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The reciprocity theorem for Dedekind-Rademacher sums

by

L. Carlitz (Durham, N.C.)*

1. Put

$$(1.1)\qquad ((x)) = \begin{cases} x-[x]-\frac{1}{2} & (x \neq \text{integer}), \\ 0 & (x = \text{integer}). \end{cases}$$

The Dedekind sum $s(h, k)$ is defined by

$$(1.2)\qquad s(h, k) = \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right).$$

The sum satisfies the reciprocity relation

$$(1.3)\qquad s(h, k)+s(k, h) = -\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right),$$

where $(h, k) = 1$. For proofs and references see [4].

Rademacher [3] has defined the more general sum

$$(1.4)\qquad s(h, k; y, x) = \sum_{r \,(\mathrm{mod}\, k)} \left(\left(h\frac{r+x}{k}+y\right)\right)\left(\left(\frac{r+x}{k}\right)\right),$$

where x, y are arbitrary real numbers. Grosswald [4] has agreed that it is appropriate to call $s(h, k; y, x)$ a Dedekind–Rademacher sum. In the paper cited, Rademacher proved that $s(h, k; y, x)$ satisfies

$$(1.5)\qquad s(h, k; y, x)+s(k, h; x, y) = ((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(x)+\frac{1}{hk}\bar{B}_2(hx+ky)+\frac{k}{h}\bar{B}_2(y)\right\},$$

where $(h, k) = 1$, x, y are not both integers and $\bar{B}_2(x) = \bar{B}_2(x-[x])$, where

$$B_2(x) = x^2-x+\tfrac{1}{6},$$

the Bernoulli polynomial of degree 2. The writer [1], [2] has proved a generalization of (1.5).

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Rademacher's proof of (1.5) is elegant but rather involved. In the present note we give a simplified proof of the result. The simplification is due mainly to using the function $x-[x]-\frac{1}{2}$ in place of $((x))$.

2. Put

$$\bar{B}_1(x) = x-[x]-\tfrac{1}{2} = B_1(x-[x]). \tag{2.1}$$

Then $\bar{B}_1(x+1) = \bar{B}_1(x)$ and

$$\bar{B}_1(kx) = \sum_{r\,(\mathrm{mod}\,k)} \bar{B}_1\left(x+\frac{r}{k}\right). \tag{2.2}$$

We now define

$$\bar{s}(h,k;y,x) = \sum_{r\,(\mathrm{mod}\,k)} \bar{B}_1\left(h\frac{r+x}{k}+y\right)\bar{B}_1\left(\frac{r+x}{k}\right). \tag{2.3}$$

Thus, using (2.2), we get

$$\bar{s}(h,k;y,x) = \sum_{r,s} \bar{B}_1\left(\frac{r+x}{k}\right)\bar{B}_1\left(\frac{r+x}{k}+\frac{s+y}{h}\right), \tag{2.4}$$

where r, s run through complete residue systems modulo h, k respectively. It is convenient to put

$$\xi = \frac{r+x}{k}, \qquad \eta = \frac{s+y}{h}, \tag{2.5}$$

so that (2.4) becomes

$$\bar{s}(h,k;y,x) = \sum_{r,s} \bar{B}_1(\xi)\bar{B}_1(\xi+\eta). \tag{2.6}$$

Hence

$$\bar{S} \equiv \bar{s}(h,k;y,x)+\bar{s}(k,h;x,y) = \sum_{r,s}\big(\bar{B}_1(\xi)+\bar{B}_1(\eta)\big)\bar{B}_1(\xi+\eta). \tag{2.7}$$

There is no loss in generality in assuming that

$$0 \leqslant x < 1, \quad 0 \leqslant y < 1 \tag{2.8}$$

and that

$$0 \leqslant r < k, \quad 0 \leqslant s < h. \tag{2.9}$$

Thus (2.7) becomes

$$\bar{S} = \sum_{r,s} (\xi+\eta-1)\bar{B}_1(\xi+\eta). \tag{2.10}$$

Put

$$T = \sum_{r,s}\big(\xi+\eta-1-\bar{B}_1(\xi+\eta)\big)^2 = S_1-2\bar{S}+S_2, \tag{2.11}$$

where

$$S_1 = \sum_{r,s}(\xi+\eta-1)^2, \qquad S_2 = \sum_{r,s}\bar{B}_1^2(\xi+\eta). \tag{2.12}$$

By direct computation

$$S_1 = \frac{z^2}{hk}-\left(\frac{1}{h}+\frac{1}{k}\right)z+\frac{1}{6}hk+\frac{h}{6k}+\frac{k}{6h}+\frac{1}{2}. \tag{2.13}$$

