A note on Fermat’s conjecture

by

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Introduction. Recently Everett [3] has proved the following theorem.

Theorem 1. Let an odd prime \( p \geq 3 \) and an integer \( c \geq 1 \) be fixed. Then there are at most a finite number of relatively prime, positive integer pairs \((x, y)\) on the line \( y = x + v \) such that \( x^p + y^p = \) the \( p\)-th power of an integer.

The proof is based on Roth’s famous theorem, stating that a real algebraic irrational is approximable to no order higher than 2. It is surprising in the proof that \( 21^p \) works as the irrational.

Some time ago, the author [3] stated the next theorem.

Theorem 2. Let \( p \) be a prime \( \geq 3 \). Then there exist at most a finite number of positive integer triples \((x, y, z)\) which satisfy the conditions

\[ x^p + y^p = z^p, \quad (x, y, z) = 1 \]

and for which some difference \(|x - y|, |x - z|, |z - y|\) is less than a given positive number \( M \).

Theorem 1 is contained in the case \(|x - y| < M\). Theorem 2 can be proved most naturally by the general method given by Inkeri and Hyrönä [5]. Because they only discussed one case (albeit a typical one), we give a complete proof in Section 2. Further we will state a generalization of this theorem. The proof of Theorem 2 given in [3] is of interest for that part which concerns the so-called first case of Fermat’s conjecture (\( p \neq 2 \)). The method used is completely elementary, and yields also an upper bound in terms of \( p \) and \( M \) for each of the numbers \( x, y, z \) of every solution. On account of Theorem 1, a footnote on p. 53 in [3] is worth mentioning.

According to this note, the proof of Theorem 2, excluding the case \( x - y < M \), \( x, y > \), can be carried out elementarily, using only Thue’s theorem concerning \( \mathbb{Q} \) (Roth’s), the approximability of an algebraic number by rational numbers. Since Roth’s result is fairly deep, we will, in Section 1, prove Theorem 1 by means of Thue’s theorem. Our proof is simpler and shorter than that of Everett. In Section 3 our results, and a result of
Baker will apply to Abell's conjecture (a special case of Fermat's conjecture).

1. Proof of Theorem 1 by Thue's theorem. Let \(x, y, z\) be any solution of Fermat's equation (1) in positive integers such that \(y - x = v \geq 1, (x, y) = 1\). Then \(0 < x < y < x, (x, y, z) = 1\) and, by Abell's well-known formulas (see [6], p. 322, Satz 1044 or [3], pp. 6–8), we may write

\[
\begin{align*}
  z - x &= b^v, \quad z - y = a^v \quad (0 < a < b)
\end{align*}
\]

where

(i) \(a\) and \(b\) are integers if \(p|xy\);
(ii) \(b^v = p^{v-1}b_0^v, a\) and \(b_0\) are integers, if \(p|y\);
(iii) \(a^v = p^{v-1}a_0^v, a_0\) and \(b\) are integers, if \(p|x\).

By subtraction it follows from (2) that

\[
\begin{align*}
  v &= y - x = b^v - a^v.
\end{align*}
\]

In Case (i) \(b^v - a^v > b^v - ab^{v-1} = (b - a)b^{v-1} > b, \) since \(b - a \geq 1\). Thus \(b\) is bounded, by (3). Putting \(a = p^{v-1} (1)\) we obtain in Case (ii) of (2) and (3),

\[
\begin{align*}
  bv &= (p_{b_0})^v - pa^v > pb_0(ab)^{v-1} - (ab)^v > |p_{b_0} - ab||a^{v-1}|
\end{align*}
\]

and hence \(a - p_{b_0}a < pv/a^v\). Here \(a\) is an algebraic number of degree \(p\) \((\geq 3)\), because the polynomial \(x^p - a\) is irreducible over the field of rational numbers. Now, by Thue's theorem ([6], p. 56, Satz 689), there are only finitely many positive integer pairs \((x, y)\) for which \(a - xy < A/y^n\), where \(A\) is a positive number. Thus we can conclude that \(b (< p_{b_0})\) is bounded in Case (ii). The same holds also in Case (iii), for it follows as above that

\[
\begin{align*}
  pv &= (p_{a_0})^v - pa^v > (ab)^v - pa_0(ab)^{v-1} > |ab - pa_0||b^{v-1},
\end{align*}
\]

whence \(a - pa_0/b < pv/b^v\).

