \sum_{\substack{q_1<q_2\leqslant v\\ q_1,q_2\notin\mathscr{S}\\ q_1,q_2\nmid m}} \left\{\frac{x}{q_1^2q_2^2} - \sum_{\substack{p\leqslant v\\ p\mid m \text{ or } p\in\mathscr{S}}} \frac{x}{q_1^2q_2^2p} + \sum_{\substack{p_1<p_2\leqslant v\\ p_i\mid m \text{ or } \in\mathscr{S}\ (i=1,2)}} \frac{x}{q_1^2q_2^2p_1p_2} - \ldots\right\} -$$

$$-\ldots+O(4^v)+O\left(\frac{x}{v}\right)$$

$$= x \prod_{\substack{p\leqslant v\\ p|m \text{ or } p\in\mathscr{S}}} (1-p^{-1}) \prod_{\substack{q\leqslant v\\ q\notin\mathscr{S}\\ q\nmid m}} (1-q^{-2})+O\left(4^v+\frac{x}{v}\right);$$

the O-constant is independent of m.

We now consider the main term. The product

$$\prod_{q\notin\mathscr{S},\, q\nmid m} (1-q^{-2})$$

is convergent, and

(22) $$1\geqslant \prod_{\substack{q>v\\ q\notin\mathscr{S}}} (1-q^{-2}) \geqslant \prod_{q>v} (1-q^{-2}) = \exp\Big(-\sum_{q>v} q^{-2}+O\Big(\sum_{q>v} q^{-4}\Big)\Big)$$

$$\geqslant \exp\left(O\left(\frac{1}{v}\right)\right) = 1+O\left(\frac{1}{v}\right).$$

Thus, on observing that we do not need to distinguish between the primes p and q any more, and on using (9), (21) and (22), we have

$$B_v(m;x) = x\prod_{p\notin\mathscr{S}} (1-p^{-2}) \prod_{\substack{p\notin\mathscr{S}\\ p|m}} \left(\frac{1-p^{-1}}{1-p^{-2}}\right) \prod_{\substack{p\leqslant v\\ p\in\mathscr{S}}} (1-p^{-1})\left(1+O\left(\frac{1}{v}\right)\right)+$$

$$+O\left(4^v+\frac{x}{v}\right)$$

$$= C_2(m)\left(1+O\big((\log v)^{-1}\big)\right)\frac{x}{(\log v)^{\lambda}}+O\left(4^v+\frac{x}{v}\right),$$

where $C_2(m)$ is given by (16). It follows from the proof that the O-constants are independent of m. The inequality (17) follows immediately.

Definition. Let $D_p(m;x)$ denote the number of squarefree positive integers $n\leqslant x$ with $(n,m)=1$ and with p the least prime divisor in $\mathscr{S}$ of n.

Corollary. *If m has no prime divisor exceeding p, then in the notation of the lemma (with $v=p$)*

$$D_p(m;x) = B_p\left(m;\frac{x}{p}\right) = C_2(m)\left(1+O\big((\log p)^{-1}\big)\right)\frac{x}{p(\log p)^{\lambda}}+O\left(\frac{x}{p^2}+4^p\right).$$

4. Proof of Theorem 1. Every integer n can be written uniquely in the form

(23) $n = n_1n_2$ where $(n_1,n_2)=1$, n_1 squarefree, n_2 squarefull;

these meanings for n_1 and n_2 will be assumed throughout the rest of the paper. Since f is multiplicative,

$$1 = (n,f(n)) = (n_1n_2, f(n_1)f(n_2))$$

if and only if

$$1 = (n_1,f(n_1))\,(n_1,f(n_2))\,(n_2,f(n_1))\,(n_2,f(n_2)).$$

Hence

(24) $$\sum_{\substack{n\leqslant x\\ (n,f(n))=1}} 1 = \sum_{\substack{n_2\leqslant x\\ (n_2,f(n_2))=1}} \sum_{\substack{n_1\leqslant x/n_2\\ (n_1,n_2f(n_2))=1\\ (f(n_1),n_2)=1\\ (n_1,f(n_1))=1}} 1 = \sum_{\substack{n_2\leqslant x\\ (n_2,f(n_2))=1}} \Sigma(n_2;x),$$

say, and in order to estimate the sum on the right, we must investigate further $\Sigma(n_2;x)$. We accomplish our aim by considering four different categories of integers n_2. We can certainly assume that $f(n_2)\neq 0$ for every n_2 considered, since we require $(n_2,f(n_2))=1$ in the sum over n_2.

