secondly (Lemma 10)

$$E' < DC\Pi_0 \frac{N}{(\log N)^{5/2}} (\log v) \sqrt{\left(1-\frac{1}{v}\right)} \left\{1+O\left(\frac{\log\log N}{(\log N)^{1/10}}\right)\right\}.$$

Here v is defined by the relation $z = N^{1/(2v)}$ as in (4.13). Choose v in the range $1 \leqslant v < 6/5$ permitted in Lemma 10 so as to maximise

$$g(v) = \int\limits_1^v \frac{dt}{\sqrt{t^2-t}} - \frac{C\sqrt{2}}{B}(\log v) \sqrt{\left(1-\frac{1}{v}\right)};$$

this maximum value G is positive because for $v > 1$

$$g(v) = 2\sqrt{v-1} + O\big((v-1)^{3/2}\big).$$

This gives the theorem stated in the Introduction, with

$$A = DBG\Pi_0/\sqrt{2}. \tag{6.1}$$

Here B is as in (4.13) and D is given by (2.4). The product Π_0 was defined in (4.11). Because of (4.12) we have $A > 0$ as required.

It is possible to replace the constant A of our theorem by a larger number, for example by following up the consequences of the remark made at the end of Section 3.

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Lower bounds for discriminants of number fields

by

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1. Introduction. Let K be an algebraic number field of degree $n = n_K$, with r_1 real conjugate fields and $2r_2$ complex conjugate fields, and let $D = D_K$ be the absolute value of the discriminant of K. In 1882 Kronecker [6] conjectured that

$$D > 1 \quad \text{for} \quad n > 1. \tag{1.1}$$

This very important result was first proved in 1891 by Minkowski [10] as one of the earliest applications of geometry of numbers. Subsequently, by refining his methods, Minkowski [11] showed that in fact

$$D^{1/n} \geqslant \left(\frac{\pi}{4}\right)^{2r_2/n} n^2 (n!)^{-2/n} = (e^2)^{r_1/n} \left(\frac{\pi e^2}{4}\right)^{2r_2/n} + o(1) \tag{1.2}$$
$$= (7.389\ldots)^{r_1/n}\ (5.803\ldots)^{2r_2/n} + o(1)$$

as $n \to \infty$, which is the estimate usually presented in books [8], [13]. Due to the efforts of many mathematicians, today there exists an extensive literature devoted to lower bounds for discriminants (for complete references, see [13], pp. 80–81 and [16]). Of those papers which do not use geometry of numbers methods, most prove only (1.1). Of the few which obtain lower bounds for D which are exponential in n, the best until very recently was Siegel's estimate [18], which states that for K totally real (i.e., $r_1 = n$, $r_2 = 0$),

$$D^{1/n} \geqslant 7.402\ldots + o(1)$$

as $n \to \infty$, which is slightly better than (1.2). Considerably better estimates have obtained through geometry of numbers. The best published bound for totally real K is due to Rogers [16], who showed that in this case

$$D^{1/n} \geqslant \frac{16 e^3}{\pi^2} + o(1) = 32.561\ldots + o(1)$$

as $n\to\infty$. Mulholland [12] later generalized this to prove that for any K

$$D^{1/n} \geqslant \left(\frac{16e^3}{\pi^2}\right)^{r_1/n} \left(\frac{\pi e^3}{4}\right)^{2r_2/n} + o(1) \tag{1.3}$$

$$= (32.561\ldots)^{r_1/n} (15.775\ldots)^{2r_2/n} + o(1)$$

as $n\to\infty$. Furthermore, Mulholland indicated how one can show

$$D^{1/n} \geqslant 15.887\ \ldots + o(1) \tag{1.4}$$

as $n\to\infty$.

Our purpose will be to develop a new analytical method of obtaining lower bounds for discriminants. We will prove

THEOREM 1.

$$D^{1/n} \geqslant (55)^{r_1/n}\ (21)^{2r_2/n} \tag{1.5}$$

for n sufficiently large.

Our method is based on a result of Stark [19] (our Lemma 1) which gives a relation between the discriminant of K and the zeros of the Dedekind zeta function ζ_K of K. Since our method depends on the zeros of ζ_K, about which relatively little is known, it should not be surprising that additional information about their location leads to improved estimates. Now if the Generalized Riemann Hypothesis is true, any zero $\beta+i\gamma$ of ζ_K with $0<\beta<1$ satisfies $\beta = 1/2$. We will work with a weaker condition, which we will call Hypothesis R′:

HYPOTHESIS R′: *If $\beta+i\gamma$ is a zero of the zeta function of an algebraic number field and $\beta > 1/2$, then either $|\gamma| \leqslant \frac{1}{2}(1-\beta)$ or $|\gamma| \geqslant 10$.*

We will also say that a particular field K satisfies Hypothesis R′ if the zeros of ζ_K satisfy the condition above. Hypothesis R′ is clearly implied by the Generalized Riemann Hypothesis, and it is known to be true for several number fields. We will prove

THEOREM 2. *Hypothesis* R′ *implies that*

$$D^{1/n} \geqslant (136)^{r_1/n}\ (34.5)^{2r_2/n} \tag{1.6}$$

for n sufficiently large.

In fact, an examination of the proof of Theorem 2 shows the existence of constants $c>0$ and n_0 such that if there is a sequence of fields K for which $D^{1/n}$ violates (1.6) (note that $n\to\infty$ for any such sequence), then for $n \geqslant n_0$, ζ_K has $\geqslant cn$ zeros $\beta+i\gamma$ with $\beta > 1/2$, $\frac{1}{2}(1-\beta) < |\gamma| < 10$.

The bounds of Theorems 1 and 2 can be improved further by our method. However, to avoid complicating the proofs unduly, we present only the estimates (1.5) and (1.6). Also, the conclusion of Theorem 2 can be obtained with a hypothesis somewhat weaker than Hypothesis R′, as will be indicated at the end of the proof of Theorem 2. The reason for selecting Hypothesis R′ rather than another one is that it appears to exploit our method to the fullest. To be precise, with the present method even an assumption of the Generalized Riemann Hypothesis does not seem to lead to results significantly better than those implied by Hypothesis R′. One can obtain better results by assuming, in addition to Hypothesis R′, that for fields with small discriminants there are no zeros $\frac{1}{2}+i\gamma$ with $\frac{1}{2} \leqslant |\gamma| \leqslant 2$, say, but such an assumption would be extremely implausible (cf. [9]). On the other hand, a variety of hypothetical improvements on Theorem 1 can be obtained by assuming zero-free regions smaller than those of Hypothesis R′.

In general we will be concerned only with estimates of $D^{1/n}$ as $n\to\infty$. However, this is done only to simplify the proofs. The "sufficiently large" n's of Theorems 1 and 2 are effectively computable, although very large. To obtain good estimates for n of moderate size, one has to modify our method slightly, as will be indicated in Section 3. In that section we will also give a new proof of Minkowski's basic result (1.1).

2. Applications. In this section we will discuss some applications of our estimates to the problem of infinite class field towers and to a conjecture of Serre. References to other applications may be found in [13].

For a long time it had been thought that

$$D^{1/n}\to\infty \quad \text{as} \quad n\to\infty.$$

However, in 1964 Golod and Shafarevich [4] (see also [17]) proved the existence of infinite class field towers, for which $D^{1/n}$ is constant. The smallest known values for $D^{1/n}$ corresponding to infinite class field towers come from quadratic extensions of the rationals. If t denotes the 2-rank of the class group of a quadratic field K, and K has a finite class field tower, then ([17], p. 233, Remark 4)

$$t \leqslant 5 \quad \text{if} \quad K \text{ is real,} \tag{2.1}$$

$$t \leqslant 4 \quad \text{if} \quad K \text{ is complex.} \tag{2.2}$$

Thus if $t \geqslant 6$ (K real) or $t \geqslant 5$ (K complex), K has an infinite class field tower and so there exists a sequence of fields K_i, with degrees $n_i\to\infty$, for which

$$D_{K_i}^{1/n_i} = D_K^{1/2}.$$

Moreover, if K is real, all the fields K_i are totally real, and if K is complex, all the K_i are totally complex.

