We select δ so that

$$\log\frac{1}{\delta} = 2M^2t + (\log k)^{1/3}$$

(the last term in case $M = 0$) and note that as $\xi(k)\to 0$ as x, and so $k\to\infty$, (14) is satisfied. With this choice of δ, (12) implies that (9) holds, and the proof is complete.

Remark. It may be that the Cauchy–Schwarz inequality is inefficient in deriving (12) and that the factor $(1+M^2)^{t/2}$ is not needed. However, I could not find a useful estimate for the number of bad sets $l_1, l_2, \ldots, l_t$ with a fixed u — evidently there is no uniformly good estimate of this type since if $u = t$ all the sets are bad, indeed

$$\chi_1(l_1^{\varepsilon_1} l_2^{\varepsilon_2} \ldots l_t^{\varepsilon_t}) = 1, \quad \varepsilon_j\text{'s arbitrary}.$$

Siegel's theorem gives the estimate $M = o(1)$ and it seems reasonable that rather more than this is needed.

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On the equation of Catalan

by

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1. Introduction. The following conjecture was first enunciated by Catalan [8] in 1844 but has never been proved.

The only solution in integers $p > 1$, $q > 1$, $x > 1$, $y > 1$ of the equation

$$x^p - y^q = 1 \tag{1}$$

is $p = y = 2$, $q = x = 3$.

In 1953 Cassels [6] independently made the weaker conjecture that equation (1) has only a finite number of solutions.

The equation has been shown to be impossible for some special values of p and q. In 1738 Euler [10] showed that the only solution of $x^2 - y^3 = 1$ is $x = 3$, $y = 2$. In 1850 Lebesgue [14] proved that there is no solution at all when $q = 2$ and $p \neq 3$. It was shown by Nagell [18] in 1921 that there are no solutions if $p = 3$ or if $q = 3$, $p \neq 2$. The problem of showing that there is no solution when $p = 4$ was posed by Nagell and solved by S. Selberg [20] in 1932. Since 1967 this last result has become a special case of a theorem of Chao Ko [9], that there are no solutions if $p = 2$. Hence one has $p \geqslant 5$ and $q \geqslant 5$ for all unknown solutions of (1).

In proving Catalan's conjecture one can obviously assume without loss of generality that p and q are different primes. In 1960 Cassels [7] showed that if (1) holds then $p|y$ and $q|x$. It is an easy consequence of Cassels' result that there are no three consecutive positive integers which are all perfect powers, [17].

There are several results concerning the number of solutions when some of the variables are fixed. If x and y are fixed, then there are only finitely many solutions (p, q) of (1). This follows from Gel'fond's transcendence measure for $\log x/\log y$, [11]. LeVeque [15] showed that there is at most one solution (p, q) which can be found explicitly if it exists. Cassels [6] simplified his proof. If p and q are fixed, it is an immediate consequence of a result of Siegel [21] that (1) has only finitely many solutions (x, y). See also Mahler [16]. In this case Hyyrö [12] proved that there are at most $\exp(631p^2q^2)$ solutions. An explicit upper bound for

$\max(|x|, |y|)$ was given by Baker [2]. Hyyrö [13], Satz 16, proved that there is at most one solution if p and y, or q and x, are fixed. Hyyrö [13], Satz 17, also gave a finite upper bound for the number of solutions if either x or y is given.

In this paper we prove Cassels' conjecture by an effective method:

THEOREM 1. *The equation* (1) *has only finitely many solutions in integers $p > 1$, $q > 1$, $x > 1$, $y > 1$. Effective bounds for the solutions p, q, x, y can be given.*

Dr. G. V. Čudnovskii announced in a letter to the author that he has also proved Theorem 1 and moreover, that he proved the same statement for the more general equation $x^p - y^q = k$, where k is any fixed non-zero integer.

Hyyrö [12] proved that the equation (1) has no other solutions if $x \leqslant 10^{11}$. The upper bounds obtainable by our method are much larger, and it is not likely that one can prove Catalan's original conjecture by checking the remaining values of x on a computer.

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2. Auxiliary results. The proof of Theorem 1 in § 5 is rather short, but it contains four applications of the Gel'fond–Baker method. At the end of the proof we obtain that there are absolute bounds for p and q for every solution p, q, x, y of (1). We then complete our proof by using the following result.

