Some formulas for the Riemann zeta function at odd integer argument resulting from Fourier expansions of the Epstein zeta function

by

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0. Introduction. Formulas for the Riemann zeta function at even integer argument in terms of Bernoulli numbers have long been known. The case of odd argument remains a mystery. There are some results which shed light on the problem, however, [2], [3], [4], and [6], for example.

Here we derive some similar results using the expansions of the Epstein zeta function to be found in [6] and [8]. The results are rather different from those of [3] in a surprising way. That is, we obtain:

$$\zeta(3) = \frac{9}{45} \pi^3 - 4 \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-4}(n)(2\pi^2 n^2 + \pi n + \frac{1}{2}),$$

to be compared with the result of [3]:

$$\zeta(3) = \frac{7}{180} \pi^3 - 2 \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-4}(n).$$

However Berndt has observed that both results above follow from formula (30) of his paper [2]. And indeed he obtains an infinite number of similar formulas in this way. Here

$$\sigma_k(n) = \sum_{d|n} d^k \quad \text{and} \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1.$$

The outline of the paper is as follows. In Section 1, we prove some consequences of the Selberg-Chowla formula [6], including the first formula above for \(\zeta(3)\). In Section 2, we prove some formulas for \(\zeta(3)\) resulting from the generalization of the Selberg formula to \(3 \times 3\) positive definite quadratic forms, to be found in [8]. And we also prove a formula for the Epstein zeta function of the \(2 \times 2\) identity matrix at \(3/2\).
1. Some consequences of the Selberg–Chowla formula. It is necessary to recall some definitions from [8]. Let \( S \) be the \( n \times n \) matrix of a positive definite (real) quadratic form and let \( D \) be a complex variable with \( Re D > \frac{n}{2} \). Then Epstein’s zeta function is defined by

\[
Z_n(S, D) = \frac{1}{2} \sum_{\alpha_0, \alpha_2 = 0} (\alpha_2 S_2)^{-\alpha_2},
\]

where the sum is over all column vectors with integral coordinates, not all of which are zero. Here for any matrix \( A \), \( A^t \) denotes the transpose of \( A \). Next we define the constant term in the Laurent expansion of the Epstein zeta function about the pole \( D = n/2 \) to be \( k_n(D) \), i.e.,

\[
k_n(D) = \lim_{\epsilon \to 0^+} \left( Z_n(S, D) e^{-\frac{\pi}{2} S^{1/2} (D - \frac{n}{2})^{-1/2}} \right),
\]

where \( |S| \) denotes the determinant of \( S \). And define the modified Bessel function of the second kind \( K_\nu(s) \) by

\[
K_\nu(s) = \frac{1}{2} \sum_{m=0}^\infty e^{-\pi(s+1/2)^2} (s+1/2)^{-\frac{\nu}{2}+m}, \quad \text{for} \quad |arg s| < \pi/2.
\]

Let \( \Gamma(s) \) be the gamma function and \( \psi(s) = \Gamma'(s)/\Gamma(s) \).

In our applications \( S \) will always be a diagonal matrix. In particular we may assume

\[
S = \begin{bmatrix}
T & 0 \\
0 & S_1
\end{bmatrix}
\]

where \( T \) is an \( n_1 \times n_1 \) matrix and \( S_1 \) is an \( n_2 \times n_2 \) matrix. Here \( S \) is \( n \times n \), so that \( n = n_1 + n_2 \). The elements indicate matrices of zeroes of the appropriate size. For such a matrix \( S \), then we define (cf. definition (2.2) of [8]):

\[
H_{n_1, n_2}(S, D) = \sum_{a, b, c = 0} (T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b).
\]

The sum runs over integral column vectors \( a \) and \( b \) not all of whose entries are zero. Here \( T[a] = e^{i a T a} \).

We shall need two results of [8], which we recall now.

\[
Z_n(S, D) = Z_n(S_2, D) + \frac{\pi}{D} \Gamma(D) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b).
\]

And

\[
H_{n_1, n_2}(S, D) = \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b).
\]

We first use these formulas to prove a result obtained by Ramanujan [4] (v. 1, p. 257, v. 2, p. 170).

**Proposition 1.**

\[
\sum_{n=1}^{\infty} e^{-\pi n^2} \sigma_1(n) = \frac{1}{24} - \frac{1}{8 \pi}.
\]

**Proof.** Note that \( k_1(S) = k_1(S') \), when

\[
S = \begin{bmatrix}
d & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
S' = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\]

for any \( d \geq 0 \).