More precisely, we may say that there exists a positive constant \(M\) depending only on the fixed \(p\) such that \(b < M\) for every solution \((x, y, z)\) in question. Since, by (1), \(2^{1/2}x < z\), we have, according to (2)

\[
0 < z(1 - 2^{1/3}) < z - x < M^p \quad \text{or} \quad x < M^p/(1 - 2^{1/2}),
\]

which completes the proof.

Still more quickly than above, cases (ii) and (iii) can be settled using Thue's theorem on binary forms ([6], p. 37) instead of Thue's approximation theorem. From (4) and (5) we immediately see that in Case (ii) \(u = p_{b_0}, v = a\) and in Case (iii) \(u = pa_0, v = b\) is an integer solution of the equation

\[
\begin{align*}
  |v^p - pv^n| = pv.
\end{align*}
\]

Since \(x^n - p\) is, as mentioned before, an irreducible polynomial of degree \(\geq 3\) with integer coefficients, this equation has, by Thue's theorem, at most a finite number of solutions in integers \(u, v\). After this, one can proceed as above.

2. The proof and a generalization of Theorem 2. The natural source of results like Theorems 1 and 2 (see [5], where some such results have been stated) is the following fairly general theorem.

**Theorem 3.** Let \(m > 2, n > 2, a_0 \neq 0, a_1, \ldots, a_n\) be integers such that \(m \) or \(n \) is \(\geq 3\). Then the Diophantine equation

\[
\begin{align*}
  y^n = a_0 + a_1x^{a_n} + \ldots + a_n
\end{align*}
\]

has only finitely many solutions in integers if all zeros of the polynomial on the right of (6) are simple.

Siegel ([7], pp. 155–157) has proved this theorem in the case \(m = 2\). Applying his method Inkeri and Hyyré [5] have given a complete proof. The theorem is contained in a more extensive result of LeVeque ([8], p. 210, Theorem 1). Finally, Baker [1] has obtained, using his famous new method and Siegel's techniques, an upper bound for all integer solutions of (6). For \(m > 3, n > 3\) his results implies that

\[
\begin{align*}
  \max(|x|, |y|) < \exp\exp((5m)^{10}(w_{100}A)^2),
\end{align*}
\]

where \(A = \max|a_j|\).

It is easily seen that to prove Theorem 2 (and so Theorem 1, too), it suffices to prove the following theorem. (The result with a minus sign in (8) has already been discussed in [5].)

**Theorem 4.** Each of the Diophantine equations

\[
\begin{align*}
  \pm x^p \pm a^p = y^p,
\end{align*}
\]

where \(p\) is a prime \(\geq 3\) and \(a\) a fixed positive integer, has only finitely many solutions in integers \(x, y\).

**Proof.** The degrees of the polynomials \(f(x) = (x + k)^p \pm x^p\) are \(\geq p - 1 \geq 2\). It is seen immediately that

\[
\begin{align*}
  f(x) = (x + k)^p (x)^p = -kp^{p-1}.
\end{align*}
\]

Since \(f(0) = k^p \neq 0\), we infer from this identity, that \(f(x)\) has no multiple zero. Now our theorem follows directly from Theorem 3.

To generalize Theorem 4 we consider the equation

\[
\begin{align*}
  f(x) = g(x)^p a_x (x)^p = y^p,
\end{align*}
\]

where \(a_x\) is an integer, \(g(x) = ax^p\) is a polynomial of degree \(p\) with constant term \(a\), and \(f(x) = g(x)^p a_x (x)^p = y^p\).
where \( p \) is a prime \( \geq 3 \) and \( g \) and \( h \) are distinct, non-constant polynomials with integer coefficients. Denote by \( \zeta \) a primitive \( p \)th root of unity. Then

\[
f(x) = (g(x) - h(x)) (g(x) - \zeta h(x)) \cdots (g(x) - \zeta^{p-1} h(x)).
\]

We have for the degree of the polynomial \( f(x) \)

\[
\deg f = \sum_{r=0}^{p-1} \deg (g - \zeta^r h) \geq p - 1 \geq 2,
\]

because \( \deg (g - h) \geq 0 \) and \( \deg (g - \zeta^r h) \geq 1 \) for \( 1 \leq r \leq p - 1 \).