THEOREM A (Baker [2]). *All solutions in integers x, y of the Diophantine equation*

$$y^m = a_0x^n + a_1x^{n-1} + \ldots + a_n, \tag{2}$$

where $m \geqslant 3$, $n \geqslant 3$, $a_0 \neq 0$, $a_1, \ldots, a_n$ denote rational integers and where the polynomial on the right of (2) *possesses at least two simple zeros, satisfy*

$$\max(|x|; |y|) < \exp\exp\{(5m)^{10}(n^{10n}H)^{n^2}\},$$

where $H = \max_j |a_j|$.

Earlier in the proof we shall use the following refinement of a recent result of Baker [3].

THEOREM 2. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers with degrees at most d and let the heights of $\alpha_1, \ldots, \alpha_{n-1}$ and α_n be at most A' $(\geqslant 2)$ and A $(\geqslant 2)$ respectively. Then there exists an effectively computable constant $C = C(d, n)$ such that the inequalities*

$$0 < |b_1\log\alpha_1 + \ldots + b_n\log\alpha_n| < \exp\left(-C(\log A')^{2n^2+16n}\log A\log B\right) \tag{3}$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most B $(\geqslant 2)$.

It has been assumed that the logarithms have their principal values, but the result would hold for any choice of logarithms if C were allowed to depend on their determinations. The only novelty of the theorem is the explicitly given dependence on A'. We note that in the proof of Theorem 1 one has $d = 1$ and $n = 2$ or 3.

Our proof in § 4 follows the main lines of Baker's proof and we only indicate the modifications to be made. One modification is of independent interest. Our Lemma T1 is an improvement upon Baker's Lemma 1. Lemma T1 enables one to choose ε at p. 123 more easily and to correct an inequality at p. 126 without difficulty. (Compare the footnote at p. 35 of [4].) Another modification can be found at the end of the proof of Theorem 2. The exponent of $\log A'$ in Theorem 2 would be at least $12n^2$ if we would apply the result of Fel'dman like Baker did. Instead we use the following new result of Baker.

THEOREM B (Baker [5]). *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers with degrees at most d and let the heights of $\alpha_1, \ldots, \alpha_n$ be at most $A_1, \ldots, A_n$ respectively. Put $Q^* = \log A_1 \ldots \log A_n$ and assume $Q^* \geqslant 2$. Then there exists an effectively computable constant $C = C(d, n)$ such that the inequalities*

$$0 < |b_1\log\alpha_1 + \ldots + b_n\log\alpha_n| < \exp(-CQ^*\log Q^*\log B)$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most B $(\geqslant 2)$.

3. An improvement of Baker's Lemma 1. For any integer $k \geqslant 1$ we signify by $\nu(k)$ the least common multiple of $1, \ldots, k$. We define

$$\Delta(x; k) = (x+1)\ldots(x+k)/k!$$

and we write $\Delta(x; 0) = 1$. Further for any integers $l \geqslant 0$, $m \geqslant 0$ we denote

$$\Delta(x; k, l, m) = \left(\frac{(\Delta(x; k))^l}{m!}\right)^{(m)}.$$

We prove the following improvement of [3], Lemma 1.

LEMMA T1. *Let q and qx be positive integers. Then*

$$q^{2kl}(\nu(k))^m\Delta(x; k, l, m)$$

is a positive integer and we have

$$\Delta(x; k, l, m) \leqslant 4^{l(x+k)}, \quad \nu(k) \leqslant 4^k.$$

Proof. We have

$$\Delta(x; k, l, m) = \left(\Delta(x; k)\right)^l \sum_{j_1,\ldots,j_m}^{*} \left((x+j_1)\ldots(x+j_m)\right)^{-1},$$

where $j_1, \ldots, j_m$ run through all selections of m integers from the set $1, \ldots, k$ repeated l times, and the right hand side is read as 0 if $m > kl$. Hence,

$$(4)\qquad q^{kl}\Delta(x; k, l, m) = q^m \frac{(qx+q)^l \ldots (qx+qk)^l}{(k!)^l} \sum_{j_1,\ldots,j_m} \big((qx+qj_1)\ldots(qx+qj_m)\big)^{-1}.$$

We write $q^{kl}\Delta(x; k, l, m) = r/s$, where $r, s \in \mathbf{Z}$, $(r, s) = 1$. If p is a prime with $p|s$, then, from (4), $p|k!$ and, hence, $p \leqslant k$. We distinguish two cases for the prime divisors p of s.