Applying formula (1.6), one finds that

\[
\zeta(2) + \pi k_1(d) + \pi H_{1,1}(S', 1) = d^{-1/2} \log 2
\]

\[
= d^{-1/2} \zeta(2) + \pi d^{-1/2} k_1(1) + \pi H_{1,1}(S', 1) - \pi d^{-1/2} \log 2.
\]

Now \( k_1(d) \) is the constant term in the Laurent expansion of \( Z_1(d, S) \) at \( d = 1/2 \). Thus

\[
k_1(d) = d^{-1/2} (\gamma - \log 2 d),
\]

where \( \gamma = \text{Euler’s constant} \).

Thus solving the above equation for the \( \zeta(2) \) term one obtains:

\[
\zeta(2)(1 - d^{-1}) = \pi(d^{-1/2} \log 2 + H_{1,1}(S', 1) - H_{1,1}(S, 1)).
\]

Next divide by \( d - 1 \) and let \( d \) approach 1. One obtains the result:

\[
\zeta(2) = \frac{\pi}{2} + \pi \lim_{d \to 1} (d - 1)^{-1} \left( H_{1,1}(S', 1) - H_{1,1}(S, 1) \right).
\]

Recalling the definitions of \( S \) and \( S' \) and (1.4), one sees that:

\[
H_{1,1}(S', 1) = \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b).
\]

And

\[
\lim_{d \to 1} (d - 1)^{-1} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b) \sum_{a, b, c = 0} (a T[a] S_2^{-1/2} b).
\]

\[
= -\frac{1}{2} k_1(2 \pi) - K_{1/2}(2 \pi) 2 \pi.
\]
Recall (1.1, p. 444)
\begin{equation}
K_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-\pi z} e^{-1/2} \\
\end{equation}
so that
\begin{equation}
K_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-\pi z} e^{-1/2} e^{-\pi z} e^{-1/2}.
\end{equation}
Therefore
\begin{equation}
-\frac{1}{2}K_{1/2}(z) =\sum_{a,b\geq 1}^{\infty} e^{-\pi ab} b = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_1(n).
\end{equation}

Putting the above results together, we see that:
\begin{equation}
\zeta(2) = \frac{\pi}{2} + 4\pi^2 \sum_{a,b\geq 1}^{\infty} e^{-\pi ab} b = \frac{\pi}{2} + \frac{4\pi^2}{2} \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_1(n).
\end{equation}

The proof is completed by noting that \(\zeta(2) = \pi^2/6\).

**Proposition 2.**
\begin{equation}
\zeta(2\pi) + \zeta(2\pi - 1) \pi^{1/2} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})^{-1} (1 - \phi) = -2\pi^2 \Gamma(\frac{1}{2})^{-1} \sum_{a\neq 2}^{\infty} \left(\frac{b}{a}\right)^{1/2} \{-2K_{1/2} - (2\pi ab K_{1/2} - (2\pi ab)^2)\}.
\end{equation}

**Proof.** Let \(S, S'\) be as in (1.7). Then \(Z_1(S, \phi) = Z_2(S', \phi).\) Applying formula (1.5) we see that:
\begin{equation}
\zeta(2\pi) + d^{1/2} e^{-\pi^{1/2}} \Gamma(\phi - \phi) \Gamma(\phi)^{-1} (1 - \phi) + \\
-\frac{\pi}{2} \sum_{a=b+1}^{\infty} \left(\frac{ab}{b^2}\right)^{1/2} \{-2K_{1/2} - (2\pi ab K_{1/2} - (2\pi ab)^2)\}
\end{equation}
\begin{equation}
= \zeta(2\pi) d^{1/2} e^{-\pi^{1/2}} \Gamma(\phi - \phi) \Gamma(\phi)^{-1} (1 - \phi) + \\
+\pi \sum_{a=b+1}^{\infty} \left(\frac{a^2}{d^{1/2}}\right)^{1/2} \left(\frac{a}{d^{1/2}} - 2K_{1/2} - (2\pi ab K_{1/2} - (2\pi ab)^2)\right).
\end{equation}

Solving for the terms with the zeta function we get:
\begin{equation}
\zeta(2\pi) (1 - d^{1/2}) + \zeta(2\pi - 1) \pi^{1/2} \Gamma(\phi - \phi) \Gamma(\phi)^{-1} (d^{1/2} - d^{1/2}) = 4\pi^2 \Gamma(\phi)^{-1} \sum_{a,b\geq 1}^{\infty} \left(\frac{b}{a}\right)^{1/2} \{-d^{1/2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\} - \frac{d^{1/2}}{2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\}.
\end{equation}