Case 1: $n_2>u$. We use a result of Erdös and Szekeres [3] on squarefull integers, which states that

(25) $$\sum_{n_2\leqslant x} 1 = \big(C_4+h(x)\big)x^{1/2}$$

where $C_4=\zeta(3/2)\zeta^{-1}(3)$ and $h(x)=o(1)$. Hence by partial summation

(26) $$\sum_{n_2\leqslant x}\frac{1}{n_2} = C_5-C_4x^{-1/2}+o(x^{-1/2}),$$

where

$$C_5 = 2C_4+\int_1^{\infty} h(t)t^{-3/2}\,dt,$$

the integral being convergent since $h(x)=o(1)$. Using (14), it follows that

$$\sum_{u<n_2\leqslant x}\frac{1}{n_2} = C_4u^{-1/2}+o(u^{-1/2}) = O\big((\log z)^{-\lambda-\delta/2}\big).$$

From the trivial estimate

$$\Sigma(n_2;x)\leqslant x/n_2,$$

we obtain

(27) $$\sum_{\substack{u<n_2\leqslant x\\ (n_2,f(n_2))=1}} \Sigma(n_2;x)\leqslant x\sum_{u<n_2\leqslant x}\frac{1}{n_2} = O\big(x/(\log z)^{\lambda+\delta/2}\big).$$

Case 2: $n_2 \leqslant u$ and $q \mid n_2$ for some $q \in \mathscr{S} \setminus \mathscr{S}_0$. Since $q \mid n_2$ for some $q \in \mathscr{S} \setminus \mathscr{S}_0$, we have

$$\big(n_2, f(n_1)\big) = 1 \Rightarrow q \nmid f(n_1)$$

and hence

$$\Sigma(n_2; x) \leqslant \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_2, f(n_1))=1}} 1 \leqslant \sum_{\substack{n_1 \leqslant x/n_2 \\ q \nmid f(n_1)}} 1 = o\left(\frac{x}{n_2}\left(\log\log\frac{x}{n_2}\right)^{-2}\right),$$

by Lemma 9 since certainly

$$q \leqslant u < \left(\log\log\frac{x}{n_2}\right)^{1-\varepsilon/2} \quad \text{for} \quad 1 \leqslant n_2 \leqslant u.$$

Hence by (26)

$$(28) \qquad \sum_{\substack{n_2 \leqslant u \\ (n_2, f(n_2))=1 \\ \exists q \mid n_2 : q \in \mathscr{S} \setminus \mathscr{S}_0}} \Sigma(n_2; x) = o\left(\frac{x}{(\log\log x)^2}\sum_{n_2 \leqslant u}\frac{1}{n_2}\right) = o\left(\frac{x}{(\log\log x)^2}\right).$$

Case 3: $n_2 \leqslant u$, $q \notin \mathscr{S} \setminus \mathscr{S}_0 \,\forall q \mid n_2$, and $q \leqslant y \,\forall q \mid f(n_2)$. This case will give us our main term. We first split the sum $\Sigma(n_2; x)$ (a sum over n_1) into three parts; in $\Sigma'(n_2; x)$ the least prime divisor in $\mathscr{S}$ of n_1 is less than y, in $\Sigma''(n_2; x)$ the least prime divisor p in $\mathscr{S}$ of n_1 satisfies $y \leqslant p \leqslant z$, and $\Sigma'''(n_2; x)$ contains the remaining n_1. Note that $\big(n_1, W_1(0)\big) = 1$ if n_1 contributes to $\Sigma''(n_2; x)$, or $\Sigma'''(n_2; x)$, for if $q \mid \big(n_1, W_1(0)\big)$, $q \mid n_1$ whilst $q \in \mathscr{S}$ and $q < y$ by the remark before Lemma 7. The first two sums contribute to our error term, and we deal with these first.

We have by Lemma 9, since $y \leqslant (\log\log x/u)^{1-\varepsilon/2} \leqslant (\log\log x/n_2)^{1-\varepsilon/2}$ certainly for $n_2 \leqslant u$, and since if $p \in \mathscr{S}_0$ for some $p \mid n_1$, $\big(n_1, f(n_1)\big) \geqslant \big(p, f(p)\big) = p$,

$$(29) \qquad \Sigma'(n_2; x) \leqslant \sum_{\substack{p<y \\ p \in \mathscr{S}}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, f(n_1))=1 \\ p = \text{least prime } \in \mathscr{S} \text{ and } \mid n_1}} 1 \leqslant \sum_{\substack{p<y \\ p \in \mathscr{S} \setminus \mathscr{S}_0}} \sum_{\substack{n_1 \leqslant x/n_2 \\ p \nmid f(n_1)}} 1$$

$$< y\, o\left(\frac{x}{n_2}\left(\log\log\frac{x}{n_2}\right)^{-2}\right) = \frac{1}{n_2}\, o\big(xy(\log\log x)^{-2}\big)$$

$$= \frac{1}{n_2}\, o\big(x(\log\log x)^{-1}\big),$$

since $n_2 \leqslant u$.

