für $a > \frac{2}{3}$. Somit braucht man in (12) nur jene ν zu berücksichtigen, für die

$$1-\sigma_\nu > \frac{1}{(\log N)^a}$$

gilt. Wenn wir also etwa $\eta = 10^{-4}$ setzen, so hat man für

$$\frac{1}{(\log N)^a} < 1-\sigma_\nu \leqslant \eta$$

jedenfalls

$$(13)\qquad \int_{-\delta}^{\delta} |S_1^{(\nu)}(\alpha)|^2 d\alpha \ll N^{1-\frac{1}{4(\log N)^a}} (\log N)^{21} \ll N e^{-\frac{1}{5}(\log N)^{1-a}}.$$

Insgesamt erhält man aus (4), (6), (11) und (13)

$$\int_{-\delta}^{\delta} |S_1(\alpha)|^2 d\alpha \ll N\delta^{2\eta}(\log N)^{b+8}.$$

Wie in § 1 schließt man dann, daß die Gleichung

$$|p+p'-N| < (\log N)^A$$

für alle großen N Lösungen in Primzahlen hat, sobald

$$A2\eta > b+8$$

gilt, d.h.

$$A > 5000(b+8);$$

dies alles unter der Annahme der Richtigkeit der Dichtehypothese (1).

Literaturverzeichnis

[1] Ю. В. Линник, *Некоторые условные теоремы касающиеся бинарной проблемы Гольдбаха*, Изв. АНСССР, сер. матем. 16 (1952), S. 503–520.

[2] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Berlin-Heidelberg-New York 1971.

[3] K. Prachar, *Über den Primzahlsatz von A. Selberg*, Acta Arith. 28 (1975), S. 277–297.

[4] A. Selberg, *On the normal density of primes in small intervals and the difference of consecutive primes*, Arch. for Math. og Naturv. B, 47 (1943), No. 6.

Eingegangen 29. 6. 1974

ACTA ARITHMETICA
XXIX (1976)

On the non-linear sieve

by

J. W. Porter (Cardiff)

1. Introduction. In [8] Jurkat and Richert obtained definitive bounds for what may be called the linear sieve problem, and in [10] Richert used these bounds to obtain elegant results in various applications, including the distribution of almost-primes in the sequence of values taken by an irreducible polynomial. To date the best bounds available for the non-linear sieve problem, corresponding to the investigation of reducible polynomials in the above type of application, are those due to Ankeny and Onishi [1]. The present writer [9] investigated their bounds numerically and Halberstam and Richert [6] gave a detailed treatment of the applications. Hagedorn [4] also studied the properties of their bounds. It has been realized for some time (cf. Selberg [11]), that improvements on the bounds of Ankeny and Onishi could be obtained by means of an iterative technique based on the Buchstab identity ((2.7) below), and it is the purpose of this paper to give a detailed treatment of the first step in this iterative process and investigate some of the properties of the resultant improved bounds.

2. Notation and statement of results. We follow the notation of Halberstam and Richert ([6] and [7]).

Let $\mathscr{A}$ be a finite sequence of not necessarily distinct integers. For any integer d we denote by $\mathscr{A}_d$ the subsequence of $\mathscr{A}$ consisting of those elements divisible by d. We use $|\mathscr{A}|$ and $|\mathscr{A}_d|$ to denote the number of elements of $\mathscr{A}$ and $\mathscr{A}_d$ respectively.

Further, let $\mathscr{P}$ be a set of primes and denote by $\overline{\mathscr{P}}$ the complement of $\mathscr{P}$ in the set of all primes. For any $z \geqslant 2$ we write

$$P(z) = \prod_{\substack{p\in\mathscr{P}\\ p<z}} p.$$

We define the sifting function $S(\mathscr{A}_q; \mathscr{P}, z)$ for $z \geqslant 2$ and an integer q satisfying (i) $\mu(q) \neq 0$, (ii) $(q, P(z)) = 1$, (iii) $(q, p) = 1$ for all $p \in \overline{\mathscr{P}}$ by

$$S(\mathscr{A}_q; \mathscr{P}, z) = \left|\{a \in \mathscr{A}_q;\ (a, P(z)) = 1\}\right|.$$

In words, $S(\mathscr{A}_q;\mathscr{P},z)$ is the number of elements of $\mathscr{A}_q$ which contain no prime factor belonging to $\mathscr{P}$ which is less than z.

The sieve problem, in one of its aspects, consists of finding upper and lower bounds for this quantity $S(\mathscr{A}_q;\mathscr{P},z)$.

In order to make progress it is necessary to impose on our sequence $\mathscr{A}$ some fairly stringent conditions, which, however, can be chosen so as to be valid for some interesting applications. We first introduce a number $X>1$, which is to be thought of as a convenient approximation to $|\mathscr{A}|$. Then we set

$$R_1 = |\mathscr{A}|-X,$$

and for each prime $p\in\mathscr{P}$ we choose a number $\omega(p)$ and set

$$R_p = |\mathscr{A}_p| - \frac{\omega(p)}{p}X.$$

(In practice this is always done in such a way that R_p is small.) We define $\omega(1)=1$, $\omega(p)=0$ if $p\in\overline{\mathscr{P}}$, and for a squarefree integer d put

$$\omega(d) = \prod_{p|d}\omega(p).$$

We then write

$$R_d = |\mathscr{A}_d| - \frac{\omega(d)}{d}X.$$

The conditions that we impose on our sequence $\mathscr{A}$ are most simply expressed in the form of a demand that the choice of ω should be made in such a way that the following requirements hold:

(I). There exists a constant $A_1\geqslant 1$ such that for all primes p,

$$0\leqslant \frac{\omega(p)}{p}\leqslant 1-\frac{1}{A_1}.$$

(II). There exist numbers $A_2\geqslant 1$, $L\geqslant 1$ and $\varkappa>0$ with the property that for $w>z\geqslant 2$,

$$-L\leqslant \sum_{z\leqslant p<w}\frac{\omega(p)}{p}\log p-\varkappa\log\frac{w}{z}\leqslant A_2.$$

Although this can always be done, significant results can only be obtained if it can be done in such a way that R_d is small on average, as will be apparent from (2.2) and (2.3).

It is convenient at this point to state the following consequence of (I) and (II), which is due to Halberstam and Richert [5].

LEMMA. *If, for $z>2$, we write*

$$W(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right),$$

then, under the conditions (I) *and* (II),

$$W(z) = \frac{C(\varkappa)}{\log^{\varkappa} z}\left\{1+O\left(\frac{L}{\log z}\right)\right\}, \tag{2.1}$$

where

$$C(\varkappa) = e^{-\gamma\varkappa}\prod_p\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\varkappa}.$$

(γ is Euler's constant.)

The case $\varkappa=1$ corresponds to what was referred to in the introduction as the linear sieve problem, and accordingly is excluded from further consideration in what follows, where we assume $\varkappa\geqslant 2$.

It should also be remarked that we allow the constants implied by the O-notation to depend on $\varkappa$ and the A's but not on L. (For the reason for this, see Halberstam and Richert [5].)

It has been realized for some time that in these circumstances the sieve problem resolves itself into a search for functions $F_\varkappa$ and $f_\varkappa$ for which

$$S(\mathscr{A}_q;\mathscr{P},z) \leqslant \frac{\omega(q)}{q}XW(z)\left\{F_\varkappa(\tau_z)+O\left(\frac{L(\log\log\xi)^{A_3}}{\log\xi}\right)\right\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big) \tag{2.2}$$

and

$$S(\mathscr{A}_q;\mathscr{P},z) \geqslant \frac{\omega(q)}{q}XW(z)\left\{f_\varkappa(\tau_z)+O\left(\frac{L(\log\log\xi)^{A_3}}{\log\xi}\right)\right\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big) \tag{2.3}$$

where $\tau_z=\log\xi^2/\log z$, $\nu(n)$ is the number of distinct prime divisors of n, and ξ is an arbitrary number greater than 3.