Next divide by \(d^{1/2}\) and take the limit as \(d\) approaches \(1\), to see that:
\begin{equation}
\zeta(2\pi) + \zeta(2\pi - 1) \pi^{1/2} \Gamma(\phi - \phi) \Gamma(\phi)^{-1} (1 - \phi) = 4\pi^2 \Gamma(\phi)^{-1} \sum_{a,b\geq 1}^{\infty} \left(\frac{b}{a}\right)^{1/2} \{-d^{1/2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\} - \frac{d^{1/2}}{2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\}.
\end{equation}

Now
\begin{equation}
\lim_{d\to 1} \frac{\pi}{2} \left\{d^{1/2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\} - \frac{d^{1/2}}{2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\}\right\}
\end{equation}
\begin{equation}
= -\frac{1}{2} \{2\pi ab d^{1/2} K_{1/2} - (2\pi ab d^{1/2})\}.
\end{equation}

Substituting this into the preceding formula finishes the first equality in the proposition. The second one comes from the formula for \(K'(\phi)\) which is a direct consequence of (1.3).

**Corollary.**
\begin{equation}
2 \zeta(4) - \frac{\pi}{2} \zeta(3) = 2\pi \sum_{a,b\geq 1}^{\infty} e^{-2\pi n} \sigma_4(n) \{2\pi^2 n^2 + \pi + \frac{1}{2}\}
\end{equation}
\begin{equation}
3 \zeta(6) - \frac{3}{2} \zeta(5) = \pi \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_4(n) \{2\pi^2 n^2 + 3\pi^2 n + 3\pi + \frac{3}{4}\}
\end{equation}

**Proof.** To prove the first formula, let \(\phi = 2\) in Proposition 2. This gives
\begin{equation}
2 \zeta(4) - \frac{\pi}{2} \zeta(3) = 2\pi \sum_{a,b\geq 1}^{\infty} \left(\frac{b}{a}\right)^{1/2} \left\{2\pi ab K_{1/2} + K_2(2\pi ab) - K_2(2\pi ab)\right\}.
\end{equation}

Using formulas from [1], p. 444, one sees that:
\begin{equation}
2K_1(\phi) + sK_{1/2}(\phi) - K_2(\phi) = \sqrt{\frac{\pi}{2}} e^{-\phi} \{2\pi^2 + 2\pi^2 + 2\pi^2\}
\end{equation}
\begin{equation}
= 2\pi \sum_{a,b\geq 1}^{\infty} e^{-2\pi n} \sigma_4(n) \{2\pi^2 n^2 + 3\pi^2 n + 3\pi + \frac{3}{4}\}
\end{equation}

Substituting, we obtain:
\begin{equation}
2 \zeta(4) - \frac{\pi}{2} \zeta(3) = 2\pi \sum_{a,b\geq 1}^{\infty} e^{-2\pi n} \sigma_4(n) \sigma_4(n) \{2\pi^2 n^2 + 3\pi^2 n + 3\pi + \frac{3}{4}\}
\end{equation}
\begin{equation}
= 2\pi \sum_{a,b\geq 1}^{\infty} e^{-2\pi n} \sigma_4(n) \{2\pi^2 n^2 + 3\pi^2 n + 3\pi + \frac{3}{4}\}
\end{equation}
\begin{equation}
= \pi \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_4(n) \{2\pi^2 n^2 + 3\pi^2 n + 3\pi + \frac{3}{4}\}.
\end{equation}

Letting \(ab = n\) and summing over \(n\) and \(a\) gives the formula of the corollary.
To prove the second formula of the corollary, substitute \( q = 3 \) into the formula of Proposition 2. This gives

\[
3\zeta(6) - \pi^6/4 = \pi^3 \sum_{a,b \in \mathbb{Z}, \gcd(a,b) = 1} \left( \frac{b}{a} \right)^{12} \left\{ aK_{3a}(x) + aK_{12b}(x) - K_{6b}(x) \right\}_{2 \text{mod} 6}.
\]

Using formulas from [1], p. 444, one sees that:

\[
sK_{3a}(x) + sK_{12b}(x) - K_{6b}(x) = \sqrt{\frac{\pi}{2}} e^{-a} \{ 2a^{-1/2} + 6a^{-1/2} + 12a^{-1/2} + 12a^{-3/2} \}.
\]