For $\Sigma''(n_2; x)$ we use the Corollary of Lemma 12 since we are assuming that $q \mid f(n_2)$ implies that $q \leqslant y \leqslant p$. We have

$$(30) \qquad \Sigma''(n_2; x) \leqslant \sum_{\substack{y \leqslant p \leqslant z \\ p \in \mathscr{S}}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, n_2 f(n_2))=1 \\ p = \text{least prime } \mid n_1 \text{ and } \in \mathscr{S}}} 1 = \sum_{\substack{y \leqslant p \leqslant z \\ p \in \mathscr{S}}} D_p\left(n_2 f(n_2); \frac{x}{n_2}\right)$$

$$= \sum_{\substack{y \leqslant p \leqslant z \\ p \in \mathscr{S}}} \left\{C_2\big(n_2 f(n_2)\big)\big(1 + O((\log p)^{-1})\big)\frac{x}{n_2 p (\log p)^{\lambda}} + O\left(4^p + \frac{x}{n_2 p^2}\right)\right\}$$

$$\leqslant \frac{C_3}{n_2}\big(1 + O((\log y)^{-1})\big)\frac{x}{(\log y)^{\lambda}}\sum_{\substack{y \leqslant p \leqslant z \\ p \in \mathscr{S}}}\frac{1}{p} + O\left(4^z + \frac{x}{n_2 y}\right)$$

$$= \frac{C_3}{n_2}\big(1 + O((\log z)^{-1})\big)\frac{x}{(\log z)^{\lambda}}\, o(1) + \frac{1}{n_2}\, O\big(x/(\log\log x)^{1-\varepsilon}\big).$$

We are assuming that $n_2 \leqslant u$ in calculating the error terms, and using (17), the definitions of y, z, u in (14) and the estimate for $\sum_{p \leqslant v, p \in \mathscr{S}} 1/p$ given by Theorem 2. We recall that the O-constants stand for something not dependent on n_2.

Finally we have to consider $\Sigma'''(n_2; x)$ and we show that this supplies our main term. More precisely we show first that

$$\Sigma'''(n_2; x) \sim B_z\big(n_2 f(n_2); x/n_2\big).$$

Note that $\big(n_2, f(n_1)\big) = 1$ for all n_1 contributing to $\Sigma'''(n_2; x)$; for $q \mid n_2$ implies that $q \notin \mathscr{S} \setminus \mathscr{S}_0$, and if $q \in \mathscr{S}_0$ and $q \mid n_2$, then $q \nmid n_1$ since $(n_1, n_2) = 1$, and so it easily follows that $q \nmid f(n_1) = \prod_{r \mid n_1} W_1(r)$ for any $q \mid n_2$. Let

$$(31) \qquad m = n_2 f(n_2).$$

Then

$$0 \leqslant B_z(m; x/n_2) - \Sigma'''(n_2; x) = \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, m)=1 \\ p \mid n_1, p \in \mathscr{S} \Rightarrow p > z}} 1 - \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, m)=1 \\ p \mid n_1, p \in \mathscr{S} \Rightarrow p > z \\ (n_1, f(n_1))=1}} 1$$

$$= \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, m)=1 \\ p \mid n_1, p \in \mathscr{S} \Rightarrow p > z \\ (n_1, f(n_1))>1}} 1 \leqslant N_1 = O\left(\frac{x}{n_2}\left(\log\log\frac{x}{n_2}\right)^{-\varepsilon}\right) = \frac{1}{n_2}\, O\big(x(\log\log x)^{-\varepsilon}\big)$$

by Lemma 11. Hence by Lemma 12 (since the condition $q \mid f(n_2) \Rightarrow q \leqslant y < z$ holds)

$$\Sigma'''(n_2; x) = B_z(m; x/n_2) + \frac{1}{n_2} O\big(x/(\log\log x)^s\big)$$

$$= C_2(m)\left(1 + O\left(\frac{1}{\log z}\right)\right)\frac{x}{n_2(\log z)^{\lambda}} + O\left(\frac{x}{zn_2} + 4^z\right) + \frac{1}{n_2} O\left(\frac{x}{(\log\log x)^s}\right)$$

where m is given by (31) and depends on n_2, and $C_2(m)$ is stated precisely in (16). Thus by (29) and (30), we have in case 3,

$$\Sigma(n_2; x) = \frac{C_2\big(n_2 f(n_2)\big)}{n_2}\left(1 + O\left(\frac{1}{\log z}\right)\right)\frac{x}{(\log z)^{\lambda}} + \frac{1}{n_2} o\left(\frac{x}{(\log z)^{\lambda}}\right). \tag{32}$$