Ankeny and Onishi obtained such functions defined as follows: Let $\sigma_\varkappa(u)$ denote the continuous solution of the differential-difference problem:

$$\sigma_\varkappa(u) = \frac{2^{-\varkappa}e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)}u^\varkappa\ (0\leqslant u\leqslant 2);\qquad \sigma'_\varkappa(u) = \frac{\varkappa}{u}\{\sigma_\varkappa(u)-\sigma_\varkappa(u-2)\}\ (u\geqslant 2).$$

They proved that $\sigma_\varkappa(u)\to 1$ as $u\to\infty$ in such a way that

$$\frac{1}{\sigma_\varkappa(u)}-1 = O(e^{-u/2}),$$

We may therefore write

$$\eta_\varkappa(u) = \varkappa u^{-\varkappa}\int_u^\infty t^{\varkappa-1}\left(\frac{1}{\sigma_\varkappa(t-1)}-1\right)dt.$$

Then Ankeny and Onishi proved the

THEOREM. *The following are admissible choices of $F_\varkappa$ and $f_\varkappa$:*

$$F_\varkappa(u) = \frac{1}{\sigma_\varkappa(u)} \quad \textit{and} \quad f_\varkappa(u) = 1-\eta_\varkappa(u). \tag{2.4}$$

The function $\eta_\varkappa(u)$ is continuous and decreasing monotonically to 0. Since $\eta_\varkappa(u)\to\infty$ as $u\to 1+$ for $\varkappa\geqslant 1$, it follows that, for $\varkappa\geqslant 1$, there is a unique positive solution of the equation $\eta_\varkappa(u) = 1$, denoted by $\nu_\varkappa$. This parameter is of some importance in the applications of the sieve.

In this paper we seek (and find) better functions $F_\varkappa$ and $f_\varkappa$, which satisfy (2.2) and (2.3). It is convenient here to introduce some functions which are required in the definition of our new $F_\varkappa$ and $f_\varkappa$, and mention a few of their properties. We define inductively

$$\eta_{\varkappa,r}(u) = \varkappa u^{-\varkappa}\int_u^\infty t^{\varkappa-1}\eta_{\varkappa,r-1}(t-1)\,dt, \tag{2.5}$$

interpreting $\eta_{\varkappa,1}(u)$ as $\eta_\varkappa(u)$.

This defines $\eta_{\varkappa,r}$ for $u>r$ as a continuous monotonically decreasing function of u tending to 0 as $u\to\infty$, and to $+\infty$ as $u\to r+$, if $\varkappa\geqslant r$.

Ankeny and Onishi proved that $\eta_\varkappa(u) = O(u^{-1}e^{-u/2})$. One may deduce immediately from (2.5) that

$$\eta_{\varkappa,2}(u) = O(u^{-2}e^{-u/2}) \quad \text{and} \quad \eta_{\varkappa,3}(u) = O(u^{-3}e^{-u/2}). \tag{2.6}$$

The above-mentioned lower bound of Ankeny and Onishi was obtained from their upper bound (derived by Selberg's method) by means of the Buchstab identity, valid for $2\leqslant z_1\leqslant z$,

$$S(\mathscr{A}_q;\mathscr{P},z) = S(\mathscr{A}_q;\mathscr{P},z_1) - \sum_{\substack{z_1\leqslant p<z\\ p\in\mathscr{P}}} S(\mathscr{A}_{qp};\mathscr{P},p). \tag{2.7}$$

Our method is to use the Ankeny–Onishi lower bound in (2.7) to obtain a new upper bound for $S(\mathscr{A}_q;\mathscr{P},z)$ and then to derive from this new upper bound, again by means of (2.7) a new lower bound for the same quantity.

In addition to the results of Ankeny and Onishi quoted above we require the following result of the "Fundamental Lemma" type:

$$S(\mathscr{A}_q;\mathscr{P},z) = \frac{\omega(q)}{q}XW(z)\{1+O(e^{-\tau_z})\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

Applying this with

$$z_1 = \exp\left(\frac{\log\xi}{\log\log\xi}\right) \tag{2.8}$$

we obtain

$$S(\mathscr{A}_q;\mathscr{P},z_1) = \frac{\omega(q)}{q}XW(z_1)\left\{1+O\left(\frac{1}{\log\xi}\right)\right\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big). \tag{2.9}$$

3. The upper bound. For convenience we collect together the following simple estimates, which are required several times in this and the following section:

With z_1 given by (2.8), we have for $z_1<w=O(\xi^\alpha)$ $(\alpha<2)$

$$\sum_{z_1\leqslant p<w}\frac{\omega(p)}{p}W(p) = W(z_1)-W(w); \tag{3.1}$$

$$\sum_{z_1\leqslant p<w}\frac{\omega(p)}{p}W(p)\frac{1}{\log\xi p^{-1/2}} = O\left(\frac{(\log\log\xi)^\varkappa}{\log^{\varkappa+1}\xi}\right) \tag{3.2}$$

and

$$\sum_{\substack{z_1\leqslant p<w\\ p\in\mathscr{P}}}\sum_{\substack{d<\xi^2/p\\ d|P(p)}}3^{\nu(d)}|R_{qpd}|\leqslant\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|. \tag{3.3}$$

In order to obtain an upper bound for $S(\mathscr{A}_q;\mathscr{P},z)$ from (2.7) we require a lower bound for $S(\mathscr{A}_{qp};\mathscr{P},p)$. Two lower bounds are at our disposal, viz that given by (2.4) with q replaced by qp and z by p, applied with ξ^2/p replacing ξ^2 (in order to ensure that the error term can be dealt with), and the trivial lower bound 0. The trivial lower bound is better if

$$\frac{\log\xi^2p^{-1}}{\log p}\leqslant\nu_\varkappa,\quad \text{i.e. if } p\geqslant\xi^{2/(1+\nu_\varkappa)}.$$

Writing $\theta=\min(\xi^{2/(1+\nu_\varkappa)},z)$ we have

$$\sum_{\substack{z_1\leqslant p<z\\ p\in\mathscr{P}}}S(\mathscr{A}_{qp};\mathscr{P},p)\geqslant\frac{\omega(q)}{q}X\sum_{z_1\leqslant p<\theta}\frac{\omega(p)}{p}W(p)\Big\{1-\eta_\varkappa(\tau_p-1)+$$
$$+O\left(\frac{L(\log\log\xi)^{A_3}}{\log\xi p^{-1/2}}\right)\Big\}+O\Big(\sum_{\substack{z_1\leqslant p<\theta\\ p\in\mathscr{P}}}\sum_{\substack{d<\xi^2/p\\ d|P(p)}}3^{\nu(d)}|R_{qpd}|\Big)$$

Now, by (3.1) and (2.1),

$$\sum_{z_1\leqslant p<\theta}\frac{\omega(p)}{p}W(p)\eta_\varkappa(\tau_p-1)$$

$$=\int_{z_1}^{\theta}\eta_\varkappa(\tau_t-1)\,d\big(W(z_1)-W(t)\big) = -\int_{z_1}^{\theta}\eta_\varkappa(\tau_t-1)\,dW(t)$$

$$= -W(\theta)\eta_\varkappa(\tau_\theta-1)+W(z_1)\eta_\varkappa(\tau_{z_1}-1)+\int_{z_1}^{\theta}W(t)\frac{d}{dt}\eta_\varkappa(\tau_t-1)\,dt$$

$$= -W(\theta)\eta_\varkappa(\tau_\theta-1)+W(z_1)\eta_\varkappa(\tau_{z_1}-1)+$$

$$+\int_{z_1}^{\theta}\left\{\frac{C(\varkappa)}{\log^\varkappa t}+O\left(\frac{L}{\log^{\varkappa+1}t}\right)\right\}\frac{d}{dt}\eta_\varkappa(\tau_t-1)\,dt.$$