(a) $p \nmid q$. The number of factors p in $(k!)^l$ is exactly

$$(5)\qquad l\left(\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\ldots+\left[\frac{k}{p^t}\right]\right), \quad \text{where } t = \left[\frac{\log k}{\log p}\right].$$

Since $(p, q) = 1$, the product $(qx+q)^l \ldots (qx+qk)^l$ contains at least as many factors p. Here we have not counted more than t factors p in one factor $qx+qj$ $(1 \leqslant j \leqslant k)$. Hence, if we omit m factors out of this product, the remaining product contains at least

$$(6)\qquad l\left(\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\ldots+\left[\frac{k}{p^t}\right]\right) - mt$$

factors p. It follows that

$$(7)\qquad (qx+q)^l \ldots (qx+qk)^l \sum_{j_1,\ldots,j_m}^{*} \big((qx+qj_1)\ldots(qx+qj_m)\big)^{-1}$$

contains at least (6) factors p. Thus s contains at most $mt = m\left[\frac{\log k}{\log p}\right]$ factors p. This is precisely the number of factors p of $\big(\nu(k)\big)^m$.

(b) $p|q$. In this case $(k!)^l$ also contains (5) factors p. The integer (7) might not be divisible by p at all. Since

$$\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\ldots+\left[\frac{k}{p^t}\right] \leqslant k,$$

we see that s contains at most kl factors p. Note that q^{kl} contains at least as many factors p.

The first assertion of our lemma follows immediately from (4) and cases (a) and (b). The second follows easily from Baker's estimates

$$\Delta(x; k, l, m) \leqslant \binom{[x]+k+1}{k}^l \binom{kl}{m} \leqslant 2^{(2x+k)l+kl} = 4^{l(x+k)}.$$

(Distinguish the cases $[x] = 0$ and $[x] > 0$.)

We apply the inequality $\pi(k) \leqslant 4k/(3\log k)$ (see [19], formula 3.6) to obtain the final estimate

$$\nu(k) = \prod_{p\leqslant k} p^{\left[\frac{\log k}{\log p}\right]} \leqslant \prod_{p\leqslant k} k = k^{\pi(k)} \leqslant e^{4k/3} < 4^k.$$

4. Proof of Theorem 2. In this section we give a proof of Theorem 2 by indicating the modifications to be made in Baker's paper [3]. We shall prove a slightly stronger form of Theorem 2, namely under the extra assumption that $a_1 = -1$ and with $2n^2+16n$ replaced by $2n^2+7n$ in the exponent of $\log A'$. This assertion with $n+1$ instead of n and $b_1 = 0$ implies Theorem 2, since $2(n+1)^2+7(n+1) \leqslant 2n^2+16n$. We suppose therefore that there exist rational integers $b_1, \ldots, b_n$ with $b_n \neq 0$ having absolute values at most B $(\geqslant 4)$ such that

$$(3^*)\qquad 0 < |b_1\log a_1 + \ldots + b_n \log a_n| < \exp\big(-C(\log A')^{2n^2+7n}\log A \log B\big)$$

holds, where $a_1 = -1$ and it is assumed that the logarithms have their principal values and that $C = C(d, n)$ is sufficiently large for the validity of the subsequent arguments. By $c, c_1, C_1, c_2, C_2, \ldots$ we signify numbers greater than 1 that can be specified explicitly in terms of d and n only.

We denote by k an integer exceeding a sufficiently large number c and we write

$$g = \log A, \quad g_0 = \log A', \quad h = L_{-1}+1 = [\log B],$$
$$L = L_0 = \ldots = L_{n-1} = [g_0^{2n} g k^{1-1/(4n)}], \quad L_n = [g_0^{2n+1} k^{1/2}],$$

where, as usual, $[x]$ denotes the integral part of x. Further we write $f_m(z)$ for the mth derivative of $f(z)$.