From the Corollary to Lemma 7, it follows that if $n_2 \leqslant \eta^{-2}$, every prime divisor of $f(n_2)$ is less than or equal to y if $x > x_0(\eta)$, and by (17) and (26)

$$0 \leqslant \sum_{\substack{n_2 > \eta^{-2} \\ f(n_2) \neq 0}} \frac{C_2\big(n_2 f(n_2)\big)}{n_2} \leqslant C_3 \sum_{n_2 > \eta^{-2}} \frac{1}{n_2} = O(\eta) = o(1)$$

as $x \to \infty$ since $\eta > 0$ is arbitrary. Note that $C_2(m)$ is defined for any integer $m \neq 0$, although Lemma 12 requires that m has no large prime divisors. We can write

$$\sum_{\substack{n_2 \leqslant u \\ \dots\dots}} = \sum_{\substack{n_2 \leqslant \eta^{-2} \\ \dots\dots}} + \sum_{\substack{\eta^{-2} < n_2 \leqslant u \\ \dots\dots}}, \qquad \sum_{\substack{n_2 \leqslant \eta^{-2} \\ \dots\dots}} = \sum_{\substack{n_2 = 1 \\ \dots\dots}}^{\infty} - \sum_{\substack{n_2 > \eta^{-2} \\ \dots\dots}},$$

and using these remarks and (26) after substituting (32) in the left side below, we have

$$\sum_{\substack{n_2 \leqslant u \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2 \\ q \leqslant y \forall q \mid f(n_2)}} \Sigma(n_2; x) = \frac{x}{(\log z)^{\lambda}}\left(1 + O\left(\frac{1}{\log z}\right)\right)\big(C_6 + o(1)\big) + o\left(\frac{x}{(\log z)^{\lambda}}\right) \tag{33}$$

$$= \big(C_6 + o(1)\big)\frac{x}{(\log\log\log x)^{\lambda}},$$

where by (16)

$$C_6 = \sum_{\substack{n_2 = 1 \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}}^{\infty} \frac{C_2\big(n_2 f(n_2)\big)}{n_2} \tag{34}$$

$$= C_1 \prod_{p \notin \mathscr{S}} (1 - p^{-2}) \sum_{\substack{n_2 = 1 \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}}^{\infty} \frac{1}{n_2} \prod_{\substack{p \mid n_2 f(n_2) \\ p \notin \mathscr{S}}} (1 + p^{-1})^{-1}.$$

Case 4: $n_2 \leqslant u$, $q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2$, and $\exists q \mid f(n_2)$ with $q > y$. From the Corollary to Lemma 7, we know that every n_2 in this case satisfies $n_2 > \eta^{-2}$ if $x > x_0(\eta)$. To deal with this part of the sum over n_2, we show that we can replace f there by a simpler polynomial-like multiplicative function f_1 that does not have unduly large prime divisors, but which has the same values as f at primes so that the set $\mathscr{S}$ remains unchanged; in this way we find a bound for our sum. Define f_1, a multiplicative function, by

$$f_1(p) = f(p) = W_1(p) \quad \text{for all primes } p,$$
$$f_1(p^j) = 1 \quad \text{for all primes } p \text{ and all } j \geqslant 2.$$

Then in the notation of (23),

$$f_1(n) = f_1(n_1) = f(n_1), \quad f_1(n_2) = 1 \tag{35}$$

and $f_1(n) \mid f(n)$ for all n. Hence

$$\big(n, f(n)\big) = 1 \Rightarrow \big(n, f_1(n)\big) = 1 \tag{36}$$

and $\big(n_1, n_2 f_1(n_2)\big) = (n_1, n_2)$. As we saw in case 3, the conditions $(n_1, n_2) = 1$ and $q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2$ imply that $\big(n_2, f_1(n_1)\big) = \big(n_2, f(n_1)\big) = 1$. It follows that

$$\sum_{\substack{n_2 \leqslant u \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2 \\ \exists q \mid f(n_2): q > y}} \Sigma(n_2; x) \leqslant \sum_{\substack{\eta^{-2} < n_2 \leqslant u \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, n_2 f(n_2)) = 1 \\ (n_1, f(n_1)) = 1}} 1 \leqslant \sum_{\substack{\eta^{-2} < n_2 \leqslant u \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, n_2) = 1 \\ (n_1, f_1(n_1)) = 1}} 1 \tag{37}$$