The 2-rank of the class group of a quadratic field is easy to calculate; if r is the number of finite ramified primes in K, then $t = r-1$ or $r-2$ ([2], p. 225). In Tables 1 and 2 we give, separately for the real and complex cases, the smallest D for which the corresponding quadratic field has 2-rank t. The last column in each table states whether the given field has a finite or an infinite class field tower.

Thus we see that there exist totally real fields ($r_1 = n$, $r_2 = 0$) of arbitrarily high degrees with

$$D^{1/n} = 5123.1\ldots, \tag{2.3}$$

and similarly there exist totally complex fields ($r_1 = 0$, $2r_2 = n$) of arbitrarily high degrees with

$$D^{1/n} = 296.2\ldots \tag{2.4}$$

This gives us upper bounds on how far the estimates of Theorems 1 and 2 can be improved. On the other hand, lower estimates on discriminants also give us bounds on how far the Golod–Shafarevich result can be improved, since they enable us to deduce that many fields have finite class field towers. For example, the Mulholland–Rogers estimate (1.3) shows that $Q(\sqrt{d})$ has a finite class field tower for $d = -4\cdot 3\cdot 7$ and $d = 4\cdot 3\cdot 5\cdot 13$. Theorem 1 shows that this also holds for $d = -4\cdot 3\cdot 5\cdot 7$, while by Theorem 2, Hypothesis R′ implies this for $d = 4\cdot 3\cdot 5\cdot 7\cdot 11$. The $d = -4\cdot 3\cdot 5\cdot 7$ result is especially interesting, since Koch [5] has shown that there exist complex quadratic fields with only 4 ramified primes ($t = 3$) which have infinite class field towers(¹).

The gap between (2.3) and (2.4) and the estimates of Theorem 1 (or even Theorem 2) is very wide. However, it may be possible to narrow this gap either by obtaining better lower bounds for discriminants or by showing that some fields with small discriminants have infinite class field towers. (This might be possible, for example, by applying existing estimates of the Golod–Shafarevich type to class fields or subfields of class fields of quadratic fields.) It would be particularly interesting to see whether $Q(\sqrt{d})$ with $d = -4\cdot 3\cdot 5\cdot 7\cdot 13 = -5460$ has an infinite class field tower (class number is 16, and this is the last known complex quadratic field with one class per genus).

Our second application is to a conjecture of J.-P. Serre [20]. This conjecture implies that for every prime p, there exist only finitely many finite Galois extensions K of Q which are ramified only at p, and whose Galois groups G have faithful representations $\varrho\colon G\to \mathrm{GL}(2, F)$, F any finite field of characteristic p, for which determinant (ϱ (complex conjugation)) $= -1$. In his proof of Serre's conjecture for $p = 2$, Tate [20] used class field theory to reduce to the case of G non-solvable with

$$D_K^{1/n} \leqslant 2^{5/2}, \quad n = [K:Q].$$

(¹) Added in proof. D. Shanks and R. Serafin (Math. Comp. 27 (1973), pp. 183–187) have recently constructed a complex quadratic field with an infinite class field tower which has only two ramified primes.

This led to a contradiction via Minkowski's estimate (1.2). For a few other small primes, Tate's method apparently can again be used to reduce to the problem of determining all K with non-solvable groups G for which

$$D_K^{1/n} \leqslant p^{2+1/p}.$$

For $p = 3$, the bound above is 12.980 ... and Mulholland's estimate (1.3) shows that there are only finitely many such fields. For $p = 5$, the bound is 34.493 ..., and unfortunately Theorem 1 is too weak. If Hypothesis R′ is true, however, then Theorem 2 implies that there exist only finitely many such fields.

3. Basic method. Since precise numerical factors will be very important in our work, we will use the following refinement of the O-notation: for two functions, f and g, we will write

$$f = \theta(g)$$

to mean that

$$|f| \leqslant g$$

in the indicated range. We will continue to use the notation of Section 1. The non-trivial zeros $\varrho = \beta + i\gamma$ of $\zeta_K(s)$ will be those zeros for which $0 < \beta < 1$. We recall that if ϱ is a non-trivial zero for $\zeta_K(s)$, then so are $\bar\varrho$, $1-\varrho$, and $1-\bar\varrho$. For this and other basic facts we refer to [7] or [13].

We begin our work by essentially reproving Stark's lemma ([19], p. 137).

LEMMA 1. *Let*

$$A = \sqrt{D}\,2^{-r_2}\pi^{-n/2}. \tag{3.1}$$

Then

$$\log A = -\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - r_2\frac{\Gamma'}{\Gamma}(s) - \frac{1}{s} - \frac{1}{s-1} - \frac{\zeta_K'}{\zeta_K}(s) + \sum_{\varrho}{}' \frac{1}{s-\varrho}, \tag{3.2}$$

identically in the complex variable s, where ϱ runs through the non-trivial zeros of $\zeta_K(s)$, and $\sum'$ indicates that the ϱ and $\bar\varrho$ terms are to be summed together.

Proof. Let

$$\xi(s) = s(s-1)A^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_K(s). \tag{3.3}$$

Then $\xi(s)$ is an entire function and

$$\xi(1-s) = \xi(s). \tag{3.4}$$

Since $\xi(s)$ is of order 1, Hadamard's factorization theorem gives us the expansion

$$\xi(s) = e^{a+bs}\prod_{\varrho}\left(1-\frac{s}{\varrho}\right)e^{s/\varrho}$$

for some a and b, where ϱ runs through all the zeros of $\xi(s)$, which are exactly the non-trivial zeros of $\zeta_K(s)$. Differentiating this product logarithmically gives us

$$\text{(3.5)}\qquad \frac{\xi'}{\xi}(s) = b+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right),$$

where the sum converges absolutely. But by (3.4),

$$\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(1-s),$$

and so

$$b+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right) = -b-\sum_{\varrho}\left(\frac{1}{(1-\varrho)-s}+\frac{1}{\varrho}\right).$$

Since $1-\varrho$ is a zero whenever ϱ is, we obtain

$$b = -\sum_{\varrho}{}'\frac{1}{\varrho},$$

where we now have to sum the ϱ and $\bar{\varrho}$ terms together. Thus (3.5) becomes

$$\frac{\xi'}{\xi}(s) = \sum_{\varrho}{}'\frac{1}{s-\varrho}.$$

Combining this with the definition (3.3) of $\xi(s)$, we obtain (3.2).