Substituting this into the preceding gives:

\[
3\zeta(6) - \pi^6/4 = \pi^3 \sum_{a,b \in \mathbb{Z}, \gcd(a,b) = 1} e^{-2\pi a b} \left( \frac{b}{a} \right)^{12} \{ 2(2\pi a b)^{-1/2} + 6(2\pi a b)^{-1/2} + 12(2\pi a b)^{-3/2} + 12(2\pi a b)^{-3/2} \}.
\]

Letting \( ab = n \) and summing over \( n \) and \( a \) gives the formula of the corollary.

\[ \Box \]

2. A representation for \( \zeta(3) \) using the Fourier expansion of the Epstein zeta function of a \( 3 \times 3 \) positive definite quadratic form and a formula for \( G_k(I, \frac{3}{2}) \). Let \( d > 0 \) and suppose that \( S, S' \) are the diagonal \( 2 \times 2 \) matrices obtained from \( d \) by (1.7). Then define for \( d > 0 \)

\[
(2.1) \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/d \end{pmatrix} \quad \text{and} \quad D' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Using (1.6) for \( n = 3 \) one obtains the following result about \( \zeta(3) \).

**Proposition 3.**

\[
\frac{1}{2} \zeta(3)(d^{-3/2} - 1) + \zeta(2)(d^{-1} - d^{-1/2}) = \pi(H_{11,1}(S', 1) - d^{-1/2}H_{11,1}(I, 1) + H_{21,1}(D', \frac{3}{2}) - H_{21,1}(D, \frac{3}{2})).
\]

Here \( I \) is the \( 2 \times 2 \) identity matrix.

**Proof.** Recall formula (1.8) which says that

\[ k_1(d) = d^{-1/2}(\gamma - \log d^{1/2}). \]

Using (1.6) and a formula from [1], p. 258, one sees that:

\[
(2.2) \quad k_1(S) = \zeta(2) + \pi d^{-1/2}(\gamma - \log d^{-1/2} - \log 2) + \pi H_{11,1}(S, 1) = k_1(S') = d^{-1/2}(\gamma - \log 2) + \pi H_{11,1}(S', 1).
\]

Next apply (1.6) with \( n = 3, n_1 = 2, n_2 = 1 \). This gives:

\[
k_3(D) = d^{-1/2} \zeta(3) + 2d^{-1/2} k_3(I) + 2\pi H_{11,1}(D, \frac{3}{2}) + 2\pi d^{-1/2}(\log 2 - 1) = d^{-1/2} \zeta(3) + 2d^{-1/2} \zeta(3) + 2\pi d^{-1/2}(\gamma - 1) + 2\pi d^{-1/2} H_{11,1}(I, 1) + 2\pi H_{11,1}(D, \frac{3}{2})
\]

Using formulas from [1], p. 444, one sees that:

\[
sk_k(S) + sk_k(S') - H_{k1,1}(x) = \sqrt{\frac{\pi}{2}} e^{-a} \{ 2a^{-1/2} + 6a^{-1/2} + 12a^{-1/2} + 12a^{-3/2} \}.
\]

Substituting this into the preceding gives:

\[
3\zeta(6) - \pi^6/4 = \pi^3 \sum_{a,b \in \mathbb{Z}, \gcd(a,b) = 1} e^{-2\pi a b} \left( \frac{b}{a} \right)^{12} \{ 2(2\pi a b)^{-1/2} + 6(2\pi a b)^{-1/2} + 12(2\pi a b)^{-3/2} + 12(2\pi a b)^{-3/2} \}.
\]

Letting \( ab = n \) and summing over \( n \) and \( a \) gives the formula of the corollary.

\[ \Box \]

\[
-\frac{1}{4} \zeta(3) + \frac{1}{4} \zeta(2) = -\frac{\pi}{48} + \frac{1}{16} + 2\pi \sum_{n \geq 1} e^{-2\pi n} c_n(n) + 4\pi \sum_{k, n, m \geq 1} h_k(2n^2 + m^2)(n^2 + m^2)^{-3/2} K_{1/2}(2\pi k(n^2 + m^2)^{1/2}) + 8\pi \sum_{k, n, m \geq 1} h_k(2n^2 + m^2)(n^2 + m^2)^{-1} K_{1/2}(2\pi k(n^2 + m^2)^{1/2}).
\]

