References

[1] L. Carlitz, *Invariantive theory of equations in a finite field*, Trans. Amer. Math. Soc. 75 (1953), pp. 405–427.
[2] S. R. Cavior, *Equivalence classes of functions over a finite field*, Acta Arith. 10 (1964), pp. 119–136.
[3] — *Equivalence classes of sets of polynomials over a finite field*, Journ. Reine Angew. Math. 225 (1967), pp. 191–202.
[4] L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, New York 1958.

THE PENNSYLVANIA STATE UNIVERSITY
Sharon, Pennsylvania, U.S.A.

Received on 17. 5. 1974
and in revised form on 30. 9. 1974 (575)

ACTA ARITHMETICA
XXIX (1976)

Some results on the distribution of values of additive functions on the set of pairs of positive integers, II

by

G. Jogesh Babu (Urbana, Ill.)

1. Introduction. H. DeLange [1] in 1969 defined a density for sets of pairs $[m, n]$ of positive integers and determined it for some sets defined by arithmetical properties. In this paper we give necessary and sufficient conditions for a real-valued additive arithmetic function f on the set of pairs of positive integers to have a distribution (mod 1) and generalize a result obtained in [5] to additive functions defined on the set of pairs of positive integers.

2. Notations and definitions. Throughout this paper the letters p, q with or without suffixes denote always prime numbers. The letters $m, n, r, s, \ldots$ with or without suffixes denote positive integers and t, k denote non-negative integers. If A is a set of pairs of positive integers then $N(A)$ denotes the cardinality of the pairs in A. Let E be a set of pairs $[m, n]$ of positive integers. If

$$(1/xy)N\{[m, n]\in E:\ m \leqslant x \text{ and } n \leqslant y\}$$

tends to a limit a as x and y tend to infinity independently, then we say that the set E possesses density a, see [1].

Let Z_2 denote the set of pairs of positive integers.

Definition. A real-valued function on Z_2 is said to be *additive* if

$$f(m_1m_2, n_1n_2) = f(m_1, n_1) + f(m_2, n_2)$$

whenever $(m_1n_1, m_2n_2) = 1$.

Definition. A real-valued additive function f on Z_2 is said to have *distribution* (mod 1) if there is a nondecreasing, right continuous function F on the real line such that $F(c) = 0$ if $c < 0$, $F(c) = 1$ if $c > 1$ and for all continuity points $a, b \in (0, 1)$ of F and $a < b$, the density of

$$[[m, n]:\ a < \{f(m, n)\} < b]$$

exists and equals $F(b) - F(a)$, where $\{z\}$ denotes the fractional part of z.

We put $\|x\| = \min(\{x\}, 1-\{x\})$ and $e(t) = \exp(2\pi it)$.

3. Results.

THEOREM 1. *Let f be a real-valued additive function on Z_2.*

(i) *f has uniform distribution* (mod 1) *if and only if for each non-zero integer k*

$$\sum_p \frac{1}{p}\left(\|kf(p,1)-t\log p\|^2+\|kf(1,p)-u\log p\|^2\right)=\infty, \tag{3.1}$$

for all real t and u.

(ii) *f has a non-uniform distribution* (mod 1) *if and only if each of the following three series*

$$\sum_p \frac{1}{p}\left(\|kf(p,1)\|^2+\|kf(1,p)\|^2\right),$$

$$\sum_p \frac{1}{p}\,\|kf(p,1)\|\operatorname{sgn}\left(\tfrac{1}{2}-\{kf(p,1)\}\right), \tag{3.2}$$

$$\sum_p \frac{1}{p}\,\|kf(1,p)\|\operatorname{sgn}\left(\tfrac{1}{2}-\{kf(1,p)\}\right)$$

converge, for at least one positive integer k, where

$$\operatorname{sgn}(x)=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -1 & \text{if } x<0.\end{cases}$$

Remark. This result was also obtained by Delange (personal communication) independently under an extra assumption that

$$f(2^j,2^r)=f(2^j,1)+f(1,2^r)$$

fnd

$$f(3^j,3^r)=f(3^j,1)+f(1,3^r)$$

aor all $j\geqslant 0$ and $r\geqslant 0$.

THEOREM 2. *Let f be a real-valued additive function on Z_2. Suppose there is a sequence $\{a_n\}$ of real numbers and a distribution function H on the real line such that for each of its continuity point c, the limit of*

$$n^{-2}N\{[m,m']:\ m\leqslant n,\ m'\leqslant n \text{ and } f(m,m')-a_n<c\}$$

is $H(c)$ as $n\to\infty$. Then there exist real numbers a and b and an additive function g on Z_2 such that for all $m\geqslant 1$, $n\geqslant 1$

$$f(m,n)=a\log m+b\log n+g(m,n) \tag{3.3}$$

and

$$\sum_p \frac{1}{p}\,[g(1,p)^*]^2+\sum_p \frac{1}{p}\,[g(p,1)^*]^2<\infty, \tag{3.4}$$

where, for any real number x, x^* is x or 1 according as $|x|<1$ or $|x|\geqslant 1$.