In the formulation of Lemma 5 the upper bound for the absolute values of the integers $p(\lambda_{-1}, \lambda_0, \ldots, \lambda_n)$ should be replaced by $\exp(c_3 g_0^{2n+2} ghk)$, while the upper bound for $m_0 + \ldots + m_{n-1}$ becomes $g_0^{2n+2} gk$. In the proof we have

$$M \leqslant d^n h (g_0^{2n+2} gk+1)^n < g_0^{2n^2+2n} g^n h k^{n+1/4}/2$$

and

$$N = (L_{-1}+1)\ldots(L_n+1) \geqslant g_0^{2n^2+2n+1} g^n h k^{n+1/4}.$$

Hence, $N > 2M$. The absolute value of the product over r in the definition of $V(s)$ is at most

$$(2A')^{nLh}(2A)^{L_n h} < \exp(2nLhg_0 + 2L_n hg).$$

Using

$$(8)\qquad L_n g < L g_0$$

we see that we can take $U = \exp(c_7 g_0^{2n+2} ghk)$. Hence, the integers

$p(\lambda_{-1}, \ldots, \lambda_n)$ can be chosen to have absolute values at most $NU \leqslant \exp(c_3 g_0^{2n+2} ghk)$, as required.

In Lemma 6 we consider all non-negative integers $m_0, \ldots, m_{n-1}$ with $m_0 + \ldots + m_{n-1} \leqslant g_0^{2n+2} gk$. Instead of (B4) (i.e. formula (4) of Baker's paper), we have

$$\text{(T4)} \qquad |f(z)| < \exp(c_8 g_0^{2n+2} ghk + c_9 g_0 L|z|),$$

and in place of (B5) we have

$$\text{(T5)} \qquad |f(l)| \geqslant \exp(-c_{10} g_0^{2n+2} ghk - c_{11} g_0 Ll).$$

On making obvious modifications in Baker's proof we obtain

$$\text{(T6)} \qquad |a_n - a_n'| < \exp(-\tfrac{1}{2} C g_0^{2n^2+7n} gh),$$

and

$$|A(z)| \leqslant \exp(c_{14} g_0^{2n+2} ghk + c_{13} L|z|).$$

Using the modified Lemma 5, the estimate

$$|a_1^{\lambda_1 z} \ldots a_{n-1}^{\lambda_{n-1} z}| \leqslant \exp(c g_0 L|z|),$$

Baker's estimate for $a_n'^z$ and the inequality (8) the required estimate (T4) follows easily.

To prove the second assertion it is readily verified that each conjugate of Q has absolute value at most $\exp(c_{15} g_0^{2n+2} ghk + c_{16} g_0 Ll)$. Furthermore, from Lemma T1 we see that, on multiplying Q by

$$P = a_1^{L_1 l} \ldots a_n^{L_n l} \big(\nu(h)\big)^{m_0},$$

we obtain an algebraic integer and, by (8),

$$P \leqslant \exp(c_{17} g_0 Ll + c_{18} h m_0).$$

Hence we conclude that either $Q = 0$ or

$$|Q| \geqslant \exp(-c_{19} g_0^{2n+2} ghk - c_{20} g_0 Ll).$$

But, as above, we deduce easily from (T6) that

$$|Q - f(l)| \leqslant \exp(c_{22} g_0^{2n+2} gkl - \tfrac{1}{2} C g_0^{2n^2+7n} gh),$$

and, if $l \leqslant g_0^{2n^2+4n} hk^{2n}$ and C is larger than some function of k, the number on the right hand is at most

$$\exp(-\tfrac{1}{4} C g_0^{2n^2+7n} gh) \leqslant \tfrac{1}{2}|Q|.$$

Hence, if $Q \neq 0$, we obtain $|f(l)| > \frac{1}{2}|Q|$ and this proves (T5).

In Lemma 7 where ε now depends only on d and n, we have to replace the upper bound for l by $h(g_0^{n+2}k)^{\varepsilon J}$ and the upper bound for $m_0 + \ldots + m_{n-1}$ by $2^{-J} g_0^{2n+2} gk$. We shall show that in fact a suitable value for ε is $1/8n$. (This bound is even better than Baker's value for ε and this is due to the improvement in Lemma T1.) We follow Baker's proof with

$$R_J = [h(g_0^{n+2}k)^{\varepsilon J}], \qquad S_J = [2^{-J} g_0^{2n+2} gk] \qquad (J = 0, 1, \ldots)$$