As was noticed by Stark ([19], p. 140), identity (3.2) of Lemma 1 immediately yields a good lower bound for the discriminant, since for $s = \sigma > 1$,

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) > 0,\qquad \sum_{\varrho}{}'\frac{1}{\sigma-\varrho} = \sum_{\varrho}\frac{1}{2}\left(\frac{1}{\sigma-\varrho}+\frac{1}{\sigma-\bar{\varrho}}\right) = \sum_{\varrho}\frac{\sigma-\beta}{|\sigma-\varrho|^2} > 0,$$

and so

$$\text{(3.6)}\qquad \log A > -\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}\right)-r_2\frac{\Gamma'}{\Gamma}(\sigma)-\frac{1}{\sigma}-\frac{1}{\sigma-1}.$$

Taking $\sigma = 1+n^{-1/2}$, say, and using the definition of A, we obtain

$$\frac{1}{n}\log D \geqslant \frac{r_1}{n}\left(\log\pi-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)\right)+\frac{2r_2}{n}\left(\log 2\pi-\frac{\Gamma'}{\Gamma}(1)\right)+o(1),$$

$$D^{1/n} \geqslant (22.38\ldots)^{r_1/n}(11.19\ldots)^{2r_2/n}+o(1)$$

as $n\to\infty$. Inequality (3.6) can also be used to give a new proof of Minkowski's basic result (1.1). Letting $\sigma = 3$ in (3.6) and using the definition of A, we obtain

$$\frac{1}{2}\log D > \frac{r_1}{2}\left(\log\pi-\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}\right)\right)+r_2\left(\log 2\pi-\frac{\Gamma'}{\Gamma}(\sigma)\right)-\frac{1}{\sigma}-\frac{1}{\sigma-1}$$

$$> \frac{r_1}{2}\cdot 1.1+r_2\cdot 0.91-\frac{5}{6}.$$

But if $n = r_1+2r_2\geqslant 2$, then either $r_1\geqslant 2$ or $r_2\geqslant 1$, and in either case $\frac{1}{2}\log D > 0$, which gives (1.1).

All our further work will be based on identity (3.2) of Lemma 1. In obtaining the estimates above we were content with observing that

$$\text{(3.7)}\qquad -\frac{\zeta_K'}{\zeta_K}(\sigma)+\sum_{\varrho}{}'\frac{1}{\sigma-\varrho} > 0 \quad\text{for}\quad \sigma > 1.$$

Our goal in the rest of this paper will be to show that the quantity above is in fact quite large. Also, while before we utilized (3.2) only for $s = \sigma > 1$, we will now use it for other values of s as well. In fact, we will use (3.2) to derive a system of inequalities which will allow us to show that the expression in (3.7) cannot be small.

To start with, let us define

$$\text{(3.8)}\qquad Z(\sigma) = -\frac{\zeta_K'}{\zeta_K}(\sigma).$$

Since for $\sigma = \mathrm{Re}(s) > 1$

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{P}\frac{\log N(P)}{N(P)^s-1},$$

where P runs through the prime ideals of K, we have

$$\text{(3.9)}\qquad \left|\frac{\zeta_K'}{\zeta_K}(s)\right| \leqslant -\frac{\zeta_K'}{\zeta_K}(\sigma) = Z(\sigma).$$

Also, since $N(P)\geqslant 2$ for all P and

$$\frac{1}{x^{\sigma_0}-1} \geqslant \frac{2^{\sigma}-1}{2^{\sigma_0}-1}\cdot\frac{1}{x^{\sigma}-1}\quad\text{for}\quad x\geqslant 2,\ 1\leqslant\sigma_0\leqslant\sigma,$$

we have

$$Z(\sigma_0) \geqslant \frac{2^{\sigma}-1}{2^{\sigma_0}-1} Z(\sigma) \quad \text{for} \quad 1 < \sigma_0 \leqslant \sigma. \tag{3.10}$$

Further, let us introduce real-valued functions G_r, G_i, E_r, and E_i by

$$G_r(s)+iG_i(s) = -\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - r_2\frac{\Gamma'}{\Gamma}(s), \tag{3.11}$$

$$E_r(s,\varrho)+iE_i(s,\varrho) = \frac{1}{s-\varrho}+\frac{1}{s-\bar{\varrho}}+\frac{1}{s-(1-\varrho)}+\frac{1}{s-(1-\bar{\varrho})}. \tag{3.12}$$

As we remarked earlier, if ϱ is a non-trivial zero of $\zeta_K(s)$, then so are $\bar{\varrho}$, $1-\varrho$, and $1-\bar{\varrho}$. Hence

$$\sum_{\varrho}{}' \frac{1}{s-\varrho} = \frac{1}{4}\sum_{\varrho}\{E_r(s,\varrho)+iE_i(s,\varrho)\},$$

where the second sum now converges absolutely. Hence if $\sigma = \mathrm{Re}(s) > 1$, by taking the real and imaginary parts of (3.2) we obtain

$$0 = G_i(s)+\theta\left(\frac{2}{\sigma-1}\right)+\theta\big(Z(\sigma)\big)+\frac{1}{4}\sum_{\varrho}E_i(s,\varrho), \tag{3.13}$$

$$\log A = G_r(s)+\theta\left(\frac{2}{\sigma-1}\right)+\theta\big(Z(\sigma)\big)+\frac{1}{4}\sum_{\varrho}E_r(s,\varrho),$$

and

$$\log A = G_r(\sigma)+\theta\left(\frac{2}{\sigma-1}\right)+Z(\sigma)+\frac{1}{4}\sum_{\varrho}E_r(\sigma,\varrho). \tag{3.14}$$

We next eliminate $\log A$ from these equations by subtracting the third from the second to obtain

(3.15)

$$0 = G_r(s)-G_r(\sigma)+\theta\left(\frac{4}{\sigma-1}\right)-Z(\sigma)\big(1+\theta(1)\big)+\frac{1}{4}\sum\{E_r(s,\varrho)-E_r(\sigma,\varrho)\}.$$

Define now, with $\sigma = \mathrm{Re}(s)$ (as always),

$$E_r^*(s,\varrho) = E_r(s,\varrho)-E_r(\sigma,\varrho), \tag{3.16}$$

$$G_r^*(s) = G_r(s)-G_r(\sigma), \tag{3.17}$$

and take any $\sigma_0 > 1$ (to be specified later). Rewriting (3.13) and (3.15) as a system of inequalities using (3.10), (3.16), and (3.17), we find that

for all s with $\sigma \geqslant \sigma_0$,

$$\frac{1}{4}\sum_{\varrho}E_r^*(s,\varrho) \geqslant -G_r^*(s)-\frac{4}{\sigma-1}, \tag{3.18a}$$

$$2\,\frac{2^{\sigma_0}-1}{2^{\sigma}-1}Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}-E_r^*(s,\varrho) \geqslant G_r^*(s)-\frac{4}{\sigma-1}, \tag{3.18b}$$

$$\frac{2^{\sigma_0}-1}{2^{\sigma}-1}Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}E_i(s,\varrho) \geqslant -G_i(s)-\frac{2}{\sigma-1}, \tag{3.18c}$$

$$\frac{2^{\sigma_0}-1}{2^{\sigma}-1}Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}-E_i(s,\varrho) \geqslant G_i(s)-\frac{2}{\sigma-1}. \tag{3.18d}$$

Let us now define $e_1 = E_r^*$, $e_2 = -E_r^*$, $e_3 = E_i$, $e_4 = -E_i$, $g_1 = -G_r^*$, $g_2 = G_r^*$, $g_3 = -G_i$, $g_4 = G_i$, and

$$C_1(s) = 0, \qquad C_2(s) = 2\,\frac{2^{\sigma_0}-1}{2^{\sigma}-1}, \qquad C_3(s) = C_4(s) = \tfrac{1}{2}C_2(s).$$

Then we can rewrite (3.18) more compactly as

$$C_k(s)Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}e_k(s,\varrho) \geqslant g_k(s)-\frac{4}{\sigma-1}, \tag{3.19}$$

valid for $1 \leqslant k \leqslant 4$ and all complex s with $\sigma \geqslant \sigma_0$.