In this case, we can set $a_n=a'(n,n)+\text{constant}+o(1)$, where

$$a(x,y)=\sum_{p\leqslant x}\frac{1}{p}\,g(p,1)^*+\sum_{p\leqslant y}\frac{1}{p}\,g(1,p)^*$$

and

$$a'(x,y)=a(x,y)+a\log x+b\log y.$$

If f satisfy (3.3) with g satisfying (3.4) then there exists a distribution function G such that at each of its continuity point c,

$$(1/xy)\,N\{[m,n]:\ m\leqslant x,\ n\leqslant y,\ f(m,n)-a'(x,y)<c\}$$

tends to $G(c)$ as x and y tend to infinity independently.

4. Preliminary results.

LEMMA 1 ([5]). *Let f be a real-valued additive arithmetic function. Suppose there exists a $\delta>0$ such that for each $t\in[-\delta,\delta]$*

$$\sum_p \frac{1}{p}\left[1-\operatorname{Re}\left(\exp\left(itf(p)\right)p^{-iu(t)}\right)\right]<\infty$$

for some real number $u(t)$. Then there exist a real number a and an additive arithmetic function g such that $f(m)=a\log m+g(m)$ for all $m\geqslant 1$ and

$$\sum_p \frac{1}{p}\left(g(p)^*\right)^2<\infty.$$

LEMMA 2. *Let $b_1,\ldots,b_r$ be real numbers. For $0<\varepsilon<1/2$ there exists a non-zero integer n such that*

$$\|nb_i\|<\varepsilon,\quad i=1,\ldots,r.$$

For a proof see Theorem 201 of [6], p. 170.

DEFINITION. A complex-valued function g defined on the set of pairs of positive integers is said to be *multiplicative* if $g(1,1)=1$ and

$$g(m_1m_2,n_1n_2)=g(m_1,n_1)\,g(m_2,n_2)$$

whenever $(m_1n_1,m_2n_2)=1$.

DEFINITION. A multiplicative function g is said to have a *mean value* if the limit of

$$(1/xy)\sum_{\substack{m\leqslant x\\ n\leqslant y}} g(m,n)$$

exists as x and y tend to infinity independently.

LEMMA 3 ([2]). *Let g be a multiplicative function such that $|g(m,n)|\leqslant 1$ for all m and $n\geqslant 1$. If for all real numbers u and t*

$$\sum_p \frac{1}{p}\left[2-\operatorname{Re}\left(g(p,1)p^{-iu}+g(1,p)p^{-it}\right)\right] = \infty,$$

then g possesses zero mean value. If there exist real numbers a and b such that

$$\sum_p \frac{1}{p}\left[2-\operatorname{Re}\left(g(p,1)p^{-ia}+g(1,p)p^{-ib}\right)\right] < \infty,$$

then g possesses zero mean value if

$$\left(\sum_{j,r\geqslant 0}\left(\frac{g(2^j,2^r)}{2^{j(1+ia)+r(1+ib)}}\right)\right)\left(\sum_{j,r\geqslant 0}\left(\frac{g(3^j,3^r)}{3^{j(1+ia)+r(1+ib)}}\right)\right) = 0. \tag{4.1}$$

If (4.1) does not hold, then as x and y tend to infinity independently, we have

$$(1/xy)\sum_{\substack{m\leqslant x\\ n\leqslant y}} g(m,n) = Cx^{ia}y^{ib}L_1(\log x)L_2(\log y)+o(1)$$

and C is a non-zero complex number (a multiple of the left-hand side of (4.1)) and L_1 and L_2 are functions on the positive real line defined by

$$L_1(t) = \exp\left[i\sum_{p\leqslant e^t}\frac{1}{p}\operatorname{Im}\left(g(p,1)p^{-ia}\right)\right],$$

$$L_2(t) = \exp\left[i\sum_{p\leqslant e^t}\frac{1}{p}\operatorname{Im}\left(g(1,p)p^{-ib}\right)\right]$$

for all real numbers $t>0$.

Moreover, if g has a non-zero mean value, then the series

$$\sum \frac{1}{p}\left(1-g(p,1)\right) \tag{4.2}$$

and

$$\sum \frac{1}{p}\left(1-g(1,p)\right) \tag{4.3}$$

converge. Conversely, if (4.2) and (4.3) converge, then g has a mean value.

