The reciprocity theorem for Dedekind-Rademacher sums

by

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1. Put

\[(a) = \begin{cases} a - [a] - \frac{1}{k} & (a \neq \text{integer}), \\ 0 & (a = \text{integer}). \end{cases} \]

The Dedekind sum \(s(h, k)\) is defined by

\[(1.2) \quad s(h, k) = \sum_{r \equiv (mod k)} \left( \frac{r}{k} \right) \left( \frac{hr}{k} \right). \]

The sum satisfies the reciprocity relation

\[(1.3) \quad s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12} \left( \frac{h}{k} + \frac{1}{h} + \frac{k}{h} \right), \]

where \((h, k) = 1\). For proofs and references see [4].

Rademacher [3] has defined the more general sum

\[(1.4) \quad s(h, k; y, \omega) = \sum_{r \equiv (mod k)} \left( \frac{r + \omega}{k} x + y \right) \left( \frac{r + \omega}{k} \right), \]

where \(x, y\) are arbitrary real numbers. Grosswald [4] has agreed that it is appropriate to call \(s(h, k; y, \omega)\) a Dedekind-Rademacher sum. In the paper cited, Rademacher proved that \(s(h, k; y, \omega)\) satisfies

\[(1.5) \quad s(h, k; y, \omega) + s(k, h; x, y) = \frac{(\omega)(y)}{2} + \frac{1}{h^2} B_2(x) + \frac{1}{hk} B_2(x + ky) + \frac{h}{k} B_2(y), \]

where \((h, k) = 1, x, y\) are not both integers and \(B_2(x) = B_2(x - \lfloor x \rfloor)\), where

\[B_2(x) = x^2 - x + \frac{1}{6},\]

the Bernoulli polynomial of degree 2. The writer [1, 2] has proved a generalization of (1.5).

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Rademacher's proof of (1.5) is elegant but rather involved. In the present note we give a simplified proof of the result. The simplification is due mainly to using the function \( x - [x] - \frac{1}{2} \) in place of \( (\zeta(x)) \).

2. Put

\[
B_1(x) = x - [x] - \frac{1}{2} = B_1(x - [x]).
\]

Then \( B_2(x + 1) = B_1(x) \) and

\[
B_2(x) = \sum_{r \mod k} B_1\left(\frac{x + r}{k}\right).
\]

We now define

\[
\bar{r}(h, k; y, w) = \sum_{r \mod k} B_1\left(\frac{r + x}{k} + y\right) B_1\left(\frac{r + x}{k} + \frac{s + y}{h}\right).
\]

Thus, using (2.2), we get

\[
\bar{s}(h, k; y, w) = \sum_{r, s} B_1\left(\frac{r + x}{k}\right) B_1\left(\frac{r + x}{k} + \frac{s + y}{h}\right),
\]

where \( r, s \) run through complete residue systems modulo \( h, k \) respectively.

It is convenient to put

\[
\xi = \frac{r + x}{k}, \quad \eta = \frac{s + y}{h},
\]

so that (2.4) becomes

\[
\bar{s}(h, k; y, w) = \sum_{r, s} B_1(\xi) B_1(\xi + \eta).
\]

Hence

\[
\bar{S} = \bar{s}(h, k; y, w) + \bar{s}(h, k; x, y) = \sum_{r, s} (B_1(\xi) + B_1(\eta)) B_1(\xi + \eta).
\]

There is no loss in generality in assuming that

\[
0 \leq x < 1, \quad 0 \leq y < 1
\]

and that

\[
0 \leq r < h, \quad 0 \leq s < h.
\]

Thus (2.7) becomes

\[
\bar{S} = \sum_{r, s} (\xi + \eta - 1) B_1(\xi + \eta).
\]

Put

\[
T = \sum_{r, s} (\xi + \eta - 1 - B_1(\xi + \eta))^2 = S_1 - 2 \bar{S} + S_2,
\]

where

\[
S_1 = \sum_{r, s} (\xi + \eta - 1)^2, \quad S_2 = \sum_{r, s} B_1^2(\xi + \eta).
\]