Since we are interested in lower bounds for $\log A$ and $G_r(\sigma_0)$ is easy to calculate, (3.14) shows that it will suffice to obtain lower bounds for

$$Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}E_r(\sigma_0,\varrho). \tag{3.20}$$

We will show that such bounds are implied by the system of inequalities (3.19). To accomplish this, we utilize the duality principle of linear programming [3]. Suppose that we can choose some finite collections (the finiteness condition is not essential, but this is all we will need) of numbers Y_{kj}, S_{kj}, $1 \leqslant k \leqslant 4$, $j = 1, 2, \ldots$, with $Y_{kj} \geqslant 0$ and the S_{kj} complex numbers with $\sigma_{kj} = \mathrm{Re}(S_{kj}) \geqslant \sigma_0$, such that

$$\sum_{k,j}C_k(S_{kj})Y_{kj} \leqslant 1 \tag{3.21}$$

and

$$\sum_{k,j}e_k(S_{kj},\varrho)Y_{kj} \leqslant E_r(\sigma_0,\varrho) \tag{3.22}$$

for every non-trivial zero ϱ. Then for each pair (k, j) we take the inequality in (3.19) corresponding to our k and to $s = S_{kj}$ and multiply it by Y_{kj}, thus preserving the sign of the inequality. Summing the resulting inequalities yields

$$(3.23)\qquad \sum_{k,j} g_k(S_{kj})Y_{kj} - \sum_{k,j}\frac{4}{\sigma_{kj}-1}Y_{kj} \leqslant \sum_{k,j} C_k(S_{kj})Y_{kj}Z(\sigma_0) + \frac{1}{4}\sum_{k,j}Y_{kj}\sum_{\varrho} e_k(S_{kj},\varrho) \leqslant Z(\sigma_0) + \frac{1}{4}\sum_{\varrho}\sum_{k,j} e_k(S_{kj},\varrho)Y_{kj} \leqslant Z(\sigma_0) + \frac{1}{4}\sum_{\varrho} E_r(\sigma_0,\varrho),$$

which is the desired bound.

We have shown above that we can give a lower bound for (3.20) whenever we can satisfy (3.21) and (3.22). For a given choice of the Y_{kj} and the S_{kj} it is trivial to check whether (3.21) is true. To check (3.22), however, is harder. One observation we can make is that since $E_r(s, \varrho)$ and $E_i(s, \varrho)$ are invariant under $\varrho\to\bar\varrho$, $1-\varrho$, or $1-\bar\varrho$, it suffices to check (3.22) for ϱ in the region

$$(3.24)\qquad R_0 = \{z = x+iy;\ \tfrac{1}{2}\leqslant x<1,\ 0\leqslant y\}.$$

Unfortunately, the known zero-free regions are too small for us to say that any single point of R_0 cannot be a zero of some ζ_K. Therefore to verify (3.22) in the general case we will show that it is satisfied for all points $\varrho\in R_0$. However, if we assume Hypothesis R′, then we see that (3.22) will be satisfied if it is satisfied by all points ϱ of the region

$$(3.25)\qquad R_1 = R_0 - \{z = x+iy;\ \tfrac{1}{2}(1-x)<y<10,\ \tfrac{1}{2}<x\leqslant 1\}.$$

Thus our method obtains a lower bound for (3.20), where now $Z(\sigma_0)$ can be any non-negative number and the ϱ any points of R_0 (or R_1), subject only to the conditions (3.19).

If we use the definition $\psi(s) = \Gamma'(s)/\Gamma(s)$ of the digamma function [14] and also let

$$(3.26)\qquad G_i(\sigma) = -\frac{r_1}{16}\psi''\left(\frac{\sigma}{2}\right) - \frac{r_2}{2}\psi''(\sigma),$$

then

$$(3.27)\qquad -G_r^*(\sigma+iT) = G_i(\sigma)\cdot T^2 + O(T^4 n)$$

as $T\to 0$, uniformly for $1\leqslant\sigma\leqslant 5$, say. Also, if we define, for $\sigma\geqslant 1$ and $\varrho = \beta+i\gamma\in R_0$,

$$E_i(\sigma,\varrho) = \frac{6(\sigma-\beta)\gamma^2-2(\sigma-\beta)^3}{\big((\sigma-\beta)^2+\gamma^2\big)^3} + \frac{6(\sigma-1+\beta)\gamma^2-2(\sigma-1+\beta)^3}{\big((\sigma-1+\beta)^2+\gamma^2\big)^3},$$

then as $T\to 0$ and for all σ with $1<\sigma_0\leqslant\sigma\leqslant 5$, and all $\varrho\in R_0$,

$$(3.28)\qquad E_r^*(\sigma+iT,\varrho) = T^2\cdot E_i(\sigma,\varrho) + O\big(T^4 E_r^*(\sigma_0,\varrho)\big),$$

where the implied constant depends only on σ_0.

It is now easy to obtain our basic tool for proving Theorems 1 and 2.

THEOREM 3. *Let σ_0 and σ_1 satisfy $1<\sigma_0\leqslant\sigma_1\leqslant 5$. Suppose Y_{kj} and T_{kj}, $1\leqslant k\leqslant 4$, $j = 1,\ldots,$ are finite collections of real numbers, with $Y_{kj}\geqslant 0$, and let Y be a non-negative number. Let $S_{kj} = \sigma_1+iT_{kj}$. Let $R\subseteq R_0$ be a set such that all the zeros of $\zeta_K(s)$ lying in R_0 lie in R. Finally suppose that*

$$(3.29)\qquad \sum_{k,j} C_k(S_{kj})Y_{kj}\leqslant 1$$

and

$$(3.30)\qquad YE_i(\sigma_1,z) + \sum_{k,j} e_k(S_{kj},z)Y_{kj}\leqslant E_r(\sigma_0,z)$$

for all $z\in R$. Then

$$(3.31)\qquad Z(\sigma_0) + \frac{1}{4}\sum_{\varrho} E_r(\sigma_0,\varrho)\geqslant Y\cdot G_i(\sigma_0) + \sum_{k,j} g_k(S_{kj})Y_{kj} + O(n^{1/2}),$$

where the implied constant is independent of the field K, but may depend on σ_0,σ_1, Y, the Y_{kj} and the T_{kj}.

Proof. Except for the appearance of Y and the restriction $\mathrm{Re}(S_{kj}) = \sigma_1$, this is essentially what we obtained before. To introduce Y into our scheme, define $Y' = Y\cdot n^{1/2}$, $T = n^{-1/4}$, $S = \sigma_1+iT$. Then (3.30) and (3.28) yield

$$Y'e_1(S,z) + \sum_{k,j} e_k(S_{kj},z)Y_{kj}\leqslant\big(1+O(Yn^{-1/2})\big)E_r(\sigma_0,z)$$

for all $z\in R$. Combined with (3.29) and our earlier discussion this shows that

$$\big(1+O(Yn^{-1/2})\big)\Big(Z(\sigma_0)+\frac{1}{4}\sum_{\varrho}E_r(\sigma_0,\varrho)\Big) \geqslant Y'g_1(S) + \sum_{k,j} g_k(S_{kj})Y_{kj} - \frac{4}{\sigma_1-1}\Big(Y' + \sum_{k,j} Y_{kj}\Big),$$

which together with (3.27) proves the theorem.

In looking at the above theorem we see that we have at our disposal all of the parameters $\sigma_0,\sigma_1, Y, Y_{kj}$, and T_{kj}. Finding good choices for them is in fact the greatest difficulty in applying our method. (The simplex method [3] is very useful in this situation.) In the next two sections we will present choices of these parameters which will yield Theorems 1 and 2. In other applications other choices would be better. For example, if we wanted to obtain bounds for D for relatively small n, we would choose σ_0 and σ_1 much larger (and we would be much more careful about the error terms).