LEMMA 4. *An additive function f on the set of pairs of positive integers has a distribution* (mod 1) *if and only if for each integer k, there exists a real number b_k such that the limit of*

$$(1/xy)\sum_{\substack{m\leqslant x\\ n\leqslant y}} e\left(kf(m,n)\right)$$

is b_k as x and y tend to infinity independently. Moreover, f has uniform distribution (mod 1) *if and only if $b_k=0$ for all non-zero k.*

Proof of this lemma is similar to that of Lemma 2 of [4].

5. Proofs of main results.

Proof of Theorem 1. Since

$$8\|x\|^2\leqslant \sin^2\pi x\leqslant 2\pi^2\|x\|^2$$

and

$$1-\operatorname{Re}\left(e\left(kf(p,1)\right)p^{-it}\right) = 2\sin^2\left(\pi\left(kf(p,1)-t/2\pi\right)\log p\right),$$

we clearly have

$$\sum_p \frac{1}{p}\left[1-\operatorname{Re}\left(e\left(kf(p,1)\right)p^{-it}\right)\right] < \infty,$$

if and only if

$$\sum_p \frac{1}{p}\|kf(p,1)-(t/2\pi)\log p\|^2 < \infty. \tag{5.1}$$

So, if for each non-zero integer k (3.1) is satisfied, then from Lemmas 3 and 4 it follows that f has uniform distribution (mod 1). The fact that f has a non-uniform distribution (mod 1) when the three series in (3.2) converge for some positive integer k, can be shown exactly in the similar manner as in [3], pp. 226–229, using Lemmas 3 and 4 and so the proof of this is omitted.

Now to prove the converse, suppose f has uniform distribution (mod 1). Then for each non-zero integer k the limit of

$$(1/xy)\sum_{\substack{m\leqslant x\\ n\leqslant y}} e\left(kf(m,n)\right) \tag{5.2}$$

is zero as x and y tend to infinity independently. Let k be a non-zero integer. By Lemma 3 we have, either for all t and u

$$\sum_p (1/p)\left(\|kf(p,1)-t\log p\|^2+\|kf(1,p)-u\log p\|^2\right) = \infty$$

or there exist real numbers a and b such that

$$\sum_p (1/p)\left(\|kf(p,1)-a\log p\|^2+\|kf(1,p)-b\log p\|^2\right) < \infty \tag{5.3}$$

and $h(f, k, a, b) = 0$, where

$$(5.4) \quad h(f, k, a, b) = \Big(\sum_{\substack{j\geq 0\\ r\geq 0}} \frac{e\big(kf(2^j, 2^r)\big)}{2^{j(1+2\pi ia)+r(1+2\pi ib)}}\Big)\Big(\sum_{\substack{j\geq 0\\ r\geq 0}} \frac{e\big(kf(3^j, 3^r)\big)}{3^{j(1+2\pi ia)+r(1+2\pi ib)}}\Big).$$

Suppose (5.3) holds. Observe that the set of all integers k for which there is a t such that

$$\sum_p \frac{1}{p}\,\|kf(p,1)-t\log p\|^2 < \infty$$

is a group. (This can be seen by using the inequality $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$.) By this and by Lemma 2 there exists a non-zero integer r such that $h(f, rk, ra, rb) \neq 0$ and

$$\sum_p (1/p)\big(\|krf(p,1)-ra\log p\|^2 + \|krf(1,p)-rb\log p\|^2\big) < \infty.$$

From this and the fact that

$$(5.5) \qquad (1/xy)\sum_{\substack{m\leq x\\ n\leq y}} e\big(rkf(m,n)\big)$$

tends to a limit as x and y tend to infinity independently, it follows, by Lemma 3, that (5.5) does not tend to zero. This contradiction proves that (3.1) holds for all $k \neq 0$.

Now, suppose that f has a non-uniform distribution (mod 1). Then for some positive k the limit of (5.2) is non-zero. So again, by Lemma 3, (4.2) and (4.3) converge, with $g(m,n) = e\big(kf(m,n)\big)$. This gives the convergence of all the series in (3.2).

This completes the proof of Theorem 1.

Proof of Theorem 2. Let φ be the characteristic function of H. We have for all real t

$$n^{-2}\exp(-ita_n)\sum_{m,m'\leq n}\exp\big(itf(m,m')\big)\to\varphi(t)$$

as $n\to\infty$. Since $\varphi(0) = 1$ and φ is continuous at zero, there exists a $\delta > 0$ such that for all $|t| < \delta$

$$n^{-2}\sum_{m,m'\leq n}\exp\big(itf(m,m')\big)\not\to 0$$

as $n\to\infty$. By Lemma 3, for all $|t| < \delta$, there exist real numbers $a(t)$, $b(t)$ such that

$$\sum_p \frac{1}{p}\Big[2-\mathrm{Re}\big(\exp\big(itf(p,1)\big)p^{-ia(t)}\big)-\mathrm{Re}\big(\exp\big(itf(1,p)\big)p^{-ib(t)}\big)\Big] < \infty.$$

Hence by Lemma 1, there exist real numbers a, b and additive arithmetic functions g_1 and g_2 such that

$$\sum_p \frac{1}{p}\big(g_i(p)\big)^{*2} < \infty$$

for $i = 1, 2$ and for all $m, n \geq 1$,

$$f(m,1) = a\log m + g_1(m),$$

$$f(1,n) = b\log n + g_2(n).$$

If

$$g(m,n) = f(m,n) - f(m,1) - f(1,n) + g_1(m) + g_2(n),$$

clearly g satisfies (3.3) and (3.4). This proves the first part of the theorem.