Further

$$\int_{z_1}^{\theta}\frac{1}{\log^\varkappa t}\frac{d}{dt}\eta_\varkappa(\tau_t-1)\,dt$$

$$= -\frac{1}{\log^\varkappa z_1}\eta_\varkappa(\tau_{z_1}-1)+\frac{1}{\log^\varkappa\theta}\eta_\varkappa(\tau_\theta-1)+\varkappa\int_{z_1}^{\theta}\eta_\varkappa(\tau_t-1)\frac{dt}{t\log^{\varkappa+1}t}$$

$$= -\frac{1}{\log^\varkappa z_1}\eta_\varkappa(\tau_{z_1}-1)+\frac{1}{\log^\varkappa\theta}\eta_\varkappa(\tau_\theta-1)+\varkappa(\log\xi^2)^{-\varkappa}\int_{\tau_\theta}^{\tau_{z_1}}u^{\varkappa-1}\eta_\varkappa(u-1)\,du.$$

Hence

$$\sum_{z_1\leqslant p<\theta}\frac{\omega(p)}{p}W(p)\eta_\varkappa(\tau_p-1)$$

$$=\left\{W(z_1)-\frac{C(\varkappa)}{\log^\varkappa z_1}\right\}\eta_\varkappa(\tau_{z_1}-1)-\left\{W(\theta)-\frac{C(\varkappa)}{\log^\varkappa\theta}\right\}\eta_\varkappa(\tau_\theta-1)+$$

$$+\frac{C(\varkappa)}{\log^\varkappa\theta}\eta_{\varkappa,2}(\tau_\theta)-\frac{C(\varkappa)}{\log^\varkappa z_1}\eta_{\varkappa,2}(\tau_{z_1})+O\left(\frac{L}{\log^{\varkappa+1}z_1}\right),$$

which, by (2.1) and (2.6) is equal to

$$\frac{C(\varkappa)}{\log^\varkappa\theta}\eta_{\varkappa,2}(\tau_\theta)+O\left(\frac{L}{\log^{\varkappa+1}z_1}\right)=W(\theta)\eta_{\varkappa,2}(\tau_\theta)+O\left(\frac{L}{\log^{\varkappa+1}z_1}\right).$$

Using (3.2) and (3.3) we find

$$(3.4)\qquad \sum_{\substack{z_1\leqslant p<z\\ p\in\mathscr{P}}}S(\mathscr{A}_{qp};\mathscr{P},p)$$

$$\geqslant\frac{\omega(q)}{q}X\left\{W(z_1)-W(\theta)-W(\theta)\eta_{\varkappa,2}(\tau_\theta)+O\left(\frac{L(\log\log\xi)^{A_3+\varkappa}}{\log^{\varkappa+1}\xi}\right)\right\}+$$

$$+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

It follows that, if $\theta=z$, that is, if $\tau_z\geqslant 1+\nu_\varkappa$ we have, from (2.7), (2.9) and (3.4)

$$S(\mathscr{A}_q;\mathscr{P},z)\leqslant\frac{\omega(q)}{q}XW(z)\left\{1+\eta_{\varkappa,2}(\tau_z)+O\left(\frac{L(\log\log\xi)^{A_3+\varkappa}}{\log\xi}\right)\right\}+$$

$$+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big)$$

whereas, if $\theta=\xi^{2/(1+\nu_\varkappa)}$, i.e. $\tau_z<1+\nu_\varkappa$, since

$$\frac{W(\theta)}{W(z)}=\left(\frac{\log z}{\log\theta}\right)^\varkappa\left\{1+O\left(\frac{L}{\log\theta}\right)\right\},$$

it follows that

$$S(\mathscr{A}_q;\mathscr{P},z)\leqslant\frac{\omega(q)}{q}XW(z)\Big\{(1+\nu_\varkappa)^\varkappa\tau_z^{-\varkappa}\big(1+\eta_{\varkappa,2}(1+\nu_\varkappa)\big)+$$

$$+O\left(\frac{L(\log\log\xi)^{A_3+\varkappa}}{\log\xi}\right)\Big\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

The following is therefore an admissible choice of upper bound function:

$$F_\varkappa(u)=\begin{cases}1+\eta_{\varkappa,2}(u) & (u\geqslant 1+\nu_\varkappa),\\ (1+\nu_\varkappa)^\varkappa u^{-\varkappa}\{1+\eta_{\varkappa,2}(1+\nu_\varkappa)\} & (u<1+\nu_\varkappa).\end{cases}$$

The question arises as to how this choice compares with that of Ankeny and Onishi, namely

$$F_\varkappa(u)=\frac{1}{\sigma_\varkappa(u)}.$$

As is shown in § 5,

$$\eta_{\varkappa,2}(u)<\frac{1}{\sigma_\varkappa(u)}-1$$

for sufficiently large values of u. (Indeed it is even shown that

$$\eta_{\varkappa,2}(u) = o\left(\frac{1}{\sigma_\varkappa(u)} - 1\right).)$$

Since $\eta_{\varkappa,2}(u) \to +\infty$ as $u \to 2+$ for $\varkappa \geqslant 2$, it follows that the equation

$$\eta_{\varkappa,2}(u) = \frac{1}{\sigma_\varkappa(u)} - 1 \quad (\varkappa \geqslant 2) \tag{3.5}$$

must possess at least one root. (What numerical evidence there is suggests that it possesses only one root, but it is not necessary to assume this.) Call the least root of (3.5) $\lambda_\varkappa$. The data available suggests that $\lambda_\varkappa > 1 + \nu_\varkappa$. In any case we have the following result:

THEOREM. *The following is an admissible choice in* (2.2):

$$F_\varkappa(u) = \begin{cases} 1 + \eta_{\varkappa,2}(u) & (u \geqslant \lambda_\varkappa), \\ \dfrac{1}{\sigma_\varkappa(u)} & (u < \lambda_\varkappa). \end{cases} \tag{3.6}$$

4. The lower bound. To obtain a lower bound for $S(\mathscr{A}_q; \mathscr{P}, z)$ from (2.7) we require an upper bound for

$$\sum_{\substack{z_1 \leqslant p < z \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p).$$

We use the upper bound obtained above for $S(\mathscr{A}_{qp}; \mathscr{P}, p)$ with ξ^2 replaced by $\xi^2 p^{-1}$.

Now

$$\frac{\log \xi^2 p^{-1}}{\log p} \gtrless \lambda_\varkappa$$

according as $\xi^{2/(1+\lambda_\varkappa)} \gtrless p$. Writing $\vartheta = \min(z, \xi^{2/(1+\lambda_\varkappa)})$ we therefore have

$$\sum_{\substack{z_1 \leqslant p < z \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p) = \sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p) + \sum_{\substack{\vartheta \leqslant p < z \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p). \tag{4.1}$$

Now

$$\sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p)$$

$$\leqslant \frac{\omega(q)}{q} X \left\{ \sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} \frac{\omega(p)}{p} W(p)\big(1 + \eta_{\varkappa,2}(\tau_p - 1)\big) + O\left(\frac{L(\log\log \xi)^{A_3}}{\log \xi p^{-1/2}}\right)\right\} +$$

$$+ O\Big(\sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} \sum_{\substack{d < \xi^2/p \\ d | P(p)}} 3^{\nu(d)} |R_{qpd}|\Big).$$

Precisely as in § 3, it follows that

$$\sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} \frac{\omega(p)}{p} W(p) \eta_{\varkappa,2}(\tau_p - 1) = W(\vartheta) \eta_{\varkappa,3}(\tau_\vartheta) + O\left(\frac{L}{\log^{\varkappa+1} z_1}\right).$$

By virtue of (3.2) and (3.3), we find

$$\sum_{\substack{z_1 \leqslant p < \vartheta \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p) \tag{4.2}$$

$$\leqslant \frac{\omega(q)}{q} X \left\{ W(z_1) - W(\vartheta) + W(\vartheta)\eta_{\varkappa,3}(\tau_\vartheta) + O\left(\frac{L(\log\log \xi)^{A_3+\varkappa}}{\log^{\varkappa+1} \xi}\right)\right\} +$$

$$+ O\Big(\sum_{\substack{n \leqslant \xi^2 \\ n | P(z)}} 3^{\nu(n)} |R_{qn}|\Big).$$