and some other evident changes. Formula (B9) is replaced by

$$\text{(T9)} \qquad |f_m(r)/m!| < \exp(-\tfrac{1}{4} C g_0^{2n^2+7n} gh),$$

the upper bound for θ at the bottom of p. 124 becomes

$$\theta \leqslant \exp(c_8 g_0^{2n+2} ghk + c_9 g_0 L R_{K+1} k^{1/(8n)}).$$

Formula (B12) remains unchanged. The number on its right-hand side is at least

$$2^{-K-6} n^{-1} g_0^{2n+2+(n+2)\varepsilon K} ghk^{\varepsilon K+1} \log k.$$

The number at the right-hand side of (B13) becomes in our case

$$(c_8 + c_{10}) g_0^{2n+2} ghk + (c_9 + c_{11}) g_0 L R_{K+1} k^{1/(8n)}$$
$$\leqslant (c_8 + c_{10}) g_0^{2n+2} ghk + c_{23} g_0^{2n+1+(n+2)(\varepsilon K+\varepsilon)} ghk^{1+\varepsilon K+\varepsilon-1/(8n)}.$$

Since $\varepsilon = 1/(8n)$, we have $(n+2)\varepsilon \leqslant 1$ and (B11) is untenable. The lemma follows by induction.

The formulation of Lemma 8 does not change at all. We follow Baker's proof with

$$X = [g_0^{2n^2+4n} hk^{n+1}], \qquad Y = [c_{24} g_0^{2n+2} gk], \qquad c_{24} = 2^{-16n^2}.$$

On making obvious modifications at p. 126, l. 3 and l. 5, we obtain

$$|f(l/q)| < \exp\big(c_8 g_0^{2n+2} ghk + c_9 g_0 L X k^{1/(8n)} - X(Y+1)\log(\tfrac{1}{4} k^{1/(8n)})\big) +$$
$$+ \exp(-\tfrac{1}{16} C g_0^{2n^2+7n} gh).$$

Since

$$g_0 L k^{1/(8n)} < g_0^{2n+1} gk \quad \text{and} \quad \tfrac{1}{2} c_{24} g_0^{2n^2+6n+2} ghk^{n+2} < X(Y+1) < \tfrac{1}{32} C g_0^{2n^2+7n} gh,$$

the upper bound for $|f(l/q)|$ is at most $\exp(-g_0^{2n^2+6n} ghk^{n+2})$.

We now utilize the latter estimate to confirm the validity of (B2) with l replaced by l/q. Each conjugate of Q has absolute value at most $\exp(c_{26} g_0^{2n+2} ghk^2)$. Further it follows from Lemma T1 that on multiplying Q by

$$q^{2h(L_0+1)} \big(\nu(h)\big)^{m_0} a_1^{L_1 l} \ldots a_n^{L_n l} \leqslant \exp(c_{27} g_0^{2n+1} ghk^2),$$

one obtains an algebraic integer. (We noted already that Baker's original proof is not correct at this point.) Thus, if $Q \neq 0$, we have

$$|Q| \geqslant \exp(-c_{28} g_0^{2n+2} ghk^2 q^n).$$

But it is easily seen from (T6) that

$$|Q - f(l/q)| < \exp(-\tfrac{1}{4} C g_0^{2n^2+7n} gh),$$

whence $|f(l/q)| \geqslant \frac{1}{2}|Q|$. Since $q \leqslant 2k^{1/2} g_0^{2n+1}$, it is clear that the estimate for $|Q|$ given above is inconsistent with the upper bound for $|f(l/q)|$ obtained earlier. Hence we conclude that $Q = 0$, as required.

Since we made the assumption that $a_1 = -1$ at the beginning of this section, we can now follow Section 4 of Baker's paper word for word up to p. 128, line 12. We obtain the inequalities

$$(9)\quad 0 < |b_1'\log a_1 + \ldots + b_{n-1}'\log a_{n-1} + b_n'\log a_n'| < \exp(-Cg_0^{2n^2+7n}gh),$$

where

$$b_1' = b_1 + b_n(j_1+j), \quad b_n' = pb_n, \quad b_r' = b_r + b_n j_r \quad (1 < r < n).$$

