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Some formulas for the Riemann zeta function at odd integer argument resulting from Fourier expansions of the Epstein zeta function *

by

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0. Introduction. Formulas for the Riemann zeta function at even integer argument in terms of Bernoulli numbers have long been known. The case of odd argument remains a mystery. There are some results which shed light on the problem, however, [2], [3], [4], and [6], for example.

Here we derive some similar results using the expansions of the Epstein zeta function to be found in [6] and [8]. The results are rather different from those of [3] in a surprising way. That is, we obtain:

$$\zeta(3) = \frac{2}{45}\pi^3 - 4\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n)(2\pi^2 n^2 + \pi n + \tfrac{1}{2}),$$

to be compared with the result of [3]:

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n).$$

However Berndt has observed that both results above follow from formula (30) of his paper [2]. And indeed he obtains an infinite number of similar formulas in this way. Here

$$\sigma_k(n) = \sum_{0<d|n} d^k \quad \text{and} \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1.$$

The outline of the paper is as follows. In Section 1, we prove some consequences of the Selberg–Chowla formula [6], including the first formula above for $\zeta(3)$. In Section 2, we prove some formulas for $\zeta(3)$ resulting from the generalization of the Selberg formula to 3×3 positive definite quadratic forms, to be found in [8]. And we also prove a formula for the Epstein zeta function of the 2×2 identity matrix at $3/2$.

1. Some consequences of the Selberg–Chowla formula. It is necessary to recall some definitions from [8]. Let S be the $n\times n$ matrix of a positive definite (real) quadratic form and let ϱ be a complex variable with $\operatorname{Re}\varrho > \frac{1}{2}n$. Then Epstein's zeta function is defined by

$$Z_n(S,\varrho) = \tfrac{1}{2}\sum_{a\in Z^n-0} ({}^t aSa)^{-\varrho}, \tag{1.1}$$

where the sum is over all column vectors with integral coordinates, not all of which are zero. Here for any matrix A, ${}^tA$ denotes the transpose of A. Next we define the constant term in the Laurent expansion of the Epstein zeta function about the pole $\varrho = n/2$ to be $k_n(s)$, i.e.,

$$k_n(s) = \lim_{\varrho\to n/2}\left\{Z_n(S,\varrho) - \tfrac{1}{2}\pi^{n/2}|S|^{-1/2}\left(\varrho-\frac{n}{2}\right)^{-1}\right\}, \tag{1.2}$$

where $|S|$ = determinant of S. And define the modified Bessel function of the second kind $K_\nu(z)$ by

$$K_\nu(z) = \tfrac{1}{2}\int_0^\infty e^{-z(u+1/u)/2}u^{\nu-1}\,du \quad \text{for} \quad |\arg z| < \pi/2. \tag{1.3}$$

Let $\Gamma(z)$ be the gamma function and $\psi(z) = \Gamma'(z)/\Gamma(z)$.

In our applications S will always be a diagonal matrix. In particular we may assume

$$S = \begin{pmatrix} T & 0\\ 0 & S_2\end{pmatrix}$$

where T is an $n_1\times n_1$ matrix and S_2 is an $n_2\times n_2$ matrix. Here S is $n\times n$, so that $n = n_1+n_2$. The 0's indicate matrices of zeroes of the appropriate size. For such a matrix S, then we define (cf. definition (2.2) of [8]):

$$H_{n_1,n_2}(S,\varrho) = |S_2|^{-1/2}\sum_{\substack{a\in Z^{n_1}-0\\ b\in Z^{n_2}-0}}\left(\frac{T[a]}{S_2^{-1}[b]}\right)^{\frac{1}{4}n_2-\frac{\varrho}{2}}K_{n_2/2-\varrho}\left(2\pi\left(T[a]S_2^{-1}[b]\right)^{1/2}\right). \tag{1.4}$$

The sum runs over integral column vectors a and b not all of whose entries are zero. Here $T[a] = {}^taTa$.