As for

$$\sum_{\substack{\vartheta \leqslant p < z \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p) \tag{4.3}$$

we note that if $\vartheta = z$, i.e. $\tau_z \geqslant 1 + \lambda_\varkappa$, the sum (4.3) is empty. Consideration of (2.7), (2.9), (4.1), and (4.2) shows that, for $\tau_z \geqslant 1 + \lambda_\varkappa$,

$$S(\mathscr{A}_q; \mathscr{P}, z) \geqslant \frac{\omega(q)}{q} X W(z) \left\{1 - \eta_{\varkappa,3}(\tau_z) + O\left(\frac{L(\log\log \xi)^{A_3+\varkappa}}{\log \xi}\right)\right\} + \tag{4.4}$$

$$+ O\Big(\sum_{\substack{n < \xi^2 \\ n | P(z)}} 3^{\nu(n)} |R_{qn}|\Big).$$

On the other hand, if $\tau_z < 1 + \lambda_\varkappa$, applying (3.6) with ξ^2/p replacing ξ^2 we obtain

$$\sum_{\substack{\vartheta \leqslant p < z \\ p \in \mathscr{P}}} S(\mathscr{A}_{qp}; \mathscr{P}, p) \leqslant \frac{\omega(q)}{q} X \sum_{\vartheta \leqslant p < z} \frac{\omega(p)}{p} W(p) \left\{\frac{1}{\sigma_\varkappa(\tau_p - 1)} + \right.$$

$$\left. + O\left(\frac{L(\log \xi p^{-1/2})^{2\varkappa+1}}{\log^{2\varkappa+2} p}\right)\right\} + O\Big(\sum_{\substack{\vartheta \leqslant p < z \\ p \in \mathscr{P}}} \sum_{\substack{d < \xi^2/p \\ d | P(p)}} 3^{\nu(d)} |R_{qpd}|\Big).$$

We have again

$$\sum_{\vartheta \leqslant p < z} \frac{\omega(p)}{p} W(p) \frac{1}{\sigma_\varkappa(\tau_p - 1)} = W(z)\eta_\varkappa(\tau_z) - W(\vartheta)\eta_\varkappa(\tau_\vartheta) + O\left(\frac{L}{\log^{\varkappa+1} z_1}\right),$$

so as before it follows that

(4.5) $$\sum_{\substack{\vartheta\leqslant p<z\\ p\in\mathscr{P}}} S(\mathscr{A}_{qp};\mathscr{P},p)\leqslant\frac{\omega(q)}{q}X\Big\{W(\vartheta)-W(z)+W(z)\eta_\varkappa(\tau_z)-$$
$$-W(\vartheta)\eta_\varkappa(\vartheta)+O\Big(\frac{L(\log\log\xi)^{\varkappa+1}}{\log^{\varkappa+1}\xi}\Big)\Big\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

From (2.7), (2.9), (4.1), (4.2) and (4.5) we obtain, if $\vartheta=\xi^{2/(1+\lambda_\varkappa)}$, i.e. $\tau_z<1+\lambda_\varkappa$,

$$S(\mathscr{A}_q;\mathscr{P},z)\geqslant\frac{\omega(q)}{q}XW(z)\Big\{1-\eta_\varkappa(\tau_z)+\frac{W(\vartheta)}{W(z)}\big(\eta_\varkappa(\tau_\vartheta)-\eta_{\varkappa,3}(\tau_\vartheta)\big)+$$
$$+O\Big(\frac{L(\log\log\xi)^{A_3+\varkappa}}{\log\xi}\Big)\Big\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

It follows that, since $\vartheta=\xi^{2/(1+\lambda_\varkappa)}$ and

$$\frac{W(\vartheta)}{W(z)}=\Big(\frac{\log z}{\log\vartheta}\Big)^\varkappa\Big\{1+O\Big(\frac{L}{\log\vartheta}\Big)\Big\}$$

that

(4.6) $S(\mathscr{A}_q;\mathscr{P},z)$
$$\geqslant\frac{\omega(q)}{q}XW(z)\Big\{1-\eta_\varkappa(\tau_z)+(1+\lambda_\varkappa)^\varkappa\tau_z^{-\varkappa}\big(\eta_\varkappa(1+\lambda_\varkappa)-\eta_{\varkappa,3}(1+\lambda_\varkappa)\big)+$$
$$+O\Big(\frac{L(\log\log\xi)^{A_3+\varkappa}}{\log\xi}\Big)\Big\}+O\Big(\sum_{\substack{n<\xi^2\\ n|P(z)}}3^{\nu(n)}|R_{qn}|\Big).$$

From (4.4) and (4.6) it is apparent that the following theorem holds.

THEOREM. *The following is an admissible choice in* (2.3):

$$f_\varkappa(u)=\begin{cases}1-\eta_{\varkappa,3}(u) & (u>1+\lambda_\varkappa),\\ 1-\eta_\varkappa(u)+(1+\lambda_\varkappa)^\varkappa u^{-\varkappa}\{\eta_\varkappa(1+\lambda_\varkappa)-\eta_{\varkappa,3}(1+\lambda_\varkappa)\} & (u<1+\lambda_\varkappa).\end{cases}$$

5. In this section we investigate the asymptotic behaviour of some of the functions introduced in the preceding sections. In particular we deduce the existence of the number $\lambda_\varkappa$ and show that our choice of $F_\varkappa$ and $f_\varkappa$ is, at least for some parts of the range, an improvement on that of Ankeny and Onishi. The methods we employ are essentially those of de Bruijn ([2] and [3]).

We derive first an asymptotic formula for $\sigma'_\varkappa(u)$ using the method used by de Bruijn for a function that is, in all essentials, $\sigma'_1(u)$, and derive from this, in succession, asymptotic formulae for $1-\sigma_\varkappa(u)$, $\eta_\varkappa(u)$, $\eta_{\varkappa,2}(u)$ and $\eta_{\varkappa,3}(u)$.

We quote the following lemma from de Bruijn [2]:

LEMMA (de Bruijn). *Let $K(x,t)$ be a function of x and t which is absolutely integrable with respect to x and t for $0\leqslant x\leqslant a$, $0\leqslant t\leqslant 1$, for any $a>0$, with the properties that, for $x\geqslant 1$,*

(i) $K(x,t)\geqslant 0$ $(0\leqslant t\leqslant 1)$;

(ii) $K(x,t)=0$ $(t>1$ *or* $t<0)$;

(iii) $\int_0^1 K(x,t)\,dt=1.$

Suppose further that there exists a positive constant $\gamma<1/4$ and a continuous function $\varphi(x)$ on $x\geqslant 1$ with the properties:

(i) $\varphi(x)\geqslant 0$ $(x\geqslant 1)$;

(ii) *the series*

(5.1) $$\sum_{n=1}^{\infty}\inf\{\varphi(x);\ n\leqslant x\leqslant n+2\}$$

is divergent;

(iii) *for any $x\geqslant 1$ and any measurable subset E of the interval $0\leqslant t\leqslant 1$ with measure $\geqslant\gamma$,*

(5.2) $$\int_E K(x,t)\,dt\geqslant\varphi(x).$$

Then the integral equation

$$f(x)=\int_0^1 K(x,t)f(x-t)\,dt$$

has the property that every continuous solution is convergent, i.e. $\lim_{x\to\infty} f(x)$ *exists.*

We write $\tau_\varkappa(u)=\sigma'_\varkappa(2u)$ so that

$$u\tau_\varkappa(u)=u\sigma'_\varkappa(2u)=\tfrac12\varkappa\{\sigma_\varkappa(2u)-\sigma_\varkappa(2u-2)\}=\tfrac12\varkappa\int_0^2\sigma'_\varkappa(2u-t)\,dt$$
$$=\varkappa\int_0^1\sigma'_\varkappa\big(2(u-t)\big)\,dt=\varkappa\int_0^1\tau_\varkappa(u-t)\,dt.$$

Thus $\tau_\varkappa(u)$ is a solution of the integral equation

(5.3) $$uf(u)=\varkappa\int_0^1 f(u-t)\,dt.$$

Suppose $F(u)$ is another continuous solution of this equation. Then the function

$$g(u) = \frac{F(u)}{\tau_\varkappa(u)}$$

is easily seen to satisfy the integral equation

$$g(u) = \int_0^1 \frac{\varkappa\tau_\varkappa(u-t)}{u\tau_\varkappa(u)} g(u-t)\,dt.$$

We therefore apply the theorem of de Bruijn with

$$K(x, t) = \frac{\varkappa\tau_\varkappa(x-t)}{x\tau_\varkappa(x)} \qquad (0 \leqslant t \leqslant 1).$$

Since $\tau_\varkappa(x)$ is decreasing for $x > \varkappa$, we have for $x \geqslant \varkappa+1$

$$K(x, t) \geqslant \varkappa/x$$

and take $\varphi(x) = \gamma\varkappa x^{-1}$ $(x \geqslant \varkappa+1)$. The definition chosen for $\varphi(x)$ in the interval $[1, \varkappa+1]$ cannot affect the validity of condition (ii) and it is clearly possible to define a non-negative continuous function in this interval with the property (iii).