Clearly $b_1', \ldots, b_n'$ are rational integers with absolute values at most $4ng_0^{2n+1}k^{1/2}B$. Further we observe that each conjugate of

$$a_n'^p = a_n a_1^{-j_1} \ldots a_{n-1}^{-j_{n-1}}$$

has absolute value at most $(dA')^{np}dA$, and the same estimate holds for some integer a such that $aa_n'^p$ is an algebraic integer. Thus, from Lemma B4, we deduce that the height of a_n' is at most $(2dA')^{4nD}A^{2D/p}$, where $D\ (\leqslant d^n)$ denotes the degree of K. Since $p > g_0^{2n+1}k^{1/2}$, we have $2D/p \leqslant 1/2$. So we have proved the existence of rational integers $b_1', \ldots, b_n'$ with absolute values at most $c_1 g_0^{2n+1}k^{1/2}B$ and an algebraic number a_n' in the field generated by the a's over the rationals with height at most $c_2A'^{4nD}A^{1/2}$ such that (9) holds.

The proof is now completed by induction. We can suppose that $B > c_1^3 g_0^{6n+3}k^2$ for otherwise the result holds trivially (cf. [1], Lemma 6). It follows that $h > \frac{3}{4}\log(c_1 g_0^{2n+1}k^{1/2}B)$. If also $A > c_2^4 A'^{16nD}$ then $g > \frac{4}{3}\log(c_2A'^{4nD}A^{1/2})$ and (9) clearly remains valid with

$$h' := \log(c_1 g_0^{2n+1}k^{1/2}B) \quad \text{and} \quad g' := \log(c_2A'^{4nD}A^{1/2})$$

substituted for h and g respectively. Thus we can repeat the above argument and obtain for each $s = 1, 2, \ldots$ a set of integers $b_1^{(s)}, \ldots, b_n^{(s)}$ with absolute values at most $(c_1 g_0^{2n+1}k^{1/2})^s B =: \exp(h^{(s)})$ and an element $a_n^{(s)}$ of K with height at most

$$(c_2A'^{4nD})^{1+\frac{1}{2}+\ldots+(\frac{1}{2})^{s-1}}A^{(\frac{1}{2})^s} =: \exp(g^{(s)})$$

such that

$$(10)\quad 0 < |b_1^{(s)}\log a_1 + \ldots + b_{n-1}^{(s)}\log a_{n-1} + b_n^{(s)}\log a_n^{(s)}| < \exp(-Cg_0^{2n^2+7n}gh) < \exp(-Cg_0^{2n^2+7n}g^{(s)}h^{(s)}).$$

The algorithm terminates for some $s \leqslant 2\log\log A$, say S, when the height of $a_n^{(s)}$ is at most $c_2^4 A'^{16nD}$. It follows from Theorem B that there exists an effectively computable constant $C_1 = C_1(d, n)$ such that

$$0 < |b_1^{(S)}\log a_1 + \ldots + b_{n-1}^{(S)}\log a_{n-1} + b_n^{(S)}\log a_n^{(S)}| < \exp\Big(-C_1\log\big((c_1 g_0^{2n+1}k^{1/2})^S B\big)g_0^n\big(\log(c_2^4A'^{16nD})\big)^2\Big)$$

has no solutions. Hence, for some constant $C_2 = C_2(d, n)$

$$Cg_0^{2n^2+7n}gh < C_2(\log g\log g_0 + h)g_0^{n+3}.$$

Since $\log g\log g_0 + h < C_3 h\log g\log g_0$ for some constant C_3, we obtain a constant C_4 such that

$$g_0^{2n^2+6n-3}g \leqslant C_4\log g_0\log g.$$

Thus both g_0 and g are bounded. This has been proved under the assumption that (3*) holds for some rational integers $b_1, \ldots, b_n$, made in the beginning of this section. We now apply Theorem B to our original form. If (3*) holds, then Q^* is bounded and it follows that for some constant $C_5 = C_5(d, n)$

$$|b_1\log a_1 + \ldots + b_n\log a_n| \geqslant \exp(-C_5\log B).$$

Hence, (3*) has no solutions if we take C sufficiently large. This proves Theorem 2.

5. Proof of Theorem 1. Without loss of generality we may assume that p and q are different primes. Further we assume that q is odd. This last assumption is justified by Lebesgue's result that $q \neq 2$, [14].