We shall need two results of [8], which we recall now.

$$Z_n(S,\varrho) = Z_{n_2}(S_2,\varrho) + \pi^{n_2/2}\Gamma\left(\varrho-\frac{n_2}{2}\right)\Gamma(\varrho)^{-1}|S_2|^{-1/2}Z_{n_1}\left(T,\varrho-\frac{n_2}{2}\right) + \frac{\pi^\varrho}{\Gamma(\varrho)}H_{n_1,n_2}(S,\varrho). \tag{1.5}$$

$$k_n(S) = Z_{n_2}\left(S_2,\frac{n}{2}\right) + \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n}{2}\right)^{-1}k_{n_1}(T)|S_2|^{-1/2}\pi^{n_2/2} + H_{n_1,n_2}\left(S,\frac{n}{2}\right)\pi^{n/2}\Gamma\left(\frac{n}{2}\right)^{-1} + \frac{1}{2}|S|^{-1/2}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)^{-1}\left\{\psi\left(\frac{n_1}{2}\right)-\psi\left(\frac{n}{2}\right)\right\}. \tag{1.6}$$

We first use these formulas to prove a result obtained by Ramanujan [4] (v. 1, p. 257, v. 2, p. 170).

PROPOSITION 1.

$$\sum_{n=1}^\infty e^{-2\pi n}\sigma_1(n) = \frac{1}{24}-\frac{1}{8\pi}.$$

Proof. Note that $k_2(S) = k_2(S')$, when

$$S = \begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad S' = \begin{pmatrix} 1 & 0\\ 0 & d\end{pmatrix}, \tag{1.7}$$

for any $d > 0$.

Applying formula (1.6), one finds that:

$$\zeta(2)+\pi k_1(d)+\pi H_{1,1}(S,1)-\pi d^{-1/2}\log 2 = d^{-1}\zeta(2)+\pi d^{-1/2}k_1(1)+\pi H_{1,1}(S',1)-\pi d^{-1/2}\log 2.$$

Now $k_1(d)$ is the constant term in the Laurent expansion of $Z_1(d,\varrho) = d^{-\varrho}\zeta(2\varrho)$ about the point $\varrho = 1/2$. Thus

$$k_1(d) = d^{-1/2}(\gamma-\log d^{1/2}), \tag{1.8}$$

where γ = Euler's constant.

Thus solving the above equation for the $\zeta(2)$ term one obtains:

$$\zeta(2)(1-d^{-1}) = \pi\{d^{-1/2}\log d^{1/2}+H_{1,1}(S',1)-H_{1,1}(S,1)\}.$$

Next divide by $d-1$ and let d approach 1. One obtains the result:

$$\zeta(2) = \frac{\pi}{2}+\pi\lim_{d\to1}(d-1)^{-1}\{H_{1,1}(S',1)-H_{1,1}(S,1)\}.$$

Recalling the definitions of S and S' and (1.4), one sees that:

$$H_{1,1}(S',1) = 4d^{-1/2}\sum_{a,b\geq1}d^{-1/4}(ba^{-1})^{1/2}K_{1/2}(2\pi d^{-1/2}ab),$$

$$H_{1,1}(S,1) = 4\sum_{a,b\geq1}d^{-1/4}(ba^{-1})^{1/2}K_{1/2}(2\pi d^{1/2}ab).$$

And

$$\lim_{d\to1}(d-1)^{-1}\{d^{-3/4}K_{1/2}(2\pi d^{-1/2}ab)-d^{-1/4}K_{1/2}(2\pi d^{1/2}ab)\} = -\tfrac{1}{2}K_{1/2}(2\pi ab)-K'_{1/2}(2\pi ab)2\pi ab.$$

Recall ([1], p. 444)

$$K_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2} \tag{1.9}$$

so that

$$K'_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-z}\left(-\tfrac{1}{2}z^{-3/2} - z^{-1/2}\right). \tag{1.10}$$

Therefore

$$-\tfrac{1}{2}K_{1/2}(z) - zK'_{1/2}(z) = \sqrt{\frac{\pi}{2}}\, e^{-z} z^{1/2}.$$