**Proof.** Use the formula of Proposition 3. Divide by \( d - 1 \) and take the limit as \( d \) approaches 1. This gives:

\[
-\frac{1}{4} \zeta(3) + \frac{1}{4} \zeta(2) = \lim_{d \to 1} (d - 1)^{-1}(H_{11,1}(S', 1) - d^{-1/2}H_{11,1}(I, 1)) + \lim_{d \to 1} (d - 1)^{-1}(H_{21,1}(D', \frac{3}{2}) - H_{21,1}(D, \frac{3}{2})).
\]

By (1.4), the first term on the right-hand side is

\[
4\lim_{d \to 1} (d - 1)^{-1} \sum_{n, m \geq 1} \frac{n}{m}^{1/2} \left\{ d^{-1/2}K_{1/2}(2\pi n^2 - 2\pi m^2) - d^{-1/2}K_{1/2}(2\pi mn) \right\}.
\]

Using the formulas for \( K_{1/2} \) and \( K'_{1/2} \) from (1.9) and (1.10), we see that

\[
-\frac{1}{4} K_{1/2}(x) + \frac{1}{4} sK_{1/2}(x) = \left( \frac{x}{2} \right)^{1/2} e^{-\frac{1}{2} x^{1/2} - \frac{1}{2} x^{-1/2}}.
\]

Thus the first term in the expression for \( -\frac{1}{4} \zeta(3) + \frac{1}{4} \zeta(2) \) is:

\[
4\pi \sum_{n, m \geq 1} e^{-2\pi n m} \left( \frac{1}{4\pi n m - 1} \right)^{1/2} = 2\pi e^{-2\pi n m} \sum_{n \geq 1} e^{-2\pi n m} c_n(n) - \frac{\pi}{4} \frac{1}{16}.
\]

Here we have used Proposition 1.
The second term in the expression for $-\frac{1}{2} \zeta(3) + \frac{1}{2} \zeta(2)$ is:

$$\lim_{d \to 1} \sum_{g(h,m)} \left[ k(n^2 + dm^2)^{-1/2} K \left( \frac{2 \pi k}{2 \pi d} \right) \right] -
$$

$$- d^{-1} k(n^2 + m^2)^{-1/2} K \left( \frac{2 \pi d^{-1/2}}{2 \pi d} \right) k(n^2 + m^2)^{1/2} \right].$$

We have used (1.4) here.

Taking the indicated limit and adding the 2 results for the terms of the expression for $-\frac{1}{2} \zeta(3) + \frac{1}{2} \zeta(2)$, one obtains the result of the Corollary.

Next we use the method of Proposition 1 to obtain a formula for $Z_2(I, \frac{1}{2})$, where $I$ is the $2 \times 2$ identity matrix.

**Proposition 4.**

$$Z_2(I, \frac{1}{2}) = \frac{1}{2} \pi + 4 \pi^2 \sum_{d \geq 0, \text{ prime}} \sqrt{d^2 + 1} e^{-\pi d^2}.$$

**Proof.** For $d > 0$, define

$$D' = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad S' = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

Then, since $k_d(D') = k_d(D')$, we have by formula (1.6) with $n = 3$, $n_+ = 1$, $n_0 = 2$:

$$Z_2(I, \frac{1}{2}) = Z_2(S, \frac{1}{2})$$

$$= 2\pi \left[ k_d(1) d^{-1/2} - k_d(d) + H_{12}(D', \frac{1}{2}) - H_{12}(D', \frac{1}{2}) \right].$$

Then, as usual, divide by $d - 1$, and let $d$ approach 1.

Making use of the fact that $Z_2(S, \varphi) = Z_2(S', \varphi)$, one obtains $\frac{1}{2} Z_2(I, \frac{1}{2})$ for the limit as $d$ approaches 1 of $(d - 1)^{-1/2}$ times the left-hand side of (2.3).

Then applying (3.4) and (1.9), one finds that the limit as $d$ approaches 1 of $(d - 1)^{-1/2}$ times the last two terms inside the brackets on the right-hand side of (2.3) is

$$\frac{1}{2} \pi^2 \sum_{d \geq 0, \text{ prime}} \sqrt{d^2 + 1} e^{-\pi d^2}.$$

Combining these results completes the proof.

**Corollary.**

$$\zeta(3) = \frac{1}{2} \pi + 4 \pi^2 \sum_{d \geq 0, \text{ prime}} \sqrt{d^2 + 1} e^{-\pi d^2} - \frac{\pi^2}{3} - 8\pi \sum_{d \geq 1} \frac{b}{d} K_1(2\pi db).$$

**Proof.** This is a simple application of formula (1.5) and Proposition 4.