The next two sections are devoted mostly to verifying (3.30) for the appropriate choices of σ_0, σ_1, Y, Y_{kj}, and T_{kj}, and are quite involved. We should stress, however, that somewhat weaker results can be obtained much more easily. For example, if we assume the Generalized Riemann Hypothesis and take $\sigma_0 = \sigma_1$ very close to 1, and $Y_{kj} = 0$ for all (k, j), then by making simple approximations we are essentially reduced to checking that

$$YE_t(1, \tfrac{1}{2}+iy) \leqslant E_r(1, \tfrac{1}{2}+iy)$$

for all $y \geqslant 0$. But the functions appearing above are very simple and one readily finds that the largest value of Y satisfying the above is 4/9, which already gives a better bound than (1.3) (although it depends on an unproved hypothesis, unlike (1.3)).

One noteworthy feature of our method is that all of the inequalities (3.19) we used were derived in a straightforward manner from (3.2). However, other inequalities involving the non-trivial zeros can be readily incorporated into our scheme. In fact, there are some inequalities which seem to lead to substantially improved estimates for discriminants, as we hope to show in the future.

4. Proof of Theorem 2. Before proceeding with the proof, let us mention that the few values of the digamma and the tetragamma functions we will use in this and the next section were derived from [15], p. 111, Eq. (34). Differentiation of that identity gave exact expressions for the digamma and tetragamma functions. The integrals were then estimated and the resulting approximate expressions were used to evaluate these functions at the arguments increased by 10. To get the needed values, recursion formulas derived from $\Gamma(z+1) = z\Gamma(z)$ were then applied. Whenever possible, the resulting values were verified by comparison with the tables in [14].

To prove Theorem 2, we apply Theorem 3 with $\sigma_0 = \sigma_1 = 1+\varepsilon$ (ε very small, to be chosen more exactly later) and with

$$Y = \frac{1}{M}, \quad Y_{2,1} = \frac{.295}{M}, \quad Y_{3,1} = \frac{.92}{M}, \quad Y_{3,2} = \frac{.115}{M}, \quad Y_{4,1} = \frac{.035}{M},$$
$$T_{2,1} = 1.25, \quad T_{3,1} = .75, \quad T_{3,2} = 1.25, \quad T_{4,1} = 1.75,$$

and $M = 1.667$. This choice clearly satisfies (3.29). Therefore if (3.30) is satisfied for all $z \in R_1$, then

$$(4.1) \quad Z(\sigma_0) + \frac{1}{4}\sum_{\varrho} E_r(\sigma_0, \varrho)$$
$$\geqslant \frac{1}{M}G_t(1) + \frac{.295}{M}G_r^*(1+\tfrac{5}{4}i) - \frac{.92}{M}G_i(1+\tfrac{3}{4}i) - \frac{.115}{M}G_i(1+\tfrac{5}{4}i) +$$
$$+ \frac{.035}{M}G_i(1+\tfrac{7}{4}i) + O(n\varepsilon) - C_\varepsilon n^{-1/2}$$
$$> \frac{1}{M}\big(1.5078 r_1 + 0.9482\cdot(2r_2)\big) + O(n\varepsilon) - C_\varepsilon n^{-1/2}.$$

for some constant C_ε depending only on ε. But by (3.14)

$$\log A = G_r(1) + O(n\varepsilon+\varepsilon^{-1}) + Z(\sigma_0) + \frac{1}{4}\sum_{\varrho} E(\sigma_0, \varrho).$$

Combining this with (4.1) and the definition (3.1) of A yields

$$D^{1/n} > (136.6)^{r_1/n}(34.9)^{2r_2/n} - C'_\varepsilon n^{-1/2} + O(\varepsilon).$$

This clearly implies Theorem 2, provided we can prove (3.30) for all sufficiently small ε.

We are left with the task of proving (3.30), which in this case reduces to showing

$$(4.2) \quad E_t(\sigma_0, z) - .295\,E_r^*(\sigma_0+\tfrac{5}{4}i, z) + .92\,E_i(\sigma_0+\tfrac{3}{4}i, z) +$$
$$+ .115\,E_i(\sigma_0+\tfrac{5}{4}i, z) - .035\,E_i(\sigma_0+\tfrac{7}{4}i, z) \leqslant 1.667\,E_r(\sigma_0, z)$$

for all $z \in R_1$. First of all, if

$$R_2 = R_1 - \{z\colon\ |z-1| < \tfrac{1}{2}\},$$

then for $z \in R_2$

$$E_r^*(\sigma_0+\tfrac{5}{4}i, z) = E_r^*(1+\tfrac{5}{4}i, z) + O\big(\varepsilon E_r(1, z)\big),$$

and similar estimates hold for the other functions appearing in (4.2). Since we are interested only in very small ε, to prove (4.2) for $z \in R_2$ it will suffice to prove

$$(4.3) \quad E_t(1, z) - .295\,E_r^*(1+\tfrac{5}{4}i, z) + .92\,E_i(1+\tfrac{3}{4}i, z) +$$
$$+ .115\,E_i(1+\tfrac{5}{4}i, z) - .035\,E_i(1+\tfrac{7}{4}i, z) \leqslant \tfrac{5}{3}E_r(1, z).$$

Let us first prove (4.3) for $x = \frac{1}{2}$, $y \geqslant 0$. Writing out the individual terms, substituting $y = u/4$ and simplifying we see that we have to prove

$$(4.4) \quad \frac{160}{3(u^2+4)} + 59\,\frac{12u^2-116}{(u^2+4)(u^4-42u^2+841)} - \frac{46}{25}\,\frac{24u^2-312}{u^4-10u^2+169} -$$
$$- \frac{46}{5}\,\frac{u^2-29}{u^4-42u^2+841} + \frac{98}{25}\,\frac{u^2-53}{u^4-90u^2+2809} - \frac{1536u^2-2048}{(u^2+4)^3}$$

is $\geqslant 0$ for $u \geqslant 0$. Substituting $u^2 = v+10$ and reducing to the least common denominator yields

$$\frac{4}{75}\,\frac{P(v)}{Q(v)},$$

where $Q(v)$ is a monic polynomial of degree 9, and

$$P(v) = 73v^8 - 20{,}214v^7 + 2{,}094{,}217v^6 - 92{,}766{,}496v^5 + \\ + 1{,}419{,}515{,}855v^4 + 3{,}533{,}810{,}602v^3 - \\ - 2{,}837{,}192{,}781v^2 - 29{,}267{,}901{,}572v + 237{,}342{,}960{,}316.$$

Since the denominators in (4.4) are all positive, $Q(v)$ is positive, and so it suffices to show that $P(v) \geqslant 0$ for $v \geqslant -10$. First consider the case $-10 \leqslant v \leqslant 0$. Let

$$P_1(v) = P(-v) = \sum a_i v^i,$$

where the a_i are, to within sign, the coefficients of $P(v)$, and only a_2 and a_3 are < 0. Since $|a_2| < 10a_1$,

$$a_1 u + a_2 u^2 \geqslant 0 \quad \text{for} \quad 0 \leqslant u \leqslant 10.$$

Also, $|a_3| < a_0(2.6)^{-3}$, so

$$a_0 + a_3 u^3 \geqslant 0 \quad \text{for} \quad 0 \leqslant u \leqslant 2.6.$$

On the other hand, $a_4 > 2.6|a_3|$, so

$$a_4 u^4 + a_3 u^3 \geqslant 0 \quad \text{for} \quad 2.6 \leqslant u \leqslant 10.$$