Putting the above results together, we see that:

$$\zeta(2) = \frac{\pi}{2} + 4\pi^2 \sum_{a,b\geqslant 1} e^{-2\pi ab}\, b = \frac{\pi}{2} + 4\pi^2 \sum_{n\geqslant 1} e^{-2\pi n}\sigma_1(n).$$

The proof is completed by noting that $\zeta(2) = \pi^2/6$. ■

Proposition 2.

$$\zeta(2\varrho)\varrho + \zeta(2\varrho-1)\pi^{1/2}\Gamma(\varrho-\tfrac{1}{2})\Gamma(\varrho)^{-1}(1-\varrho)$$
$$= -2\pi^{\varrho}\Gamma(\varrho)^{-1}\sum_{a,b\geqslant 1}(ba^{-1})^{\varrho-1/2}\{K_{1/2-\varrho}(2\pi ab) + 4\pi ab\, K'_{1/2-\varrho}(2\pi ab)\}$$
$$= 2\pi^{\varrho}\Gamma(\varrho)^{-1}\sum_{a,b\geqslant 1}\left(\frac{b}{a}\right)^{\varrho-1/2}\{2\pi ab\left(K_{3/2-\varrho}(2\pi ab) + K_{1/2+\varrho}(2\pi ab)\right) - $$
$$- K_{1/2-\varrho}(2\pi ab)\}.$$

Proof. Let S, S' be as in (1.7). Then $Z_2(S,\varrho) = Z_2(S',\varrho)$. Applying formula (1.5) we see that:

$$\zeta(2\varrho) + d^{1/2-\varrho}\pi^{1/2}\Gamma(\varrho-\tfrac{1}{2})\Gamma(\varrho)^{-1}\zeta(2\varrho-1) +$$
$$+ 4\pi^{\varrho}\Gamma(\varrho)^{-1}\sum_{a,b\geqslant 1}\left(\frac{da^2}{b^2}\right)^{\frac{1}{4}-\frac{\varrho}{2}} K_{1/2-\varrho}(2\pi d^{1/2}ab)$$
$$= \zeta(2\varrho)d^{-\varrho} + d^{-1/2}\pi^{1/2}\Gamma(\varrho-\tfrac{1}{2})\Gamma(\varrho)^{-1}\zeta(2\varrho-1) +$$
$$+ 4\pi^{\varrho}\Gamma(\varrho)^{-1}d^{-1/2}\sum_{a,b\geqslant 1}\left(\frac{a^2}{d^{-1}b^2}\right)^{\frac{1}{4}-\frac{\varrho}{2}} K_{1/2-\varrho}(2\pi d^{1/2}ab).$$

Solving for the terms with the zeta function we get:

$$\zeta(2\varrho)(1-d^{-\varrho}) + \zeta(2\varrho-1)\pi^{1/2}\Gamma(\varrho-\tfrac{1}{2})\Gamma(\varrho)^{-1}(d^{1/2-\varrho} - d^{-1/2})$$
$$= 4\pi^{\varrho}\Gamma(\varrho)^{-1}\sum_{a,b\geqslant 1}\left(\frac{b}{a}\right)^{\varrho-1/2}\{d^{-\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{-1/2}) - d^{\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{1/2})\}.$$

Next divide by $d-1$ and take the limit as d approaches 1, to see that:

$$\zeta(2\varrho) + \zeta(2\varrho-1)\pi^{1/2}\Gamma(\varrho-\tfrac{1}{2})\Gamma(\varrho)^{-1}(1-\varrho)$$
$$= 4\pi^{\varrho}\Gamma(\varrho)^{-1}\lim_{d\to 1}(d-1)^{-1}\sum_{a,b\geqslant 1}\left(\frac{b}{a}\right)^{\varrho-1/2}\{d^{-\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{-1/2}) -$$
$$- d^{\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{1/2})\}.$$