By (10), a multiple zero of \( f \) is either a multiple zero of some factor on the right of (10), or a common zero at least of two factors and thus also of the polynomials \( g \) and \( h \). Now we can conclude that all zeros of \( f \) are simple, provided that the discriminants of the factors on the right of (10) and the resultant of \( g \) and \( h \) satisfy the conditions

\[
\text{(11)} \quad D(g - \zeta^r h) \neq 0 \quad (r = 0, 1, \ldots, p-1), \quad R(g, h) \neq 0.
\]

(For a polynomial \( f(x) = a \) of degree 0 we define \( D(f) = a^{-2} \).)

The conditions (11) may be replaced by the following

\[
\text{(12)} \quad R(f(x), g(x) h'(x) - g'(x) h(x)) \neq 0.
\]

Indeed, a simple calculation shows that

\[
\text{(13)} \quad pf(g' h'(x) - f'(x) g(x)) = ph(x)^{p-1} (g(x) h'(x) - g'(x) h(x)).
\]

A common zero of \( f \) and \( h \) would be, by (9), a zero of \( g \) and thus also of the last factor on the right of (13). But, because of (12), this factor and \( f \) have no common zero. Now it follows from (13) that the same also holds for \( f \) and \( f' \), whence all zeros of \( f \) must be simple.

By the way, the equivalence of (11) and (12) is evident from the decompositions

\[
R(f, gh' - g'h) = e_1 \prod_{r=0}^{p-1} R(g - \zeta^r h, (g - \zeta^r h) h' - (g' - \zeta^r h') h) = e_2 R^2(g, h) \prod_{r=1}^{p-1} D(g - \zeta^r h),
\]

where the \( e_i \)'s are numbers \( \neq 0 \). To verify these one can employ (10) and the following well-known properties of the resultant:

\[
R(fg, h) = e_1 R(f, h) R(g, h), \quad R(f, g) = \pm R(g, f), \quad R(f, g + hf') = e_2 R(f, g),
\]

where \( f, g, h \) are polynomials and the \( e_i \)'s are numbers \( \neq 0 \).

By Theorem 3, we may sum up the above results as

**Theorem 5.** Let \( g(x) \) and \( h(x) \) be two different, non-constant polynomials, with integer coefficients. If \( g \) and \( h \) satisfy the conditions (11) (or (12)), then the Diophantine equation (9) has at most a finite number of solutions in integers \( x, y \).

This implies as a special case Theorem 4, for if \( g(x) = x + k, \ h(x) = \pm x, \) we have \( g'h' - g' h = \mp k \) and hence (12) is valid.

As another example we consider the case \( g(x) = a x^3 + b x + c, \ h(x) = d x^2 + e, \) where the coefficients are integers and \( ad \neq 0 \). We have

\[
R(g, h) = a e - b d e + c d^2, \quad D(g - \zeta^r h) = b^2 - 4 a e + (4 a e - 2 b d) \zeta^r + d^2 \zeta^{2r}.
\]

If \( r \equiv 0 \pmod{p} \), no two of the numbers 0, \( r, 2r \) are congruent \( \pmod{p} \). Obviously, an equation

\[
\sum_{r=1}^{p-1} \zeta_r^r = 0,
\]

where the \( \zeta_r \)'s are integers, holds if \( c_0 = c_1 = \ldots = c_{p-1} \). Thus the conditions (11) are fulfilled if \( a e - b d e + c d^2 \neq 0 \), the numbers \( b^2 - 4 a e, \ a e - 2 b d, \ d^2 \) are not all equal, and moreover the sum of these numbers is \( \neq 0 \). For instance, to mention one special case, these three conditions are valid if \( d^2 \neq a e, \ d \equiv 0 \pmod{p} \).