We conclude that $P_1(u) \geqslant 0$ for $0 \leqslant u \leqslant 10$, and so $P(v) \geqslant 0$ for $-10 \leqslant v \leqslant 0$. Now for $v \geqslant 0$, we have $P(10v) \geqslant 73\cdot 10^7 P_2(v)$, where

$$P_2(v) = 10v^8 - 277v^7 + 2{,}868v^6 - 12{,}708v^5 + 19{,}445v^4 + 4{,}840v^3 - \\ - 389v^2 - 401v + 325,$$

and so it suffices to prove $P_2(v)$ is $\geqslant 0$ for $v \geqslant 0$. For $0 \leqslant v \leqslant \frac{1}{2}$

$$389v^2 + 401v < 325, \quad 12{,}708v^5 < 19{,}445v^4, \quad 277v^7 \leqslant 2{,}868v^6,$$

and so $P_2(v) > 0$ in this case. On the other hand, for $\frac{1}{2} \leqslant v \leqslant 1$,

$$401v < 325 + 440v^3, \quad 389v^2 < 4{,}400v^3, \\ 12{,}708v^5 \leqslant 19{,}445v^4, \quad 277v^7 \leqslant 2{,}868v^6,$$

and so $P_2(v) > 0$ is this case also. Next we expand $P_2(v)$ around 2 and 5; if $w = v-2$, $t = v-5$, and $P_2(v) = P_3(w) = P_4(t)$, then

$$P_3(w) = 10w^8 - 177w^7 + 110w^6 + 2{,}920w^5 - 1{,}915w^4 - 26{,}240w^3 - \\ - 1{,}213w^2 + 98{,}523w + 91{,}807,$$

$$P_4(t) = 10t^8 + 123t^7 + 173t^6 - 2{,}093t^5 + 2{,}870t^4 + 77{,}365t^3 + \\ + 188{,}336t^2 + 96{,}834t + 112{,}345.$$

It is now easy to show, using crude estimates of the coefficients similar to those we used above, that $P_3(w) \geqslant 0$ for $-1 \leqslant w \leqslant 1.5$, and $P_4(t) \geqslant 0$ for $t \geqslant -1.5$. Combining these results we see that $P_2(v) \geqslant 0$ for $v \geqslant 0$, which finishes the proof of (4.3) for $x = \frac{1}{2}$.

It remains to prove (4.3) for $y \geqslant 10$ and (4.2) for $0 \leqslant y \leqslant \frac{1}{2}(1-x)$. We will use the following lemma, which will also be utilized in the next section.

Lemma 2. *Suppose* $\sigma \geqslant 1$, $t > 0$, $\frac{1}{2} \leqslant x \leqslant 1$, $y \geqslant 0$. *Let* $z = x+iy$ *and* $s = \sigma + it$. *Then*

$$E_l(\sigma, z) \leqslant 0 \quad \textit{for} \quad y \leqslant \frac{1}{\sqrt{3}}(\sigma - x), \tag{4.5}$$

$$E_l(\sigma, z) \leqslant \frac{6(2\sigma-1)}{y^4}, \tag{4.6}$$

$$E_i(s, z) \leqslant 0 \quad \textit{for} \quad y \leqslant t, \tag{4.7}$$

$$|E_i(s, z)| \leqslant \frac{2}{|y-t|} + \frac{2}{|y+t|}, \tag{4.8}$$

$$0 > E_i(s, z) - \frac{4t}{y^2 - t^2} > -2\,\frac{3\sigma^2 t y^2 + \sigma^2 t^3 + \sigma^4 t}{(y^2-t^2)^3} \quad \textit{for} \quad y > t, \tag{4.9}$$

$$E_r^*(s, z) \geqslant -E_r(\sigma, z), \tag{4.10}$$

$$E_r^*(s, z) \geqslant 0 \quad \textit{for} \quad y^2 \geqslant \tfrac{1}{3}(\sigma^2 + t^2), \tag{4.11}$$

$$E_r^*(s, z) \leqslant \frac{6t^2(2\sigma-1)}{(y^2-t^2)^2}, \tag{4.12}$$

$$E_r^*(s, z) \leqslant 0 \quad \textit{for} \quad y \leqslant \frac{1}{\sqrt{3}}(\sigma - x), \tag{4.13}$$

$$E_r(\sigma, z) \geqslant 2\,\frac{2\sigma-1}{\sigma^2+y^2}, \tag{4.14}$$

$$E_r(\sigma, z) \leqslant \frac{2}{\sigma - x} + \frac{2}{\sigma - 1 + x}. \tag{4.15}$$

Proof. (4.5): We have

$$E_l(\sigma, z) = \frac{6(\sigma-x)y^2 - 2(\sigma-x)^3}{((\sigma-x)^2+y^2)^3} + \frac{6(\sigma-1+x)y^2 - 2(\sigma-1+x)^3}{((\sigma-1+x)^2+y^2)^3},$$

and both the numerators are $\leqslant 0$ in the indicated range.

(4.6): From the expression above, we see that

$$E_t(\sigma, z) \leqslant \frac{6(\sigma-x)y^2}{y^6} + \frac{6(\sigma-1+x)y^2}{y^6} = \frac{6(2\sigma-1)}{y^4}.$$

(4.7):

$$E_i(s, z) = \frac{y-t}{(\sigma-x)^2+(y-t)^2} + \frac{-y-t}{(\sigma-x)^2+(y+t)^2} + \frac{y-t}{(\sigma-1+x)^2+(y-t)^2} + \frac{-y-t}{(\sigma-1+x)^2+(y+t)^2},$$

and all the numerators are $\leqslant 0$ for $y \leqslant t$.

(4.8): The expansion above gives

$$|E_i(s, z)| \leqslant 2\frac{|y-t|}{(y-t)^2} + 2\frac{|-y-t|}{(y+t)^2} = \frac{2}{|y-t|} + \frac{2}{|y+t|}.$$

(4.9): If we let a denote either $\sigma-x$ or $\sigma-1+x$, then

$$\frac{y-t}{a^2+(y-t)^2} + \frac{-y-t}{a^2+(y+t)^2} - \frac{2t}{y^2-t^2} = \frac{-2t(3a^2y^2+a^2t^2+a^4)}{(y^2-t^2)\big(a^2+(y-t)^2\big)\big(a^2+(y+t)^2\big)}.$$

Now the numerator on the right side is $\leqslant 0$, and is also

$$\geqslant -2t(3\sigma^2y^2+\sigma^2t^2+\sigma^4),$$

which gives the desired result.

(4.10): $E_r(s, z) \geqslant 0$, so

$$E_r^*(s, z) = E_r(s, z) - E_r(\sigma, z) \geqslant -E_r(\sigma, z).$$

(4.11)–(4.13): These follow from the identity

$$\frac{1}{a^2+(y-t)^2} + \frac{1}{a^2+(y+t)^2} - \frac{2}{a^2+y^2} = \frac{2t^2(3y^2-t^2-a^2)}{(a^2+y^2)\big(a^2+(y-t)^2\big)\big(a^2+(y+t)^2\big)}.$$

(4.14):

$$E_r(\sigma, z) \geqslant 2\frac{\sigma-x}{(\sigma-x)^2+y^2} + 2\frac{\sigma-1+x}{(\sigma-1+x)^2+y^2} \geqslant 2\frac{\sigma-x}{\sigma^2+y^2} + 2\frac{\sigma-1+x}{\sigma^2+y^2} = 2\frac{2\sigma-1}{\sigma^2+y^2}.$$

(4.15): Similar to the proof of (4.14), except that we disregard the contribution of y.