Conversely, suppose f satisfies (3.3) with g satisfying (3.4). From the proof of Theorem 1 of [7] it follows that there exists a distribution function Q such that for each of its continuity point c,

$$(1/xy)N\{[m,n]:\ m\leq x,\ n\leq y:\ g(m,n)-a(m,n) < c\}$$

tends to $Q(c)$ as x and y tend to infinity independently. Let φ be the characteristic function of Q. Summing by parts and simplifying, we obtain

$$\sum_{\substack{m\leq x\\ n\leq y}}\exp\Big(it\big(f(m,n)-a\log x-b\log y-a(x,y)\big)\Big) = \big(xy\varphi(t)/(1+ita)(1+itb)\big)+o(xy)$$

as x and y tend to infinity independently, since $f(m,n) = a\log m + b\log n + g(m,n)$. This proves the theorem.

Acknowledgements. The author wishes to thank Professor J. K. Ghosh for many valuable discussions he had with him during the preparation of this paper and Professor H. Delange, whose remarks and comments helped the author to improve the results.

References

[1] H. Delange, *On some sets of pairs of positive integers*, Journ. Number Theory, 1 (1969), pp. 261–279.

[2] — *Sur les fonctions multiplicatives de plusieurs entiers*, Enseignement Math. 16 (3–4) (1970), pp. 219–246.

[3] — *On the distribution modulo 1 of additive functions*, Journ. Indian Math. Soc. 34 (1970), pp. 215–235.

[4] P. D. T. A. Elliott, *On the limiting distribution of additive functions* (mod 1), Pacific Journ. Math. 38 (1971), pp. 49–59.

[5] P. D. T. A. Elliott and C. Ryavec, *The distribution of the values of additive arithmetical functions*, Acta Math. 126 (1971), pp. 143–164.
[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, third edition, Oxford 1954.
[7] G. Jogesh Babu, *Some results on the distribution of values of additive functions on the set of pairs of positive integers I*, Acta Arith. 29 (1975), pp. 171–179.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OREGON
Eugene, Oregon, U.S.A.

Received on 17. 6. 1974
and in revised form on 25. 9. 1974 (589)

ACTA ARITHMETICA
XXIX (1976)

Über die Anwendung einer Methode von Linnik

von

K. Prachar (Wien)

Herrn Professor E. Hlawka zum 60ten Geburtstag gewidmet

Einleitung. Vom Verfasser wurde in [3] eine Methode von Linnik [1] angewendet um ein gewisses Analogon eines von A. Selberg [4] herrührenden Satzes über Primzahlen in kleinen Intervallen zu beweisen. Diese Arbeit wird im folgenden mit I bezeichnet. Hier soll zunächst ein Satz aus [1] ein wenig verbessert werden.

In § 2 geben wir unter Verwendung einer neueren Entdeckung von Halász und Turán einen Beweis für einen Satz, den Linnik [1] mit anscheinend unrichtigem Beweis veröffentlicht hat.

1. Mittels des Vorganges von Linnik beweisen wir nun den gegenüber Linnik ein wenig verbesserten

Satz 1. *Unter der Annahme der Richtigkeit der Riemann'schen Vermutung für alle L-Funktionen modulo q, $q < N(\log N)^{-C}$, q fest, hat die Gleichung $p_1+p_2 = N+hq$, wobei für gerades N auch q gerade sei, stets Lösungen in Primzahlen p_1, p_2 und ganzen Zahlen h, $0 \leqslant h \leqslant (\log N)^B$ für $3 < B < C$.*

Beweis. Sei

$$T(\alpha) = \sum_{0\leqslant h<H} e^{2\pi i q h\alpha}, \tag{1}$$

wobei H später genauer bestimmt wird.

Wenn dann für alle $a = 0, 1, \ldots, q-1$

$$\left|\alpha - \frac{a}{q}\right| > \frac{1}{qM} \tag{2}$$

gilt mit $M = [(\log N)^b]$, $b > 3$, so hat man

$$|T(\alpha)| \leqslant M \sim H/(\log N)^{B-b}, \tag{3}$$

wenn $H = [(\log N)^B]$ gesetzt wird, $B > b$.