Now

$$\lim_{d\to 1}(d-1)^{-1}\{d^{-\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{-1/2}) - d^{\frac{1}{4}-\frac{\varrho}{2}}K_{1/2-\varrho}(2\pi ab\, d^{1/2})\}$$
$$= -\tfrac{1}{2}K_{1/2-\varrho}(2\pi ab) - 2\pi ab\, K'_{1/2-\varrho}(2\pi ab).$$

Substituting this into the preceeding formula finishes the first equality in the proposition. The second one comes from the formula for $K'(z)$ which is a direct consequence of (1.3). ■

Corollary.

$$2\zeta(4) - \frac{\pi}{2}\zeta(3) = 2\pi\sum_{n\geqslant 1} e^{-2\pi n}\sigma_{-3}(n)(2\pi^2n^2 + \pi n + \tfrac{1}{2}).$$

$$3\zeta(6) - \tfrac{3}{4}\pi\zeta(5) = \pi\sum_{n\geqslant 1} e^{-2\pi n}\sigma_{-5}(n)(2\pi^3n^3 + 3\pi^2n^2 + 3\pi n + \tfrac{3}{2}).$$

Proof. To prove the first formula, let $\varrho = 2$ in Proposition 2. This gives

$$2\zeta(4) - \zeta(3)\frac{\pi}{2}$$
$$= 2\pi^2\sum_{a,b\geqslant 1}\left(\frac{b}{a}\right)^{3/2}\{2\pi ab\left(K_{1/2}(2\pi ab) + K_{5/2}(2\pi ab)\right) - K_{3/2}(2\pi ab)\}.$$

Using formulas from [1], p. 444, one sees that:

$$zK_{1/2}(z) + zK_{5/2}(z) - K_{3/2}(z) = \sqrt{\frac{\pi}{2}}\, e^{-z}\{2z^{1/2} + 2z^{-1/2} + 2z^{-3/2}\}.$$

Substituting, we obtain:

$$2\zeta(4) - \frac{\pi}{2}\zeta(3) = 4\pi^3\sum_{a,b\geqslant 1} e^{-2\pi ab}a^{-1}b^2 +$$
$$+ 2\pi^2\sum_{a,b\geqslant 1} e^{-2\pi ab}a^{-2}b + \pi\sum_{a,b\geqslant 1} e^{-2\pi ab}a^{-3}.$$

Letting $ab = n$ and summing over n and a gives the formula of the corollary.

To prove the second formula of the corollary, substitute $\varrho = 3$ into the formula of Proposition 2. This gives

$$3\zeta(6)-\pi^{\frac{3}{4}}\zeta(5) = \pi^3\sum_{a,b\geqslant 1}\left(\frac{b}{a}\right)^{5/2}\{zK_{3/2}(z)+zK_{1/2}(z)-K_{5/2}(z)\}_{z=2\pi ab}.$$

Using formulas from [1], p. 444, one sees that:

$$zK_{3/2}(z)+zK_{1/2}(z)-K_{5/2}(z) = \sqrt{\frac{\pi}{2}}\,e^{-z}\{2z^{1/2}+6z^{-1/2}+12z^{-3/2}+12z^{-5/2}\}.$$

Substituting this into the preceeding gives:

$$\begin{aligned}3\zeta(6)&-\pi^{\frac{3}{4}}\zeta(5)\\ &=\pi^{7/2}2^{-1/2}\sum_{a,b\geqslant 1}e^{-2\pi ab}\left(\frac{b}{a}\right)^{5/2}\{2(2\pi ab)^{1/2}+6(2\pi ab)^{-1/2}+12(2\pi ab)^{-3/2}+\\ &\qquad\qquad +12(2\pi ab)^{-5/2}\}\\ &=\sum_{a,b\geqslant 1}e^{-2\pi ab}\{2\pi^4a^{-2}b^3+3\pi^3a^{-3}b^2+3\pi^2a^{-4}b+\tfrac{3}{2}\pi a^{-5}\}.\end{aligned}$$

Letting $ab = n$ and summing over n and a gives the formula of the corollary. ■

2. A representation for $\zeta(3)$ using the Fourier expansion of the Epstein zeta function of a 3×3 positive definite quadratic form and a formula for $Z_2(I,\frac{3}{2})$. Let $d>0$ and suppose that S, S' are the diagonal 2×2 matrices obtained from d by (1.7). Then define for $d>0$

$$(2.1)\qquad D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&d\end{pmatrix}\quad\text{and}\quad D'=\begin{pmatrix}1&0&0\\0&d&0\\0&0&1\end{pmatrix}.$$

Using (1.6) for $n=3$ one obtains the following result about $\zeta(3)$.