We have

$$x^p = y^q + 1 = (y+1)(y^{q-1} - y^{q-2} + \ldots + 1).$$

Let $d = (y+1, y^{q-1} - y^{q-2} + \ldots + 1)$, where (a, b) denotes the g.c.d. of a and b. Then $y \equiv -1 \pmod d$ and, hence, $y^{q-1} - y^{q-2} + \ldots + 1 \equiv q \pmod d$. It follows that $d|q$, and therefore $d = 1$ or $d = q$. Since the product of $y+1$ and $y^{q-1} - y^{q-2} + \ldots + 1$ is a pth power, we find that there is a $\delta_2 \in \{-1, 0, 1\}$ and a positive integer σ such that

$$(11)\quad y + 1 = q^{\delta_2}\sigma^p.$$

In a similar way we derive from

$$y^q = x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \ldots + 1)$$

that there are integers $\delta_1 \in \{-1, 0, 1\}$ and $\varrho > 0$ such that

$$(12)\quad x - 1 = p^{\delta_1}\varrho^q.$$

(The exact values of δ_1 and δ_2 can be computed, but their exact values are immaterial for our further arguments.) On substituting (11) and (12) in (1) we obtain

$$(13)\quad (p^{\delta_1}\varrho^q + 1)^p - (q^{\delta_2}\sigma^p - 1)^q = 1.$$

This equation is almost symmetrical in (p, ϱ, δ_1) and (q, σ, δ_2). Since we have to distinguish the cases $p > q$ and $p < q$ and the proofs in both cases are similar in virtue of this symmetry, we assume $p > q$ in the sequel.

We shall first prove that there exist two absolute constants c_1 and c_2 such that

(14) $$q \leqslant c_1(\log p)^{c_2}.$$

We distinguish two cases, (a) and (b).

(a) $\varrho = 1$ or $\sigma = 1$. The following argument shows that $x \leqslant p^2$ in this case. Indeed, if $\sigma = 1$, then, from (11), $\delta_2 = 1$ and $y = q-1$ and it follows from (1) and $p > q$ that $x < y < q < p$; if $\varrho = 1$, then we have from (12) that either $x = 2$ or $x = p+1$. By (1)

$$0 < |p\log x - q\log y| \leqslant \left|\frac{x^p}{y^q} - 1\right| \leqslant \exp(-q\log y).$$

We apply Theorem 2 with $A = y$, $A' = p^2$ and $B = p$. This gives

$$|p\log x - q\log y| \geqslant \exp\left(-c_3\log y(\log p)^{c_4}\right)$$

for some absolute constants c_3 and c_4. The combination of both inequalities yields (14) in case (a).

(b) $\varrho > 1$ and $\sigma > 1$. It follows from (13) and $(x-1)^p < y^q+1 < (y+1)^q$ that

$$1 > p^{\delta_1 p}\varrho^{pq}q^{-\delta_2 q}\sigma^{-pq} = \left(1+\frac{1}{p^{\delta_1}\varrho^q}\right)^{-p}\{(1-q^{-\delta_2}\sigma^{-p})^q + q^{-\delta_2 q}\sigma^{-pq}\}.$$

Using the estimates $|\log(1+a)| \leqslant a$ for $a > 0$ and

(15) $$|\log\{(1-a)^q + a^q\}| \leqslant -q\log(1-a) \leqslant 2aq$$

for $0 \leqslant a \leqslant \frac{1}{2}$, we find

$$\left|\delta_1 p\log p - \delta_2 q\log q + pq\log\frac{\varrho}{\sigma}\right| \leqslant \frac{p}{p^{\delta_1}\varrho^q} + \frac{2q}{q^{\delta_2}\sigma^p}.$$

Since $p > q$, we have $y > x$ by (1) and hence, by (11) and (12), $q^{\delta_2}\sigma^p > p^{\delta_1}\varrho^q$. It follows that

$$\left|\delta_1 p\log p - \delta_2 q\log q + pq\log\frac{\varrho}{\sigma}\right| \leqslant \frac{3p^2}{\varrho^q}.$$

We want to prove (14). We may therefore assume that

(16) $$q > 10\log p.$$

Hence, from $\varrho \geqslant 2$,

$$\varrho^{q/2} > \varrho^{5\log p} = p^{5\log\varrho} > 3p^2.$$

Thus,

(17) $$0 < \left|\delta_1 p\log p - \delta_2 q\log q + pq\log\frac{\varrho}{\sigma}\right| < \exp(-\tfrac{1}{2}q\log\varrho).$$