It follows therefore that there exists a constant $C(F, \varkappa)$ such that $g(u) \sim C$, i.e. that

$$\tau_\varkappa(u) \sim \frac{1}{C} F(u). \tag{5.4}$$

We can therefore obtain the asymptotic behaviour of $\tau_\varkappa$ by constructing a solution $F(u)$ of (5.3) whose asymptotic behaviour we are able to determine.

Following de Bruijn, we seek a solution of the form

$$F(u) = \int e^{-uz} p(z)\,dz \tag{5.5}$$

the integral being taken along some path in the complex plane. Differentiating (5.3) and making a formal substitution of (5.5) yields

$$\int e^{-uz} p(z)\{uz - \varkappa e^z + \varkappa - 1\}\,dz = 0.$$

This suggests that we choose $p(z)$ so that

$$\frac{d}{dz}\{-e^{-uz} z p(z)\} = e^{-uz} p(z)\,(uz - \varkappa e^z + \varkappa - 1),$$

i.e.

$$e^{-uz} z p'(z) = e^{-uz} p(z)\varkappa(e^z - 1)$$

or

$$\frac{p'(z)}{p(z)} = \varkappa\frac{e^z-1}{z}.$$

We therefore take

$$p(z) = \exp\left(\varkappa\int_0^z \frac{e^s-1}{s}\,ds\right).$$

Since the function $p(z)$ tends to 0 very rapidly along the lines $z = \pm\pi i + t$, we take

$$F(u) = \frac{1}{2\pi i}\int_C \exp\left\{-uz + \varkappa\int_0^z \frac{e^s-1}{s}\,ds\right\} dz, \tag{5.6}$$

where C is the contour consisting of the line $y = -\pi$ from $x = +\infty$ to $x = 0$, the y-axis from $-\pi$ to π and the line $y = \pi$ from $x = 0$ to $x = +\infty$. That (5.6) is indeed a solution of (5.3) follows from the fact that the differentiation of (5.6) under the integral sign is valid by virtue of the continuity of the integrand and the uniformity of the convergence of the integral for $u > 0$.

To determine the asymptotic behaviour of $F(u)$ we proceed as follows. We write

$$\psi(z) = -uz + \varkappa\int_0^z \frac{e^s-1}{s}\,ds.$$

Now

$$\psi'(z) = -u + \varkappa\left(\frac{e^z-1}{z}\right).$$

Thus $\psi'(z) = 0$ if and only if $\varkappa(e^z-1) = uz$. This equation has one real positive root for $u > \varkappa$, as is clear from the graphs of the functions $\varkappa(e^x-1)$ and ux. Call this root $\xi = \xi(u)$ so that

$$\varkappa(e^\xi - 1) = u\xi.$$

Now

$$\psi''(z) = \varkappa e^z z^{-1} - \varkappa(e^z-1)z^{-2}$$

so that

$$\psi''(\xi) = \varkappa e^\xi \xi^{-1} - \varkappa(e^\xi-1)\xi^{-2} = (u\xi+\varkappa)\xi^{-1} - u\xi^{-1}.$$

We now replace the path C by a path C' consisting of the line $y = -\pi$ from $x = +\infty$ to $x = \xi$, the line $x = \xi$ from $y = -\pi$ to $y = \pi$, and the line $y = \pi$ from $x = \xi$ to $x = +\infty$. Thus

$$\begin{aligned} F(u) &= \frac{1}{2\pi i}\int_{C'} \exp\{\psi(z)\}\,dz \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{\psi(\xi+it)}\,i\,dt + \frac{1}{2\pi i}\int_\xi^\infty e^{\psi(x+i\pi)}\,dx + \frac{1}{2\pi i}\int_\xi^\infty e^{\psi(x-i\pi)}\,dx \\ &= I_1 + I_2 + I_3, \quad \text{say}. \end{aligned}$$

Now

$$\psi(x+i\pi) = -u(x+i\pi)+\varkappa\int_0^{x+i\pi}\frac{e^s-1}{s}\,ds$$

$$= -u(x+i\pi)+\varkappa\int_0^{\pi}\frac{e^{it}-1}{t}\,dt+\varkappa\int_0^{x}\frac{e^{-\sigma}-1}{\sigma+i\pi}\,d\sigma.$$

Hence

$$|e^{\psi(x+i\pi)}| = e^{\operatorname{Re}\psi(x+i\pi)} = O(e^{-ux}),$$

so that

(5.7) $$I_2 = O(e^{-u\xi}\xi^{-1}).$$

Similarly,

(5.8) $$I_3 = O(e^{-u\xi}\xi^{-1}).$$

As regards I_1, we note that given $\varepsilon > 0$, there exists a $\delta > 0$ independent of ξ such that if $|t| < \delta$,

(5.9) $$\psi''(\xi+it) = \psi''(\xi)\{1+\theta\}$$

with $|\theta| < \varepsilon$. This becomes clear if we note that

$$\frac{\psi''(\xi+it)}{\psi''(\xi)} = \frac{e^{\xi+it}(\xi+it)^{-1}-(e^{\xi+it}-1)(\xi+it)^{-2}}{e^{\xi}\xi^{-1}-(e^{\xi}-1)\xi^{-2}}$$

$$\sim e^{it} \quad \text{as} \quad \xi\to\infty.$$

With δ' to be chosen later so as to satisfy $0 < \delta' < \delta$, we write

$$I_1 = \frac{1}{2\pi}\Big\{\int_{-\pi}^{-\delta'}+\int_{-\delta'}^{\delta'}+\int_{\delta'}^{\pi}\Big\}e^{\psi(\xi+it)}\,dt$$

$$= \frac{1}{2\pi}\{I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\}, \quad \text{say}.$$

Consider first $I_1^{(3)}$.

$$|I_1^{(3)}| \leqslant \int_{\delta'}^{\pi} e^{\operatorname{Re}\psi(\xi+it)}\,dt = e^{\psi(\xi)}\int_{\delta'}^{\pi} e^{\operatorname{Re}(\psi(\xi+it)-\psi(\xi))}\,dt.$$

Now

$$\psi(\xi+it)-\psi(\xi) = -u(\xi+it)+\varkappa\int_0^{\xi+it}\frac{e^s-1}{s}\,ds+u\xi-\varkappa\int_0^{\xi}\frac{e^s-1}{s}\,ds$$

$$= -uit+\varkappa\int_{\xi}^{\xi+it}\frac{e^s-1}{s}\,ds = -uit+\varkappa\int_0^{t}\frac{e^{\xi+i\tau}-1}{\xi+i\tau}\,i\,d\tau.$$

It follows that

$$\operatorname{Re}\{\psi(\xi+it)-\psi(\xi)\} = \varkappa\int_0^{t}\frac{e^{\xi}\tau\cos\tau-\xi e^{\xi}\sin\tau-\tau}{\xi^2+\tau^2}\,d\tau.$$

Since the integrand is negative for $0 < t < \pi$

$$\sup_{\delta'\leqslant t\leqslant\pi}\operatorname{Re}\{\psi(\xi+it)-\psi(\xi)\} = \varkappa\int_0^{\delta'}\frac{e^{\xi}\tau\cos\tau-\xi e^{\xi}\sin\tau-\tau}{\xi^2+\tau^2}\,d\tau.$$