As for S_2, we have

$$\begin{aligned} S_2 &= \sum_{t\,(\mathrm{mod}\,hk)} \bar{B}_1^2\left(\frac{t+z}{hk}\right) \quad (z = hx+ky)\\ &= \sum_{t=0}^{hk-1} \bar{B}_1^2\left(\frac{t+z_0}{hk}\right) \quad (z_0 = z-[z])\\ &= \sum_{t=0}^{hk-1}\left(\frac{t+z_0}{hk}-\frac{1}{2}\right)^2, \end{aligned}$$

so that

$$S_2 = \frac{1}{6hk}(hk-1)(2hk-1)+(hk-1)\left(\frac{z_0}{hk}-\frac{1}{2}\right)+hk\left(\frac{z_0}{hk}-\frac{1}{2}\right)^2. \tag{2.14}$$

By (2.1) and (2.11)

$$T = \sum_{r,s}([\xi+\eta]-\tfrac{1}{2})^2.$$

Since $[\xi+\eta] = 0$ or 1, it follows at once that

$$T = \tfrac{1}{4}hk. \tag{2.15}$$

Substituting from (2.13), (2.14), (2.15) in (2.11), we get

$$\bar{S} = (x-\tfrac{1}{2})(y-1)+\frac{1}{2}\left\{\frac{h}{k}B_2(x)+\frac{1}{hk}B_2(z_0)+\frac{k}{h}B_2(y)\right\}. \tag{2.16}$$

Finally, removing the restriction (2.8), we state the following

THEOREM. *The sum* $\bar{s}(h,k;y,x)$ *satisfies*

$$\begin{aligned} &\bar{s}(h,k;y,x)+\bar{s}(k,h;x,y)\\ &\quad = \bar{B}_1(x)\bar{B}_1(y)+\frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(x)+\frac{1}{hk}\bar{B}_2(hx+ky)+\frac{k}{h}\bar{B}_2(y)\right\}, \end{aligned} \tag{2.17}$$

where $(h,k) = 1$ *and* x, y *are arbitrary real numbers.*

3. For x, y both integral, it is evident that (2.17) reduces to (1.3). For x = integer, $y \neq$ integer,

$$\bar{s}(h, k; y, x) = \sum_{r \pmod{k}} \bar{B}_1\left(\frac{hr}{k}+y\right)\bar{B}_1\left(\frac{r}{k}\right),$$

$$\bar{s}(k, h; x, y) = \sum_{s \pmod{h}} \bar{B}_1\left(k\frac{s+y}{h}\right)\bar{B}_1\left(\frac{s+y}{h}\right).$$

If for some pair of integers r_0, s_0, we have

$$hr_0+k(s_0+y) = 0, \tag{3.1}$$

then clearly

$$\bar{B}_1\left(\frac{hr_0}{k}+y\right) = \bar{B}_1\left(k\frac{s_0+y}{h}\right) = \bar{B}_1(0) = -\frac{1}{2},$$

while

$$\bar{B}_1\left(\frac{s_0+y}{h}\right) = \bar{B}_1\left(-\frac{r_0}{k}\right) = -\bar{B}_1\left(\frac{r_0}{k}\right).$$

Hence, if we put

$$S = s(h, k; y, x)+s(k, h; x, y), \qquad \bar{S} = \bar{s}(h, k; y, x)+\bar{s}(k, h; x, y),$$

we have

$$S-\bar{S} = \tfrac{1}{2}\bar{B}_1(y). \tag{3.2}$$

Moreover this holds even when (3.1) is not satisfied. It follows that (2.16) and (1.5) are in agreement in this case (x = integer, $y \neq$ integer). By symmetry this holds also for $x \neq$ integer, $y \neq$ integer.

Finally assume

$$x \neq \text{integer}, \qquad y \neq \text{integer}. \tag{3.3}$$

If for some pair of integers r_0, s_0, we have

$$h(r_0+x)+k(s_0+y) = 0, \tag{3.4}$$

then

$$\bar{B}_1\left(h\frac{r_0+x}{k}+y\right) = \bar{B}_1(-s_0) = -\frac{1}{2},$$

$$\bar{B}_1\left(k\frac{s_0+y}{h}+x\right) = \bar{B}_1(-r_0) = -\frac{1}{2},$$

$$\bar{B}_1\left(\frac{s_0+y}{h}\right) = B_1\left(-\frac{r_0+x}{k}\right) = -\bar{B}_1\left(\frac{r_0+x}{k}\right),$$

so that

$$S-\bar{S} = 0. \tag{3.5}$$

Moreover (3.5) holds even when (3.4) is not satisfied. It follows again that (2.16) and (1.5) are in agreement in this case.

Thus (2.16) contains both (1.3) and (1.5)

References

[1] L. Carlitz, *Generalized Dedekind sums*, Math. Zeitschr. 85 (1964), pp. 83–90.
[2] — *A theorem on generalized Dedekind sums*, Acta Arith. 11 (1965), pp. 253–260.
[3] H. Rademacher, *Some remarks on certain generalized Dedekind sums*, ibid. 9 (1964), pp. 97–105.
[4] H. Rademacher and E. Grosswald, *Dedekind sums*, The Mathematical Association of America, 1972.

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