We now prove (4.3) for $y \geqslant 10$. Applying (4.6) to $E_t(1, z)$, (4.11) to $E_r^*(1+\frac{5}{4}i, z)$, (4.9) to the E_i terms, and (4.14) to $E_r(1, z)$, we see that it suffices to show

$$\frac{10}{3}\,\frac{1}{1+y^2} \geqslant \frac{6}{y^4} + .92\frac{3}{y^2-\frac{9}{16}} + .115\frac{5}{y^2-\frac{25}{16}} - .035\frac{7}{y^2-\frac{49}{16}} + .07\frac{\frac{21}{4}y^2+\frac{7}{4}\cdot\frac{65}{16}}{(y^2-\frac{49}{16})^3}.$$

But for $y \geqslant 10$, we have

$$y^2-\frac{49}{16} \geqslant \frac{1}{1.04}y^2, \quad y^2-\frac{25}{16} \geqslant \frac{1}{1.02}y^2, \quad y^2-\frac{9}{16} \geqslant \frac{1}{1.01}y^2,$$

$$y^2-\tfrac{49}{16} \leqslant y^2, \quad 1+y^2 \leqslant 1.01y^2,$$

and so it is easily seen that the inequality above is in fact satisfied.

Finally we prove (4.2) for $0 \leqslant y \leqslant \frac{1}{2}(1-x)$ and all sufficiently small ε. We apply (4.5) to $E_t(\sigma_0, z)$, (4.10) to $E_r^*(\sigma_0+\frac{5}{4}i, z)$, (4.7) to $E_i(\sigma_0+\frac{3}{4}i, z)$ and $E_i(\sigma_0+\frac{5}{4}i, z)$, and (4.8) to $E_i(\sigma_0+\frac{7}{4}i, z)$. We find that it suffices to show

$$1.372\,E_r(\sigma_0, z) \geqslant .035\left(\frac{2}{|y-\frac{7}{4}|} + \frac{2}{|y+\frac{7}{4}|}\right).$$

But for $0 \leqslant y \leqslant \frac{1}{2}(1-x)$ and $\frac{1}{2} \leqslant x \leqslant 1$,

$$\frac{1}{|\frac{7}{4}-y|} + \frac{1}{|\frac{7}{4}+y|} = \frac{7/2}{\frac{49}{16}-y^2} \leqslant \frac{7}{6},$$

and so, applying (4.14) to $E_r(\sigma_0, z)$, it will suffice to show

$$\frac{2.7}{\sigma_0^2+y^2} \geqslant .082,$$

which is clearly true in our range if $\sigma_0 < 2$, say.

The final parts of our proof of Theorem 2 used very crude estimates, since the main point was to show that zeros far from the real axis as well as real zeros do not affect our result. With additional work one can show that if ε is sufficiently small, (4.2) is in fact satisfied by all $x+iy \in R_0$ for which $x > .55$ implies either $y \geqslant 3$ or $0 \leqslant y \leqslant \frac{1}{2}(1-x)$. This would then show that the estimate (1.6) follows also from

HYPOTHESIS R''. *If $\varrho = \beta+i\gamma$ is a zero of the zeta function of some number field and $\beta > .55$, then either $|\gamma| \geqslant 3$ or $|\gamma| \leqslant \frac{1}{2}(1-x)$.*

We should also mention that (4.2) can be proved with 1.667 replaced by a somewhat smaller number, thus leading to a slight improvement of Theorem 2. Substantially better results, however, require more elaborate choices of the Y_{kj} and the T_{kj}.

5. Proof of Theorem 1. We again apply Theorem 3, but this time with $\sigma_0 = 1.01$, $\sigma_1 = 1.25$, and

$$Y = 1/M, \quad Y_{1,1} = 5, \quad Y_{2,1} = .75, \quad Y_{3,1} = .5, \quad Y_{3,2} = 4,$$
$$T_{1,1} = .5, \quad T_{2,1} = 1, \quad T_{3,1} = .5, \quad T_{3,2} = .75,$$

where $M = 6.65$. Since

$$\frac{2^{\sigma_0}-1}{2^{\sigma_1}-1} = \frac{2^{1.01}-1}{2^{1.25}-1} = .7355\ldots,$$

this choice satisfies (3.29). Therefore if (3.30) is satisfied for all $z \in R_0$, then

$$Z(1.01) + \frac{1}{4}\sum_{\varrho} E_r(1.01, \varrho) \geqslant \frac{1}{M} G_i(1.25) - \frac{5}{M} G_r^*(1.25+0.5i) +$$
$$+ \frac{3}{4M} G_r^*(1.25+i) - \frac{1}{2M} G_i(1.25+0.5i) - \frac{4}{M} G_i(1.25+0.75i) + O(n^{1/2})$$
$$> \frac{1}{M}\big(3.1088 r_1 + 2.1691\cdot(2r_2)\big) + O(n^{-1/2}).$$

But then (3.14) and (3.1) yield

$$D^{1/n} > (55.6)^{r_1/n}(21.1)^{2r_2/n} + O(n^{-1/2}),$$

which proves Theorem 1.

To prove (3.30), we see that we have to show

$$(5.1)\quad F(x, y) = -5\cdot E_r^*\left(\frac{5}{4}+\frac{i}{2}, z\right) + \frac{3}{4} E_r^*\left(\frac{5}{4}+i, z\right) - \frac{1}{2} E_i\left(\frac{5}{4}+\frac{i}{2}, z\right) -$$
$$- 4E_i\left(\frac{5}{4}+\frac{3i}{4}, z\right) - E_i\left(\frac{5}{4}, z\right) + 6.65 E_r(1.01, z)$$

is $\geqslant 0$ for all $z = x+iy \in R_0$. The cases $y \geqslant 10$ and $y \leqslant \frac{1}{\sqrt{3}}(\frac{5}{4}-x)$ will be disposed of at the end, by simple estimates of the kind used in the preceding section. The intermediate range of y, on the other hand, will be checked numerically, necessitating the use of a computer or a programmable calculator. This computation is not strictly necessary, since the proof could be carried out by bounding the summands in (5.1) somewhat similarly to the case $y \geqslant 10$ or $y \leqslant \frac{1}{\sqrt{3}}(\frac{5}{4}-x)$. However, such a proof would be extremely involved, and so it seems much simpler conceptually to use a computer. In what follows, we use very crude bounds, the objective being not so much to cut down on the number of calculations, as to simplify the task of programming.

If $R = \{(x, y)\colon \frac{1}{2} \leqslant x \leqslant 1, y \geqslant 0\}$, then $F(x, y)$ is a C^∞ function on an open set containing R, and so whenever $(x, y) \in R$ and $(x+h_x, y+h_y) \in R$, we have ([1], p. 124)

$$(5.2)\quad F(x+h_x, y+h_y) = F(x, y) + h_x \frac{\partial}{\partial x} F(x, y) + h_y \frac{\partial}{\partial y} F(x, y) +$$
$$+ \theta\left((|h_x| + |h_y|)^2 \cdot \frac{1}{2}\cdot Q\right),$$

where Q is the maximum of the absolute values of all the second order partial derivatives of F on the line segment from (x, y) to $(x+h_x, y+h_y)$. Suppose now that all the second partials of F in the rectangle

$$R^* = ([x-\delta_x; x+\delta_x] \times [y-\delta_y, y+\delta_y]) \cap R$$

are $\leqslant q$ in absolute value. If $(x+h_x, y+h_y) \in R^*$,

$$\tfrac{1}{2} F(x, y) + h_x \frac{\partial}{\partial x} F(x, y) \geqslant h_x^2 q,$$

and

$$\tfrac{1}{2} F(x, y) + h_y \frac{\partial}{\partial y} F(x, y) \geqslant h_y^2 q,$$

then by (5.2)

$$F(x+h_x, y+h_y) \geqslant h_x^2 q + h_y^2 q - \tfrac{1}{2}(|h_x|+|h_y|)^2 q \geqslant 0.$$