on using (35) and (36). Since the sets $\mathscr{S}$, $\mathscr{S}_0$ are the same for f_1 as for f and $f_1(n_2) = 1$, f_1 fulfils the conditions of case 3 above, and so for the n_2 under consideration here, (32) holds with f replaced by f_1. Then (37) gives

$$\sum_{\substack{n_2 \leqslant u \\ (n_2, f(n_2)) = 1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2 \\ \exists q \mid f(n_2): q > y}} \Sigma(n_2; x) \leqslant \sum_{\substack{\eta^{-2} < n_2 \leqslant u \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}} \frac{C_2(n_2) + o(1)}{n_2} \frac{x}{(\log z)^{\lambda}} \tag{38}$$

$$\leqslant \big(C_3 + o(1)\big)\left(\sum_{n_2 > \eta^{-2}} \frac{1}{n_2}\right)\frac{x}{(\log z)^{\lambda}} = o\left(\frac{x}{(\log z)^{\lambda}}\right)$$

as $x \to \infty$ since $\eta > 0$ is arbitrary and (26) holds.

Combining (24), (27), (28), (33), (38), we obtain

$$\sum_{\substack{n \leqslant x \\ (n, f(n)) = 1}} 1 = \big(C_6 + o(1)\big)\frac{x}{(\log\log\log x)^{\lambda}}$$

as $x \to \infty$, where C_6 is given by (34), which is Theorem 1.

5. Some special cases of the theorem. In some cases, C_6 has a simple expression, and next we investigate some instances of this. We also consider some special functions f. We state our results as Corollaries to Theorem 1. To prove Corollaries 1, 2, 3 below directly, the above argument may be simplified; for example, for Corollary 1, we need not write $n = n_1 n_2$ as was done in § 4, and it is sufficient to evaluate $B_v^*(1; x)$ instead of establishing the more general Lemma 12. However the main ideas remain the same.

COROLLARY 1. *If f is strongly or completely multiplicative, then*

$$C_6 = C_1.$$

Proof. The argument in the two cases is essentially the same since in both cases $(n, f(n)) = 1$ if and only if $p \nmid W_1(q)$ for all primes p, q dividing n; for we have, for all primes p and $j = 1, 2, \ldots$,

$$f(p^j) = W_1(p) \text{ or } (W_1(p))^j$$

according as f is strongly or completely multiplicative.

By definition of $\mathscr{S}$, $q \mid f(n) \Rightarrow q \in \mathscr{S}$; also if $q \mid n_2$ and $q \in \mathscr{S}_0$, so $q \mid W_1(0)$, then $(n_2, f(n_2)) > 1$, whilst if $q \mid n_2 \Rightarrow q \notin \mathscr{S}$, then $(n_2, f(n_2)) = 1$. Hence by (34),

$$C_6 = C_1 \prod_{p \notin \mathscr{S}} (1-p^{-2}) \sum_{\substack{n_2=1 \\ q \notin \mathscr{S} \forall q \mid n_2}}^{\infty} \frac{1}{n_2} \prod_{q \mid n_2} (1+q^{-1})^{-1}.$$

Since the sum is absolutely convergent by (26) and the terms are multiplicative, we have (on recalling that n_2 is squarefull) that

$$\sum_{\substack{n_2=1 \\ q \notin \mathscr{S} \forall q \mid n_2}}^{\infty} \frac{1}{n_2} \prod_{q \mid n_2} (1+q^{-1})^{-1} = \prod_{q \notin \mathscr{S}} \{1+(1+q^{-1})^{-1}(q^{-2}+q^{-3}+\ldots)\}$$
$$= \prod_{q \notin \mathscr{S}} (1-q^{-2})^{-1}.$$

Hence, as claimed,

$$C_6 = C_1.$$

COROLLARY 2. *If $\mathscr{S} \setminus \mathscr{S}_0$ is the set of all primes, then*

$$C_6 = C_1 = e^{-\gamma} \quad \textit{and} \quad \lambda = 1.$$

Proof. In this case, from (16), $C_2(m) = C_1$ (independent of m), and the sum over n_2 for C_6, given in (34), has one term only, namely $n_2 = 1$. Hence, since

$$\prod_{p \leqslant x} (1-p^{-1}) \sim \frac{e^{-\gamma}}{\log x},$$

we have the result of the corollary.