Noting that

$$\int_0^{\delta'}\frac{\tau\cos\tau}{\xi^2+\tau^2}\,d\tau \leqslant \frac{1}{\xi^2}\cdot\frac{1}{2}\delta'^2,$$

$$\int_0^{\delta'}\frac{\sin\tau}{\xi^2+\tau^2}\,d\tau \geqslant \frac{1}{\xi^2+\delta'^2}(1-\cos\delta')$$

and

$$\int_0^{\delta'}\frac{\tau}{\xi^2+\tau^2}\,d\tau > 0,$$

we obtain

$$\sup_{\delta'\leqslant\tau\leqslant\pi}\operatorname{Re}\{\psi(\xi+it)-\psi(\xi)\} \leqslant \varkappa\Big\{e^{\xi}\cdot\frac{\delta'^2}{2\xi^2}-\frac{\xi e^{\xi}}{\xi^2+\delta'^2}(1-\cos\delta')\Big\},$$

so that

(5.10) $$|I_1^{(3)}| \leqslant \pi e^{\psi(\xi)}\exp\{\varkappa e^{\xi}(\tfrac{1}{2}\delta'^2\xi^{-2}-\xi(\xi^2+\delta'^2)^{-1}(1-\cos\delta'))\}.$$

A similar analysis shows that the same inequality holds with $I_1^{(3)}$ replaced by $I_1^{(1)}$.

To deal with $I_1^{(2)}$ we note that for each t satisfying $|t| < \delta'$ there exists a t_1 satisfying $|t_1| < t$, with the property that

$$\psi(\xi+it) = \psi(\xi)-\tfrac{1}{2}t^2\psi''(\xi+it_1).$$

Hence, by (5.9), if $|t| < \delta'$,

$$\psi(\xi+it) = \psi(\xi)-\tfrac{1}{2}t^2\psi''(\xi)(1+\theta)$$

with $|\theta| < \varepsilon$.

We therefore have

$$I_1^{(2)} = e^{\psi(\xi)}\int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1+\theta)}\,dt.$$

If $\theta = \theta_1 + i\theta_2$ we have

$$\operatorname{Im} I_1^{(2)} = e^{\psi(\xi)} \int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1+\theta_1)} \sin\{-\tfrac{1}{2}\theta_2 t^2 \psi''(\xi)\}\, dt$$

and

$$\operatorname{Re} I_1^{(2)} = e^{\psi(\xi)} \int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1+\theta_1)} \cos\{\tfrac{1}{2}\theta_2 t^2 \psi''(\xi)\}\, dt.$$

Now

$$\begin{aligned}
\psi''(\xi)\,\theta_2 &= \operatorname{Im}\psi''(\xi)\,\theta = \operatorname{Im}\big(\psi''(\xi+it) - \psi''(\xi)\big) = \operatorname{Im}\psi''(\xi+it)\\
&= \varkappa \operatorname{Im}\left\{\frac{e^{\xi+it}}{\xi+it} - \frac{e^{\xi+it}-1}{(\xi+it)^2}\right\}\\
&= \varkappa \operatorname{Im}\{e^{\xi+it}(\xi - it)(\xi^2+t^2)^{-1} - (e^{\xi+it}-1)(\xi^2 - t^2 - 2i\xi t)(\xi^2+t^2)^{-2}\}\\
&= \varkappa\left\{\frac{e^{\xi}\xi\sin t - e^{\xi} t\cos t}{\xi^2+t^2} - \frac{e^{\xi}(\xi^2-t^2)\sin t - 2\xi t(e^{\xi}\cos t - 1)}{(\xi^2+t^2)^2}\right\}.
\end{aligned}$$

Therefore

$$|\psi''(\xi)| = O(e^{\xi}\xi^{-1}|t|).$$

It follows that

$$|\operatorname{Im} I_1^{(2)}| = O\left(e^{\psi(\xi)} e^{\xi} \xi^{-1} \delta'^3 \int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1-\varepsilon)}\, dt\right)$$

and

$$\operatorname{Re} I_1^{(2)} = e^{\psi(\xi)} \int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1+\theta_1)}\, dt \{1 + O(e^{2\xi}\xi^{-2}\delta'^6)\}.$$

Now

$$\int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1+\theta_1)}\, dt$$

lies between the two quantities

$$\int_{-\delta'}^{\delta'} e^{-\frac{1}{2}t^2\psi''(\xi)(1\pm\varepsilon)}\, dt = \sqrt{\frac{2}{\psi''(\xi)(1\pm\varepsilon)}} \int_{-\delta'\sqrt{\psi''(\xi)(1\pm\varepsilon)/2}}^{\delta'\sqrt{\psi''(\xi)(1\pm\varepsilon)/2}} e^{-u^2}\, du$$

which are asymptotic to

$$\sqrt{\frac{2\pi}{\psi''(\xi)(1\pm\varepsilon)}}$$

if δ' is chosen so that $\delta'\sqrt{\psi''(\xi)} \to \infty$.

Therefore $I_1^{(2)}$ is sandwiched between two quantities

$$e^{\psi(\xi)}\sqrt{\frac{2\pi}{\psi''(\xi)(1\pm\varepsilon)}}\{1 + O(e^{2\xi}\xi^{-2}\delta'^6) + O(e^{\xi}\xi^{-1}\delta'^3)\}.$$

Since $\psi''(\xi) \sim e^{\xi}\xi^{-1}$, if we choose

$$\delta' = e^{-\frac{5}{12}\xi},$$

we shall have

$$\delta'\sqrt{\psi''(\xi)} \sim e^{\frac{1}{12}\xi}\xi^{-\frac{1}{2}} \to \infty$$

and

$$e^{\xi}\xi^{-1}\delta'^3 = e^{-\frac{1}{4}\xi}\xi^{-1} \to 0.$$

Also, with this choice of δ', from (5.10)

$$\begin{aligned}
|I_1^{(3)}| &\leqslant \frac{\pi e^{\psi(\xi)}}{\sqrt{\psi''(\xi)}} \exp\{\varkappa e^{\xi}(\tfrac{1}{2}\delta'^2\xi^{-2} - \xi(\xi^2+\delta'^2)^{-1}\tfrac{1}{4}\delta'^2) + \tfrac{1}{2}\xi\}\\
&= \frac{\pi e^{\psi(\xi)}}{\sqrt{\psi''(\xi)}} \exp\{\varkappa e^{\frac{1}{6}\xi}(\tfrac{1}{2}\xi^{-2} - \tfrac{1}{4}\xi(\xi^2+\delta'^2)^{-1}) + \tfrac{1}{2}\xi\}\\
&= o\big(e^{\psi(\xi)}/\sqrt{\psi''(\xi)}\big),
\end{aligned}$$

and similarly for $I_1^{(1)}$.

Finally it is easy to see from (5.7) and (5.8) that

$$I_2 = o\left(\frac{e^{\psi(\xi)}}{\sqrt{\psi''(\xi)}}\right) \quad \text{and} \quad I_3 = o\left(\frac{e^{\psi(\xi)}}{\sqrt{\psi''(\xi)}}\right).$$

Collecting these results and letting $\varepsilon \to 0$, we obtain

$$F(u) \sim \frac{e^{\psi(\xi)}}{\sqrt{2\pi\psi''(\xi)}},$$

so that, from (5.4)

$$\tau_\varkappa(u) \sim \frac{1}{C}\cdot\frac{e^{\psi(\xi)}}{\sqrt{2\pi\psi''(\xi)}}.$$

Noting that $\psi''(\xi) \sim u$ and that

$$\begin{aligned}
\psi(\xi) &= -u\xi + \varkappa\int_0^{\xi}\frac{e^s-1}{s}\, ds = -\varkappa(e^{\xi}-1) + \varkappa\int_0^{\xi}\frac{e^s-1}{s}\, ds\\
&= -\varkappa\int_0^{\xi}\frac{se^s - e^s + 1}{s}\, ds,
\end{aligned}$$

if we make the transformation to the variable η defined by

$$\varkappa(e^s - 1) = \eta s \tag{5.11}$$

so that $s = 0$ corresponds to $\eta = \varkappa$ and $s = \xi$ to $\eta = u$ and

$$\varkappa e^s ds = \eta ds + s d\eta,$$

i.e.

$$\left(\varkappa e^s - \varkappa \frac{e^s - 1}{s}\right) ds = s d\eta,$$

we obtain

$$\tau_\varkappa(u) \sim \frac{1}{C\sqrt{2\pi u}} \exp\left\{-\int_\varkappa^u s\, d\eta\right\},$$

i.e.

$$\sigma'_\varkappa(u) \sim \frac{1}{C\sqrt{\pi u}} \exp\left\{-\int_\varkappa^{\frac12 u} s\, d\eta\right\}. \tag{5.12}$$

It is convenient to interpolate here a few of the properties of the function s defined by (5.11). These are required in the subsequent calculations but also shed more light on the behaviour of the functions we are investigating than does an estimate of the type (5.12), though this is in some ways more convenient.