PROPOSITION 3.

$$\begin{aligned}\tfrac{1}{2}\zeta(3)(d^{-3/2}-1)&+\zeta(2)(d^{-1/2}-d^{-1})\\ &=\pi\{H_{1,1}(S',1)-d^{-1/2}H_{1,1}(I,1)+H_{2,1}(D',\tfrac{3}{2})-H_{2,1}(D,\tfrac{3}{2})\}.\end{aligned}$$

Here I is the 2×2 identity matrix.

Proof. Recall formula (1.8) which says that

$$k_1(d) = d^{-1/2}(\gamma-\log d^{1/2}).$$

Using (1.6) and a formula from [1], p. 258, one see that:

$$\begin{aligned}(2.2)\qquad k_2(S)&=\zeta(2)+\pi d^{-1/2}(\gamma-\log d^{1/2}-\log 2)+\pi H_{1,1}(S,1)\\ &=k_2(S')=d^{-1}\zeta(2)+\pi d^{-1/2}(\gamma-\log 2)+\pi H_{1,1}(S',1).\end{aligned}$$

Next apply (1.6) with $n=3$, $n_1=2$, $n_2=1$. This gives:

$$\begin{aligned}k_3(D)&=d^{-3/2}\zeta(3)+2d^{-1/2}k_2(I)+2\pi H_{2,1}(D,\tfrac{3}{2})+2\pi d^{-1/2}(\log 2-1)\\ &=d^{-3/2}\zeta(3)+2d^{-1/2}\zeta(2)+2\pi d^{-1/2}(\gamma-1)+2\pi d^{-1/2}H_{1,1}(I,1)+\\ &\qquad\qquad+2\pi H_{2,1}(D,\tfrac{3}{2})\\ &=k_3(D')=\zeta(3)+2k_2(S')+2\pi H_{2,1}(D',\tfrac{3}{2})+2\pi d^{-1/2}(\log 2-1)\\ &=\zeta(3)+2d^{-1}\zeta(2)+2\pi d^{-1/2}(\gamma-1)+2\pi H_{1,1}(S',1)+2\pi H_{2,1}(D',\tfrac{3}{2}).\end{aligned}$$

The proof is completed by solving for the terms involving $\zeta(2)$ and $\zeta(3)$. ■

COROLLARY.

$$\begin{aligned}-\tfrac{3}{4}\zeta(3)+\tfrac{1}{2}\zeta(2)=-\frac{\pi}{48}&+\tfrac{1}{16}+2\pi^2\sum_{n\geqslant 1}e^{-2\pi n}\sigma_1(n)+\\ &+4\pi\sum_{k,n,m\geqslant 1}k(2n^2+m^2)(n^2+m^2)^{-3/2}K_1(2\pi k(n^2+m^2)^{1/2})+\\ &+8\pi^2\sum_{k,n,m\geqslant 1}k^2(2n^2+m^2)(n^2+m^2)^{-1}K_1'(2\pi k(n^2+m^2)^{1/2}).\end{aligned}$$

Proof. Use the formula of Proposition 3. Divide by $d-1$ and take the limit as d approaches 1. This gives:

$$\begin{aligned}-\tfrac{3}{4}\zeta(3)+\tfrac{1}{2}\zeta(2)&=\lim_{d\to 1}(d-1)^{-1}\{H_{1,1}(S',1)-d^{-1/2}H_{1,1}(I,1)\}+\\ &\quad+\lim_{d\to 1}(d-1)^{-1}\{H_{2,1}(D',\tfrac{3}{2})-H_{2,1}(D,\tfrac{3}{2})\}.\end{aligned}$$