Taking $f = \varphi$, we obtain (4). This corollary is a special case of

COROLLARY 3. *If for each n_2, $q \mid f(n_2) \Rightarrow q \in \mathscr{S} \setminus \mathscr{S}_0$, then*

$$C_6 = C_1 \prod_{p \in \mathscr{S}_0} \left(1 + \frac{1}{p(p-1)}\right).$$

Proof. If for $j = 2, 3, \ldots$

$$\mathscr{S}_j = \{p : \exists x \text{ with } p \mid W_j(x)\},$$

then for this corollary,

$$\bigcup_{j \geqslant 2} \mathscr{S}_j \subseteq \mathscr{S} \setminus \mathscr{S}_0.$$

If $q \mid n_2 \Rightarrow q \notin \mathscr{S} \setminus \mathscr{S}_0$, it follows that $q \nmid f(n_2)$ and $(n_2, f(n_2)) = 1$. Hence by (34), since $q \notin \mathscr{S} \setminus \mathscr{S}_0$ means that $q \notin \mathscr{S}$ or $q \in \mathscr{S}_0$,

$$C_6 = C_1 \prod_{p \notin \mathscr{S}} (1-p^{-2}) \sum_{\substack{n_2=1 \\ q \notin \mathscr{S} \setminus \mathscr{S}_0 \forall q \mid n_2}}^{\infty} \frac{1}{n_2} \prod_{\substack{q \mid n_2 \\ q \notin \mathscr{S}_0}} (1+q^{-1})^{-1}$$
$$= C_1 \prod_{p \notin \mathscr{S}} (1-p^{-2}) \prod_{q \notin \mathscr{S}} \{1+(1+q^{-1})^{-1}(q^{-2}+q^{-3}+\ldots)\} \times$$
$$\times \prod_{q \in \mathscr{S}_0} (1+q^{-2}+q^{-3}+\ldots)$$
$$= C_1 \prod_{q \in \mathscr{S}_0} \left(1 + \frac{1}{q(q-1)}\right).$$

In particular, this corollary holds if $f(n_2) = 1$ for all squarefull n_2 (see the function f_1 of case 4 of § 4 and Corollary 4 below).

Let $C_6(f)$ denote the constant given by (34) for the particular function f. Next we compare the results for general f with those for three related multiplicative functions f_1, f_2, f^*, where f_1 was introduced in the proof of case 4 of § 4. For all primes p, define

$$f_1(p) = f_2(p) = f(p) = W_1(p),$$
$$f_1(p^j) = 1 \quad \text{and} \quad f_2(p^j) = 0 \quad \text{for } j = 2, 3, \ldots,$$

and let f^* be the strongly or completely multiplicative function given by

$$f^*(p) = f(p) = W_1(p) \quad \text{for all primes } p;$$

as in Corollary 1, it does not matter which of the two possible functions f^* we select. We observe that the sets $\mathscr{S}$, $\mathscr{S}_0$ and the constant λ of Theorem 2 are the same for f, f_1, f_2, f^* since $W_1(x)$ plays the same role for all four functions. If f satisfies the conditions of Theorem 1, so do f_1, f_2, f^* (with the same λ) and we have

COROLLARY 4.

$$C_3 = C_6(f_2) \leqslant C_6(f) \leqslant C_6(f_1) = C_6(f^*) \prod_{p \in \mathscr{S}_0} \left(1 + \frac{1}{p(p-1)}\right).$$

In particular if $\mathscr{S}_0 = \emptyset$,

$$C_6(f) \leqslant C_6(f^*) = C_1.$$

Proof. As we remarked above, $C_6(f_1)$ is given by Corollary 3, and $C_6(f^*)$ by Corollary 1, so the relation between them is immediate. The inequality $C_6(f) \leqslant C_6(f_1)$ follows from applying (34) to f and to f_1 and comparing the corresponding sums; for clearly $f_1(n_2) = 1$, $(n_2, f(n_2)) = 1 \Rightarrow (n_2, f_1(n_2)) = 1$, and

$$\prod_{\substack{p \mid n_2 f(n_2) \\ p \notin \mathscr{S}}} (1+p^{-1})^{-1} \leqslant \prod_{\substack{p \mid n_2 \\ p \notin \mathscr{S}}} (1+p^{-1})^{-1}.$$

Note that $C_6(f) < C_6(f_1)$ if for some n_2 contributing to the sum in $C_6(f_1)$, $f(n_2)$ has a prime divisor $\notin \mathscr{S}$ or $(n_2, f(n_2)) > 1$. Since $f_2(n_2) = 0$ for all $n_2 > 1$, the sum in $C_6(f_2)$ consists of the term $n_2 = 1$ only, and so by (17), $C_6(f_2) = C_3$; hence the left hand inequality of the corollary follows from (34), and this inequality will be strict if the sum in $C_6(f)$ has a non-zero contribution from some $n_2 > 1$.