3. Abel's conjecture and Baker's estimate. Abel has asserted that (1) (with an integer \( p \geq 3 \)) has no solution in positive integers \( x, y, s \) such that any of the integers \( x, y, s \) is a power of a prime. The background and the present state of this problem become apparent from the papers [3], p. 8 and pp. 51-68, in particular, Satz XV and its proof, and [5], p. 6. If \( x, y, s \) is such a solution, the following conditions hold:

(i) \( p \) is a prime;

(ii) only the smallest of numbers \( x, y, s \) (any \( s \)) can be a prime power;

(iii) \( s - y = 1; \ x = y \) and \( y = 0 \) or \( -1 \pmod{p} \).

In addition to these, we have the following fact: There exist only a finite number of solutions of (1) in positive integers \( x, y, s \) such that one of \( x, y, s \) is a prime power. This follows immediately from (iii) and Theorem 4 \( (k = 1) \).

By (iii), we can write (1) in the form

\[
(x + 1)^p - y^p = \sum_{k=1}^{y} C_k y^{p-k} = x^p
\]

for any solution in question. We now apply Baker's result (7). By Stirling's formula, the binomial coefficients \( C_k \) satisfy the conditions

\[
C_k < \frac{1}{2^{2k-p-1}} \quad \text{and} \quad C_k \sim \left( \frac{1}{\pi} \right)^{1/2} 2^{k-p-1} \quad (p = (p-1)/2, \ p \to \infty).
\]
Without essential restriction, let \( p > 5 \). Then
\[
(5p)^{13} (p^{-1/3})(p-1)^2 < 1.
\]
Further, by (7), where now \( n = p - 1, m = p, A = C_p^2 \), we have the estimates
\[
x < y < \exp\exp\left(2^p(p-1)^{10(n-3)}(p-1)^{1/2}\right) < \exp\exp\left(2^p 3^{10}\right)^{1/2}.
\]
By virtue of estimates given by the author [4] it follows that
\[
x > p^{3p-4}, \quad y > p^{3p-1},
\]
since \( p|\varphi(y+1) \). Recalling that Fermat's conjecture has been proved for \( p < 25600 \), the magnitude of each of these bounds is fairly large. However, the differences between the above upper bound and these lower bounds are enormous.

References


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On the representation of a number in the form
\[ a^2 + b^2 + p^2 + q^2 \]
where \( p, q \) are odd primes

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1. Introduction. This paper shows that every sufficiently large natural number \( N \) that satisfies the necessary condition of ineqnongrce, modulo 8, to 0, 1 or 5 is representable in the form stated in the title. The interest of this result lies partially in the fact that (so far as the author is aware) there is no immediate prospect of any solution of the corresponding "Waring-Goldbach" problem in which the numbers \( a, b \) would also be restricted to prime values.

The proof depends on a combination of the mean value theorem of Barban [1] (as re-discovered shortly afterwards by Davenport and Halberstam [3]) with the \( \frac{1}{2} \)-residue sieve method developed by Rosser (unpublished). An account of this method appears in Iwaniec's paper [7]. Barban's theorem is used in the way described in the author's paper [5] to estimate, with sufficient accuracy for our purposes, the number of pairs of primes \( p, q \) that satisfy
\[ p^2 + q^2 = X \mod l, \quad p \leq Z, \quad q \leq Z \]
for a modulus \( l \) not exceeding \( Z/\log^2 Z \). Such an estimate is the essential starting point for applications of the sieve method to binary problems involving primes. In this paper the \( \frac{1}{2} \)-residue sieve method is used in obtaining a positive lower estimate of how often \( N - p^2 - q^2 \) is free of prime factors \( \bar{a} \leq 3 \mod 4 \), and hence is of the form \( a^2 + b^2 \).

It is, perhaps, worthy of comment that the \( \frac{1}{2} \)-residue sieve is sufficiently powerful to establish the existence of numbers that are sums of two squares and lie in a suitable sequence, whereas the \( 1 \)-residue sieve has not yet been successfully used to establish the existence of primes in any sequence at all.

Iwaniec used his results in his treatment of the number of primes \( p \) not exceeding \( Z \) that are representable in the form \( a^2 + y^2 + 4 \) (where, indeed, \( a^2 + y^2 \) was replaced by an arbitrary quadratic form, positive if