In practice, evaluating $F(x, y)$ and its first partials is straightforward. The second partials we bound by the sum of the bounds for the second partials of the summands in (5.1). Since (for $z = x+iy$)

$$E_r(s, z) = \operatorname{Re}\left\{\frac{1}{s-z} + \frac{1}{s-\bar{z}} + \frac{1}{s-1+z} + \frac{1}{s-1+\bar{z}}\right\},$$
$$\frac{\partial^2}{\partial x^2} E_r(s, z) = \operatorname{Re}\left\{\frac{2}{(s-z)^3} + \frac{2}{(s-\bar{z})^3} + \frac{2}{(s-1+z)^3} + \frac{2}{(s-1+\bar{z})^3}\right\}$$
$$= \theta\left(\frac{2}{|s-z|^3} + \frac{2}{|s-\bar{z}|^3} + \frac{2}{|s-1+z|^3} + \frac{2}{|s-1+\bar{z}|^3}\right),$$

and the same estimate holds for the other second partials of E_r and also for those of E_i. Similarly

$$E_i(\sigma, z) = -2\operatorname{Re}\left\{\frac{1}{(\sigma-z)^3} + \frac{1}{(\sigma-1+z)^3}\right\}$$

implies:

$$\frac{\partial^2}{\partial x^2} E_i(\sigma, z) = \theta\left(\frac{24}{|\sigma - z|^5} + \frac{24}{|\sigma - 1 + z|^5}\right),$$

and the same estimate holds for the other second partials. An estimate for the maximum absolute value of the second partial derivatives on a rectangle is now obtained by noting that for

$$(x', y') \in R^* = ([x - \delta_x, x + \delta_x] \times [y - \delta_y, y + \delta_y]) \cap R$$

and $s = \sigma + it$ with $\sigma > 1$ we have

$$|s - (x' + iy')|^2 \geqslant (\max(\sigma - 1, \sigma - x - \delta_x))^2 + (\max(0, |t - y| - \delta_y))^2,$$

$$|s - 1 + (x' + iy')|^2 \geqslant (\max(0, \sigma - 1 + x - \delta_x))^2 + (\max(0, |t + y| - \delta_y))^2,$$

and similarly for the other terms.

To illustrate the use of these bounds, consider $x = 1$, $y = 0.35$, $\delta_x = 0.07$, and $\delta_y = 0.03$. We find

$$F(x, y) = 0.25286\ldots, \quad \frac{\partial}{\partial x} F(x, y) = -276.24\ldots,$$

$$\frac{\partial}{\partial y} F(x, y) = -0.16377\ldots, \quad \frac{1}{2} q = 1967.7\ldots$$

Applying our earlier discussion we conclude that $F \geqslant 0$ on $[.93, 1] \times \times [.345, .355]$. Another example (somewhat more typical) is obtained by considering $x = 0.57$, $y = 1.5$, $\delta_x = \delta_y = 0.15$. Then

$$F(x, y) = 1.836\ldots, \quad \frac{\partial}{\partial x} F(x, y) = -1.092,$$

$$\frac{\partial}{\partial y} F(x, y) = -5.085\ldots, \quad \text{and} \quad \frac{1}{2} q = 48.54\ldots$$

In this case we conclude that $F \geqslant 0$ on $[.5, .66] \times [1.38, 1.57]$.

To conclude that $F(x, y) \geqslant 0$ for all (x, y) with

$$\frac{1}{2} \leqslant x \leqslant 1, \quad \frac{1}{\sqrt{3}}(\tfrac{5}{4} - x) \leqslant y \leqslant 10,$$

the above method was applied at the following points (if $\frac{c - a}{b} = n$ is a positive integer, $a(b)c$ denotes the finite sequence $a, a + b, a + 2b, \ldots \ldots, a + nb = c$):

$x = .57$, $y = .5(.2)1.5(.15)2.4, 2.5(.125)10$,
$x = .7$, $y = .35(.15)2.45, 2.5(.125)10$,
$x = .83$, $y = .2(.1)2.5(.125)10$,
$x = .95$, $y = .17, 2.5(.125)10$,
$x = 1$, $y = .17, .21, .24, .26, .28(.01).5(.025).6(.05).8$
$.8(.025).9(.01).95(.025)1.1(.05)2(.1)2.5$.

Proceeding with a little more care, one can obtain the same result by considering considerably fewer points.

To conclude the proof, we have to consider $0 \leqslant y \leqslant \frac{1}{\sqrt{3}}(\frac{5}{4} - x)$ and $y \geqslant 10$. First let us consider the former situation. In the definition (5.1) of $F(x, y)$ we apply (4.13) to $E_r^*\left(\frac{5}{4} + \frac{i}{2}, z\right)$, (4.10) and then (4.15) to $E_r^*\left(\frac{5}{4} + i, z\right)$, (4.7) to the E_i terms, and (4.5) to $E_i(\frac{5}{4}, z)$, and discover

$$F(x, y) \geqslant 6.65 E_r(1.01, z) - \frac{3}{2}\left(\frac{1}{\frac{5}{4} - x} + \frac{1}{\frac{5}{4} + x}\right) \geqslant 6.65 E_r(1.01, z) - \frac{36}{5}.$$

But by (4.14)

$$E_r(1.01, z) \geqslant \frac{2.04}{(1.01)^2 + y^2} \geqslant \frac{2}{1.3},$$

since $y \leqslant 1/2$, and so

$$F(x, y) \geqslant 0 \quad \text{for} \quad 0 \leqslant y \leqslant \frac{1}{\sqrt{3}}(\tfrac{5}{4} - x).$$

It only remains to show $F(x, y) \geqslant 0$ for $\frac{1}{2} \leqslant x \leqslant 1$, $y \geqslant 10$. This time in (5.1) we apply (4.12) to $E_r^*\left(\frac{5}{4} + \frac{i}{2}, z\right)$, (4.11) to $E_r^*\left(\frac{5}{4} + i, z\right)$, (4.9) to the E_i terms, (4.6) to $E_i(1.25, z)$, and (4.14) to $E_r(1.01, z)$ to obtain

$$F(x, y) \geqslant \frac{13.566}{(1.01)^2 + y^2} - \frac{9}{y^4} - \frac{12}{y^2 - \frac{9}{16}} - \frac{1}{y^2 - \frac{1}{4}} - \frac{12}{(y^2 - \frac{1}{4})^2}$$

$$\geqslant \frac{13.56}{1.03 + y^2} - \frac{13}{y^2 - \frac{9}{16}} - \frac{21}{(y^2 - \frac{1}{4})^2} \geqslant \frac{.56 y^2 - 22}{(1.03 + y^2)(y^2 - \frac{9}{16})} - \frac{21}{(y^2 - \frac{1}{4})^2}$$

which is clearly > 0 for $y \geqslant 10$, thus completing the proof.

Table 1. K real, $K = Q(\sqrt{D})$

t	D	$D^{1/2}$	
0	5	2.23...	finite
1	8·5	6.32...	finite
2	8·5·13	22.80...	finite
3	4·3·5·7·11	67.97...	*
4	4·3·5·7·11·13	245.07...	?
5	4·3·5·7·11·13·17	1010.4...	?
6	4·3·5·7·11·13·19·23	5123.1...	infinite

* finite if Hypothesis R′ is true.

Table 2. K complex, $K = Q(\sqrt{-D})$

t	D	$D^{1/2}$	
0	3	1.73...	finite
1	3·5	3.87...	finite
2	4·3·7	9.16...	finite
3	4·3·5·7	20.49...	finite
4	4·3·5·7·13	73.89...	?
5	4·3·5·7·11·19	296.2...	infinite

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Added in proof. For further results, see also the following papers by this author:

1. *Some analytic estimates of class numbers and discriminants*, Invent. Math. 29 (1975), pp. 275–286.

2. *Lower bounds for discriminants of number fields. II*, to appear in Tôhoku Math. J.

3. *On conductors and discriminants*, to appear in the Proceedings of the Durham Symposium on Algebraic Number Theory.

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