We easily deduce by taking logarithms in (5.11) that

$$\log\varkappa + s = \log(\varkappa + \eta s) = \log\eta + \log s + O\left(\frac{1}{\eta s}\right) \tag{5.13}$$

so that

$$s \sim \log\eta. \tag{5.14}$$

Also, differentiating

$$\varkappa e^s s' = s'\eta + s,$$

i.e.

$$s' = \frac{s}{\varkappa e^s - \eta} = \frac{s}{\varkappa + \eta s - \eta} \sim \frac{1}{\eta} \tag{5.15}$$

and, differentiating again,

$$\varkappa e^s s'' + \varkappa e^s (s')^2 = 2s' + s''\eta,$$

so that

$$s'' = \frac{2s' - (s')^2(\eta s + \varkappa)}{\eta s + \varkappa - \eta} \sim -\frac{1}{\eta^2}. \tag{5.16}$$

We require also the remark that

$$s\left(\tfrac12(u-1)\right) = s\left(\tfrac12 u\right) - \frac{1}{u} + O\left(\frac{1}{u\log u}\right) \tag{5.17}$$

which follows from the Mean-Value Theorem and (5.15). We also deduce that

$$\log s\left(\tfrac12(u-1)\right) = \log s\left(\tfrac12 u\right) + O\left(\frac{1}{u\log u}\right). \tag{5.18}$$

We can now obtain a more enlightening expression for $\sigma'_\varkappa(u)$ than (5.12). For

$$\int_\varkappa^{\frac12 u} s\, d\eta = \tfrac12 u s\left(\tfrac12 u\right) - \int_\varkappa^{\frac12 u} \eta s'\, d\eta.$$

It is easy to see from (5.15) that

$$\int_\varkappa^{\frac12 u} \eta s'\, d\eta = O(u),$$

so that from (5.13)

$$\int_\varkappa^{\frac12 u} s\, d\eta = \tfrac12 u\log u + O(u\log\log u)$$

and

$$\sigma'_\varkappa(u) \sim \exp\{-\tfrac12 u\log u + O(u\log\log u)\}.$$

More accurate expressions may also be obtained (cf. de Bruijn).

Finally we note that

$$\begin{aligned}\int_{\frac12(u-1)}^{\frac12 u} s\, d\eta &= \tfrac12 u s\left(\tfrac12 u\right) - \tfrac12(u-1)s\left(\tfrac12(u-1)\right) - \int_{\frac12(u-1)}^{\frac12 u} \eta s'\, d\eta \\ &= \tfrac12 u\left\{s\left(\tfrac12 u\right) - s\left(\tfrac12(u-1)\right)\right\} + \tfrac12 s\left(\tfrac12(u-1)\right) - \int_{\frac12(u-1)}^{\frac12 u} \{1+o(1)\}\, d\eta \\ &= \tfrac12 + O\left(\frac1u\right) + \tfrac12 s\left(\tfrac12 u\right) + O\left(\frac1u\right) - \tfrac12 + o(1) = \tfrac12 s\left(\tfrac12 u\right) + o(1).\end{aligned} \tag{5.19}$$

We now prove two lemmas which will assist in the transition to asymptotic formulae for $\eta_\varkappa$, $\eta_{\varkappa,2}$ and $\eta_{\varkappa,3}$.

LEMMA 1. *Let $f(u)$ be a positive function such that the infinite integral $\int_u^\infty f(t)\,dt$ converges. Suppose that $f(u) \sim g(u)$. Then $\int_u^\infty g(t)\,dt$ converges and*

$$\int_u^\infty f(t)\,dt \sim \int_u^\infty g(t)\,dt.$$

Proof. We may write

$$g(u) = f(u)\{1+\varepsilon(u)\}$$

where $\varepsilon(u)\to 0$.

It is clear that $\int_u^\infty \varepsilon(t)f(t)\,dt$ exists and therefore that $\int_0^\infty g(t)\,dt$ exists. Moreover

$$\int_u^\infty g(t)\,dt = \int_u^\infty f(t)\,dt + \int_u^\infty f(t)\varepsilon(t)\,dt$$

and

$$\Big|\int_u^\infty \varepsilon(t)f(t)\,dt\Big| \leqslant \sup_{t\geqslant u}|\varepsilon(t)| \int_u^\infty f(t)\,dt = o\Big(\int_u^\infty f(t)\,dt\Big).$$

Therefore

$$\int_u^\infty g(t)\,dt \sim \int_u^\infty f(t)\,dt.$$

LEMMA 2. *If h is a twice differentiable function with the property that $\int_u^\infty e^{h(t)}dt$ is convergent; and if the following two conditions are satisfied:*

(i) $e^{h(u)}/h'(u)\to 0$ *as* $u\to\infty$,

(ii) $h''(u)/\{h'(u)\}^2\to 0$ *as* $u\to\infty$;

then

$$\int_u^\infty e^{h(t)}dt \sim \frac{e^{h(u)}}{\{-h'(u)\}}.$$

Proof. Integrating by parts

$$\int_u^\infty e^{h(t)}dt = \int_u^\infty \frac{1}{h'(t)}\,d(e^{h(t)}) = \frac{e^{h(u)}}{\{-h'(u)\}} + \int_u^\infty e^{h(t)}\frac{h''(t)}{\{h'(t)\}^2}\,dt.$$

It is easy to see that

$$\int_u^\infty e^{h(t)}\frac{h''(t)}{\{h'(t)\}^2}\,dt = o\Big(\int_u^\infty e^{h(t)}dt\Big)$$

and the result follows.

We can now deduce asymptotic expressions in turn for $1-\sigma_\varkappa(u)$, $\eta_\varkappa(u)$, $\eta_{\varkappa,2}(u)$ and $\eta_{\varkappa,3}(u)$.

Writing

$$h(u) = -\int_\varkappa^{\frac12 u} s\,d\eta - \tfrac12\log u$$

we have from (5.12) and Lemma 1

$$1-\sigma_\varkappa(u) = \int_u^\infty \sigma_\varkappa'(t)\,dt \sim \frac{1}{C\sqrt{\pi}}\int_u^\infty e^{h(t)}dt.$$

Since

$$h'(u) = -\tfrac12 s(\tfrac12 u) - \frac{1}{2u}$$

and

$$h''(u) = -\tfrac14 s'(\tfrac12 u) + \frac{1}{2u^2}$$

it follows from (5.14) and (5.15) that the conditions of Lemma 2 are satisfied. We therefore deduce that

$$1-\sigma_\varkappa(u) \sim \frac{1}{C\sqrt{\pi}}\cdot\frac{2}{s(\frac12 u)}\exp\Big\{-\int_\varkappa^{\frac12 u} s\,d\eta - \tfrac12\log u\Big\} \tag{5.20}$$

$$= \frac{2}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac12 u} s\,d\eta - \tfrac12\log u - \log s(\tfrac12 u)\Big\}.$$

This shows that $1-\sigma_\varkappa(u)$ tends to zero more rapidly than $\sigma_\varkappa'(u)$, by a factor which is roughly $u^{1/2}\log u$.