By direct computation

\[
S_1 = \frac{x^2}{hk} - \left(\frac{1}{h} + \frac{1}{k}\right) x + \frac{1}{6} \frac{h}{h} x + \frac{1}{6} \frac{k}{k} y + \frac{1}{2}.
\]

As for \( S_2 \), we have

\[
S_2 = \sum_{r \mod h} B_1^2\left(\frac{r + x}{h}\right),
\]

so that

\[
S_2 = \frac{1}{6} \frac{h}{h} k (h^2 k - (2 h k - 1) + (h k - 1) \left(\frac{x}{h} \frac{1}{h} - \frac{1}{2}\right) + \frac{x}{h} \frac{y}{k} - \frac{1}{2}).
\]

By (2.1) and (2.11)

\[
T = \sum_{r, s} (\xi + \eta - 1)^2.
\]

Since \( [\xi + \eta] = 0 \) or 1, it follows at once that

\[
T = \frac{1}{h^2 k}.
\]

Substituting from (2.13), (2.14), (2.15) in (2.11), we get

\[
\bar{S} = (x - \frac{1}{2}) (y - \frac{1}{2}) + \frac{1}{h} B_1(x) + \frac{1}{k} B_1(y).
\]

Finally, removing the restriction (2.8), we state the following theorem. The sum \( \bar{s}(h, k; y, w) \) satisfies

\[
\bar{s}(h, k; y, w) = \frac{1}{2} \left(\frac{1}{h} B_1(x) + \frac{1}{k} B_1(y)\right),
\]

where \( (h, k) = 1 \) and \( x, y \) are arbitrary real numbers.
3. For \( x, y \) both integral, it is evident that (2.17) reduces to (1.3). For \( x = \text{integer}, y \neq \text{integer} \),

\[
\bar{s}(k, h; y, x) = \sum_{r \equiv y \mod h} B_1\left(\frac{hr}{k} + y\right) B_1\left(\frac{r}{k}\right),
\]

\[
\bar{s}(k, h; x, y) = \sum_{s \equiv x \mod h} B_1\left(\frac{s + y}{k}\right) B_1\left(\frac{s}{k}\right).
\]

If for some pair of integers \( r_0, s_0 \), we have

\[
hr_0 - h(s_0 + y) = 0,
\]

then clearly

\[
B_1\left(\frac{hr_0}{k} + y\right) = B_1\left(\frac{s_0 + y}{k}\right) = B_1(0) = -\frac{1}{2},
\]

while

\[
B_1\left(\frac{s_0 + y}{k}\right) = B_1\left(-\frac{r_0}{k}\right) = -B_1\left(\frac{r_0}{k}\right).
\]

Hence, if we put

\[
S = s(h, k; y, x) + s(k, h; x, y), \quad \bar{S} = \bar{s}(h, k; y, x) + \bar{s}(k, h; x, y),
\]

we have

\[
S - \bar{S} = \frac{1}{2} B_1(y).
\]

Moreover this holds even when (3.1) is not satisfied. It follows that (2.16) and (1.5) are in agreement in this case \((x = \text{integer}, y \neq \text{integer})\). By symmetry this holds also for \( x \neq \text{integer}, y \neq \text{integer} \).

Finally assume

\[
x \neq \text{integer}, \quad y \neq \text{integer}.
\]

If for some pair of integers \( r_0, s_0 \), we have

\[
h(r_0 + x) + h(s_0 + y) = 0,
\]

then

\[
B_1\left(\frac{r_0 + x}{k} + y\right) = B_1(-s_0) = -\frac{1}{2},
\]

\[
B_1\left(\frac{s_0 + y}{k} + x\right) = B_1(-r_0) = -\frac{1}{2},
\]

\[
B_1\left(\frac{s_0 + y}{h}\right) = B_1\left(-\frac{r_0 + x}{k}\right) = -B_1\left(\frac{r_0 + x}{k}\right),
\]

so that

\[
S - \bar{S} = 0.
\]

Moreover (3.5) holds even when (3.4) is not satisfied. It follows again that (2.16) and (1.5) are in agreement in this case.

Thus (2.16) contains both (1.3) and (1.5)

References


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