If $\mathscr{S}_0 = \emptyset$, so that whenever (13) has a zero solution, it also has a non-zero solution, the given result follows immediately from above and Corollary 1.

We consider next the divisor functions σ_ν defined in (6).

COROLLARY 5. *If* ν *is odd*,

$$\sum_{\substack{n \leqslant x \\ (n, \sigma_\nu(n)) = 1}} 1 \sim \frac{e^{-\gamma} x}{\log\log\log x}.$$

If ν *is even and* $2^\beta \| \nu$,

$$\sum_{\substack{n \leqslant x \\ (n, \sigma_\nu(n)) = 1}} 1 \sim \frac{C_6 x}{(\log\log\log x)^{2^{-\beta}}},$$

where $C_6 = C_6(\sigma_\nu)$ *is the constant given by* (34).

Proof. For $f(n) = \sigma_\nu(n)$ (ν a positive integer), we have

$$W_j(x) = x^{\nu j} + x^{\nu(j-1)} + \ldots + x^\nu + 1.$$

Write $\nu = 2^\beta \nu_1$ where ν_1 is odd and $\beta \geqslant 0$. Then the set $\mathscr{S}$ consists of the prime 2 and all primes p of the form $p \equiv 1 (\mathrm{mod}\, 2^{\beta+1})$, and $\mathscr{S}_0 = \emptyset$; for

$$W_1(x) = x^\nu + 1 \equiv 0 (\mathrm{mod}\, p)$$

has a solution (necessarily non-zero) if and only if $p = 2$ or $(\nu, p-1) \mid \frac{1}{2}(p-1)$. We observe that if ν is odd, so $\beta = 0$, $\mathscr{S}$ is the set of all primes and so the result follows by Corollary 2. If ν is even, the result follows by a well known case of Theorem 2, and Theorem 1.

Finally, as mentioned in § 1, we investigate what happens when the degree of W_1 is zero, for example in the case $f = \tau$ (see (6)). This situation was excluded in the above discussion, but we can apply our method to obtain the corresponding result. We have

THEOREM 3. *If* $W_1(x)$ *is a non-zero constant, then as* $x \to \infty$

$$\sum_{\substack{n \leqslant x \\ (n, f(n)) = 1}} 1 = (C_7 + o(1)) x$$

where C_7 *is a constant satisfying* $0 < C_7 \leqslant 1$.

Proof. Let

$$W_1(x) = k \neq 0.$$

If n_1 is squarefree,

$$f(n_1) = k^{\omega(n_1)}$$

where $\omega(n_1)$ is the number of prime divisors of n_1. Consider the sum in (24); since for $n_1 \neq 1$, $(n_1 n_2, f(n_1)) = 1$ if and only if $(n_1 n_2, k) = 1$, we have

$$\sum_{\substack{n \leqslant x \\ (n, f(n)) = 1}} 1 = \sum_{\substack{n_2 \leqslant x \\ (n_2, k f(n_2)) = 1}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, k n_2 f(n_2)) = 1}} 1 + O(x^{1/2}) \tag{39}$$

where by (25), the term $O(x^{1/2})$ accounts for the fact that when $n_1 = 1$ the condition $(n_2, k) = 1$ need not hold. With $\eta > 0$ and arbitrary, we split the outer sum into two parts according as $n_2 \leqslant \eta^{-2}$ or $\eta^{-2} < n_2 \leqslant x$; by the Corollary to Lemma 7, if $n_2 \leqslant \eta^{-2}$ and $x > x_0(\eta)$, $f(n_2)$ has no prime divisor exceeding y. We shall assume that x is sufficiently large for $x > \max(\eta^{-4}, x_0(\eta)) = x_1(\eta)$, and then $\eta^{-2} < x^{1/2}$. By (26)

$$\sum_{\substack{\eta^{-2} < n_2 \leqslant x \\ (n_2, k f(n_2)) = 1}} \sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, k n_2 f(n_2)) = 1}} 1 \leqslant \sum_{\eta^{-2} < n_2 \leqslant x} \frac{x}{n_2} = O(x\eta) = o(x) \tag{40}$$

as $x \to \infty$ since η is arbitrary. To deal with the inner sum of (39) when $n_2 \leqslant \eta^{-2}$, so $f(n_2)$ has no prime divisor exceeding y, we use the ideas of Lemma 12 with $v = y$, $m = k n_2 f(n_2)$ and $\mathscr{S} = \emptyset$; then by (21) and (22) we have here

$$\sum_{\substack{n_1 \leqslant x/n_2 \\ (n_1, k n_2 f(n_2)) = 1}} 1 = \frac{x}{n_2} \prod_p (1-p^{-2}) \prod_{p \mid k n_2 f(n_2)} (1+p^{-1})^{-1} \left(1 + O\left(\frac{1}{y}\right)\right) + O\left(4^y + \frac{x}{y n_2}\right).$$