An application of (5.18) and (5.19) shows that

$$1-\sigma_\varkappa(u-1) \sim \frac{2}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac12 u} s\,d\eta + \tfrac12 s(\tfrac12 u) - \tfrac12\log u - \log s(\tfrac12 u)\Big\}$$

and so it follows that

$$u^{\varkappa-1}\Big(\frac{1}{\sigma_\varkappa(u-1)}-1\Big) \sim \frac{2}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac12 u} s\,d\eta + \tfrac12 s(\tfrac12 u) - \log s(\tfrac12 u) + (\varkappa-\tfrac32)\log u\Big\}.$$

We must now check whether we can apply Lemma 2 with

$$h(u) = -\int_\varkappa^{\frac12 u} s\,d\eta + \tfrac12 s(\tfrac12 u) - \log s(\tfrac12 u) + (\varkappa-\tfrac32)\log u.$$

In fact

$$h'(u) = -\tfrac12 s(\tfrac12 u) + \tfrac14 s'(\tfrac12 u) - \tfrac12\frac{s'(\frac12 u)}{s(\frac12 u)} + \frac{\varkappa-\frac32}{u} \sim -\tfrac12\log u$$

and

$$h''(u) = -\tfrac14 s'(\tfrac12 u) + \tfrac18 s''(\tfrac12 u) - \tfrac14\frac{s(\frac12 u)s''(\frac12 u) - \{s'(\frac12 u)\}^2}{\{s(\frac12 u)\}^2} - \frac{\varkappa-\frac32}{u^2}$$

so that (5.14), (5.15) and (5.16) show that the conditions of Lemma 2 are satisfied. We deduce that

(5.21)
$$\eta_\varkappa(u) \sim \varkappa u^{-\varkappa}\cdot\frac{2}{C\sqrt{\pi}}\cdot\frac{2}{s(\frac{1}{2}u)}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}s(\tfrac{1}{2}u)-\log s(\tfrac{1}{2}u)+(\varkappa-\tfrac{3}{2})\log u\Big\}$$
$$=\frac{4\varkappa}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}s(\tfrac{1}{2}u)-2\log s(\tfrac{1}{2}u)-\tfrac{3}{2}\log u\Big\}.$$

This again shows an improvement of roughly $u^{1/2}\log u$ over $1-\sigma_\varkappa(u)$.

We may now prove by induction on r that

(5.22)
$$\eta_{\varkappa,r}(u) \sim \frac{2^{r+1}\varkappa^r}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}rs(\tfrac{1}{2}u)-(r+1)\log s(\tfrac{1}{2}u)-(r+\tfrac{1}{2})\log u\Big\}.$$

For the case $r=1$ is simply (5.21) and we may deduce in turn precisely as in the passage from (5.20) to (5.21), that

$$\eta_{\varkappa,r}(u-1) \sim \frac{2^{r+1}\varkappa^r}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}(r+1)s(\tfrac{1}{2}u)-(r+1)\log s(\tfrac{1}{2}u)-(r+\tfrac{1}{2})\log u\Big\}$$

and that Lemma 2 is applicable with

$$h(u) = -\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}(r+1)s(\tfrac{1}{2}u)-(r+1)\log s(\tfrac{1}{2}u)+(\varkappa-r-\tfrac{3}{2})\log u.$$

It follows that

$$\eta_{\varkappa,r+1}(u) \sim \frac{2^{r+2}\varkappa^{r+1}}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta+\tfrac{1}{2}(r+1)s(\tfrac{1}{2}u)-(r+2)\log s(\tfrac{1}{2}u)-(r+\tfrac{3}{2})\log u\Big\},$$

which completes the proof of (5.22).

An application of (5.13) enables us to deduce that

(5.23)
$$\eta_{\varkappa,r}(u) \sim \frac{2^{r/2+1}\varkappa^{r/2+1}}{C\sqrt{\pi}}\exp\Big\{-\int_\varkappa^{\frac{1}{2}u} s\,d\eta-(\tfrac{1}{2}r+1)\log s(\tfrac{1}{2}u)-\tfrac{1}{2}(r+1)\log u\Big\}.$$

Comparison of (5.20) with (5.23) for $r=2$ shows the truth of the assertion of § 3, viz that

$$\eta_{\varkappa,2}(u) = o\Big(\frac{1}{\sigma_\varkappa(u)}-1\Big).$$

Also putting $r=1$ and $r=3$ yields the corresponding result for lower bound functions, that is

$$\eta_{\varkappa,3}(u) = o\big(\eta_\varkappa(u)\big),$$

thus showing the superiority of our bounds to those of Ankeny and Onishi.

Of particular interest from the point of view of the applications of the sieve is the improved value of what may be called the sieving limit, i.e. the root of the equation $f_\varkappa(u)=0$. The improved values are tabulated below for small values of $\varkappa$, but unfortunately the improvement over the values of Ankeny and Onishi does not seem to be large enough to yield any interesting new results in the applications.

$\varkappa$	Ankeny–Onishi sieving limit	Our sieving limit
2	4.42	4.38
3	6.85	6.82
4	9.32	9.30
5	11.80	11.78
6	14.28	14.27
7	16.77	16.76
8	19.25	19.25
9	21.74	21.73

References

[1] N. C. Ankeny and H. Onishi, *The general sieve*, Acta Arith. 10 (1964), pp. 31–62.
[2] N. G. de Bruijn, *Some Volterra integral equations all solutions of which are convergent*, Proc. Kon. Ned. Akad. Wetensch. 52 (1950), pp. 813–821 = Indag. Math. 12 (1950), pp. 257–265.
[3] — *The asymptotic behaviour of a function occurring in the theory of primes*, J. Ind. Math. Soc. 15 (1951), pp. 25–32.
[4] H. W. Hagedorn, *Sieve methods and polynomial sequences*, Acta Arith. 28 (1975), pp. 245–252.
[5] H. Halberstam and H.-E. Richert, *Mean value theorems for a class of arithmetical functions*, ibid. 18 (1971), pp. 243–256.
[6] — — *The distribution of polynomial sequences*, Mathematika 19 (1972), pp. 25–50.
[7] — — *Sieve Methods*, Academic Press, New York and London 1974.

[8] W. B. Jurkat and H.-E. Richert, *An improvement of Selberg's sieve method I*, Acta Arith. 11 (1965), pp. 217–240.

[9] J. W. Porter, *Some numerical results in the Selberg sieve method*, ibid. 20 (1972), pp. 417–421.

[10] H.-E. Richert, *Selberg's sieve with weights*, Mathematika 16 (1969), pp. 1–22.

[11] A. Selberg, *Sieve methods*, Proceedings of the Stony Brook Number Theory Conference, 1969, American Mathematical Society.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY COLLEGE
Cardiff, Wales

Received on 12. 7. 1974 (595)

ACTA ARITHMETICA
XXIX (1976)

An asymptotic formula for the property $(n, f(n)) = 1$ for a class of multiplicative functions

by

E. J. SCOURFIELD* (London)

1. Introduction. A number of authors have investigated the problem of estimating the sum

$$\sum_{\substack{1 \leqslant n \leqslant x \\ (n, f(n)) = 1}} 1 \tag{1}$$

for certain types of integer-valued arithmetic functions f. If the arithmetic properties of n and $f(n)$ are more or less unrelated, probabilistic arguments lead one to expect that the sum in (1) is asymptotic to $6x/\pi^2$, and some results in this direction have been obtained, in particular by Watson [15], Erdös and Lorentz [2] and Hall [6], [7], [8]. There is marked contrast between Hall's result for certain strongly additive functions, later improved in some cases by Faĭnleĭb [4], and the result to be derived in this paper for a related class of multiplicative functions. In [6] and [8] Hall considered the strongly additive function given by

$$f(n) = \sum_{p \mid n} p, \tag{2}$$

and in [7] he investigated a class of functions that includes functions of the type

$$f(n) = \sum_{p \mid n} g(p), \tag{3}$$

where g is a polynomial with integer coefficients satisfying some further conditions. Taking $g(x) = x$ in (3) gives (2), and in both cases the sum (1) is asymptotic to $6x/\pi^2$; in fact a very much more precise result was obtained for (2), using a combination of elementary and analytical argu-

*The author is indebted to the Polish Ministry of Science, Technology and Higher Education for financial support during a period of three months, in the autumn of 1973, spent at the Institute of Mathematics of the University of Wrocław, where much of the work for this paper was done.