It is an easy consequence of (16) and (17) that

(18) $$\left|\log\frac{\varrho}{\sigma}\right| \leqslant \frac{2\log p}{q} + 1 < 2.$$

Hence, $\sigma < e^2\varrho < \varrho^4$. We can therefore apply Theorem 2 to the left-hand side of (17) with $A = \varrho^4$, $A' = p$, $B = p^2$. So we see that absolute constants c_5 and c_6 exist such that

(19) $$\left|\delta_1 p\log p - \delta_2 q\log q + pq\log\frac{\varrho}{\sigma}\right| > \exp\left(-c_5(\log p)^{c_6}\log\varrho\right).$$

The combination of (17) and (19) yields inequality (14) in case (b). This completes the proof of (14).

Subsequently we show that there is an absolute constant C such that $p \leqslant C$ for every solution x, y, p, q of (1). Again we distinguish two cases, (a) and (b).

(a) $\sigma = 1$. We see from (11) that $\delta_2 = 1$ and $y = q-1$. By (1) we obtain

$$p\log 2 \leqslant p\log x < q\log y + 1 < q\log q + 1.$$

It now follows from (14) that

$$p < 2q\log q < c_7(\log p)^{c_8}$$

for some absolute constants c_7 and c_8. Hence, there is an absolute upper bound C_1 for p in this case.

(b) $\sigma > 1$. It follows from (13) that

$$(p^{\delta_1}\varrho^q + 1)^p q^{-\delta_2 q}\sigma^{-pq} = \left(1 - \frac{1}{q^{\delta_2}\sigma^p}\right)^q + \frac{1}{q^{\delta_2 q}\sigma^{pq}} < 1.$$

Using the estimate (15), we obtain

$$\left|\delta_2 q\log q - p\log\frac{p^{\delta_1}\varrho^q + 1}{\sigma^q}\right| \leqslant \frac{2q}{q^{\delta_2}\sigma^p} < \frac{2q^2}{\sigma^p}.$$

If $p \geqslant 32$, then $2q^2 < 2p^2 < 2^{p/2} \leqslant \sigma^{p/2}$, and

(20) $$0 < \left|\delta_2 q\log q - p\log\frac{p^{\delta_1}\varrho^q + 1}{\sigma^q}\right| \leqslant \exp\left(-\frac{1}{2}p\log\sigma\right).$$

It follows from this inequality in combination with (14) that

$$\left|\log\frac{p^{\delta_1}\varrho^q + 1}{\sigma^q}\right| \leqslant \frac{q\log q}{p} + \frac{1}{p} \leqslant \frac{2c_1^2(\log p)^{2c_2}}{p} \leqslant 1,$$

if $p \geqslant p_0$, where p_0 is some absolute constant. Since we want to prove that p is bounded, we can assume that $p \geqslant 32$ and $p \geqslant p_0$ without loss of generality. Hence, we can apply Theorem 2 with

$$A = \max(p^{\delta_1}\varrho^q+1, \sigma^q) \leqslant e\sigma^q \leqslant \sigma^{2q}, \qquad A' = q, \qquad B = p.$$

On using (14) we obtain absolute constants c_9 and c_{10} such that

(21) $$\left|\delta_2 q\log q - p\log\frac{p^{\delta_1}\varrho^q+1}{\sigma^q}\right| \geqslant \exp\left(-c_9(\log p)^{c_{10}}\log\sigma\right).$$

The combination of (20) and (21) yields

$$p \leqslant 2c_9(\log p)^{c_{10}}.$$

Hence, in both cases (a) and (b) there exists an effectively computab upper bound for p. By (14) this gives at the same time an effectively con putable upper bound for q. The case $q > p$ leads similarly to effective upp bounds for p and q.

The number of pairs (p, q) for which (1) has a solution $p, q, x,$ is therefore finite. It is an immediate consequence of Theorem A th for fixed p, q the solutions x, y of (1) can be effectively bounded. Henc the total number of solutions of (1) is finite and there exists an effective computable number C_0 such that $p \leqslant C_0$, $q \leqslant C_0$, $x \leqslant C_0$, $y \leqslant C_0$.

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