Hence by (26), we have since $4^y\eta^{-2} = o(x/y)$ for $\eta^{-2} < x^{1/2}$,

$$(41)\qquad \sum_{\substack{n_2\leqslant\eta^{-2}\\ (n_2,kf(n_2))=1}} \sum_{\substack{n_1\leqslant x/n_2\\ (n_1,kn_2f(n_2))=1}} 1$$

$$= \frac{x}{\zeta(2)} \sum_{\substack{n_2\leqslant\eta^{-2}\\ (n_2,kf(n_2))=1}} \frac{1}{n_2} \prod_{p|kn_2f(n_2)} (1+p^{-1})^{-1} + O(x/y)$$

$$= \frac{6x}{\pi^2} \sum_{\substack{n_2=1\\ (n_2,kf(n_2))=1}}^{\infty} \frac{1}{n_2} \prod_{p|kn_2f(n_2)} (1+p^{-1})^{-1} + O(x\eta) + O(x/y)$$

$$= \big(C_7 + o(1)\big)x$$

as $x\to\infty$ since η is arbitrary. The result now follows from (39), (40), (41). We observe that C_7 satisfies

$$0 < \frac{6}{\pi^2}\prod_{p|k}(1+p^{-1})^{-1} \leqslant C_7 \leqslant \prod_{p|k}(1-p^{-1}) \leqslant 1$$

since the sum in C_7 does not exceed

$$\sum_{\substack{n_2=1\\ (n_2,k)=1}}^{\infty} \frac{1}{n_2}\prod_{p|kn_2}(1+p^{-1})^{-1} = \prod_{p|k}(1+p^{-1})^{-1}\prod_{p\nmid k}(1-p^{-2})^{-1}.$$

Applying this result to the divisor function τ (defined in (6)), in which case $W_j(x) = j+1$ $(j = 1, 2, \ldots)$, we have, on taking $k = 2$ in Theorem 3, the

COROLLARY.

$$\sum_{\substack{n\leqslant x\\ (n,\tau(n))=1}} 1 = \big(C_7+o(1)\big)x \quad \text{where} \quad \frac{4}{\pi^2} \leqslant C_7 \leqslant \tfrac{1}{2}.$$

References

[1] P. Erdös, *Some asymptotic formulas in number theory*, J. Indian Math. Soc. (New Series) 12 (1948), pp. 75–78.

[2] P. Erdös and G. G. Lorentz, *On the probability that n and g(n) are relatively prime*, Acta Arith. 5 (1958), pp. 35–44.

[3] P. Erdös und G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Scientiarum Mathematicarum, Szeged, 7 (1934), pp. 95–102.

[4] A. S. Faĭnleĭb, *The mutual primality of n and f(n)*, Mat. Zametki 11 (1972), pp. 259–268 (Russian); English translation: Math. Notes 11 (1972), pp. 163–168.

[5] H. Halberstam and H.-E. Richert, *Brun's method and the Fundamental Lemma*, Acta Arith. 24 (1973), pp. 113–133.

[6], [7], [8] R. R. Hall, *On the probability that n and f(n) are relatively prime, I; Corrigendum; II; III*; Acta Arith. 17 (1970), pp. 169–183; 19 (1971), pp. 203–204; 19 (1971), pp. 175–184; 20 (1972), pp. 267–289.

[9] E. Landau, *Algebraische Zahlen*, Chelsea 1949.

[10] — *Vorlesungen über Zahlentheorie I*, Chelsea 1950.

[11] H. B. Mann, *Introduction to Algebraic Number Theory*, Columbus, Ohio, 1955.

[12], [13] V. Schulze, *Die Primteilerdichte von ganzzahligen Polynomen, I; II*; J. Reine Angew. Math. 253 (1972), pp. 175–185; 256 (1972), pp. 153–162.

[14] E. C. Titchmarsh, *A divisor problem*, Rend. di Palermo 54 (1930), pp. 414–429.

[15] G. L. Watson, *On integers n relatively prime to [αn]*, Canadian J. Math. 5 (1953), pp. 451–455.

[16] H. Weber, *Lehrbuch der Algebra II*, 3rd ed., Chelsea.

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