By (1.4), the first term on the right-hand side is

$$\begin{aligned}4\lim_{d\to 1}(d-1)^{-1}&\sum_{n,m\geqslant 1}\left(\frac{n}{m}\right)^{1/2}\{d^{-1}K_{1/2}(2\pi d^{-1/2}nm)-d^{-1/2}K_{1/2}(2\pi nm)\}\\ &=4\sum_{n,m\geqslant 1}\left(\frac{n}{m}\right)^{1/2}\left\{-\frac{1}{2}K_{1/2}(2\pi nm)-\pi nmK_{1/2}'(2\pi nm)\right\}.\end{aligned}$$

Using the formulas for $K_{1/2}$ and $K_{1/2}'$ from (1.9) and (1.10), we see that

$$-\tfrac{1}{2}K_{1/2}(z)-\tfrac{1}{2}zK_{1/2}'(z)=\left(\frac{\pi}{2}\right)^{1/2}e^{-z}(\tfrac{1}{2}z^{1/2}-\tfrac{1}{4}z^{-1/2}).$$

Thus the first term in the expression for $-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)$ is:

$$4\pi\sum_{n,m\geqslant 1}e^{-2\pi nm}(\tfrac{1}{2}\pi n-\tfrac{1}{8}m^{-1})=2\pi^2\sum_{n\geqslant 1}e^{-2\pi n}\sigma_1(n)-\frac{\pi}{48}+\frac{1}{16}.$$

Here we have used Proposition 1.

The second term in the expression for $-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)$ is:

$$8\lim_{d\to 1}(d-1)^{-1}\sum_{k,n,m\geqslant 1}\left\{k(n^2+dm^2)^{-1/2}K_1\big(2\pi k(n^2+dm^2)^{1/2}\big)-\right.$$
$$\left.-d^{-1}k(n^2+m^2)^{-1/2}K_1\big(2\pi d^{-1/2}k(n^2+m^2)^{1/2}\big)\right\}.$$

We have used (1.4) here.

Taking the indicated limit and adding the 2 results for the terms of the expression for $-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)$, one obtains the result of the Corollary. ■

Next we use the method of Proposition 1 to obtain a formula for $Z_2(I,\frac{3}{2})$, where I is the 2×2 identity matrix.

Proposition 4.

$$Z_2(I,\tfrac{3}{2})=\tfrac{4}{3}\pi+4\pi^2\sum_{\substack{a\geqslant 1\\ b,c\neq 0,0}}\sqrt{b^2+c^2}\,e^{-2\pi a\sqrt{b^2+c^2}}.$$

Proof. For $d>0$, define

$$D''=\begin{pmatrix}d&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad D'=\begin{pmatrix}1&0&0\\0&d&0\\0&0&1\end{pmatrix},\quad S=\begin{pmatrix}d&0\\0&1\end{pmatrix},\quad S'=\begin{pmatrix}1&0\\0&d\end{pmatrix}.$$

Then, since $k_3(D'')=k_3(D')$, we have by formula (1.6) with $n=3$, $n_1=1$, $n_2=2$:

$$(2.3)\qquad Z_2(I,\tfrac{3}{2})-Z_2(S,\tfrac{3}{2})=2\pi[k_1(1)d^{-1/2}-k_1(d)+H_{1,2}(D',\tfrac{3}{2})-H_{1,2}(D'',\tfrac{3}{2})].$$

Then, as usual, divide by $d-1$, and let d approach 1.

Making use of the fact that $Z_2(S,\varrho)=Z_2(S',\varrho)$, one obtains $\frac{3}{4}Z_2(I,\frac{3}{2})$ for the limit as d approaches 1 of $(d-1)^{-1}$ times the left-hand side of (2.3).

Via formula (1.8) one obtains 1/2 for the limit as d approaches 1 of $(d-1)^{-1}$ times the first two terms inside the brackets on the right-hand side of (2.3).

Then applying (1.4) and (1.9), one finds that the limit as d approaches 1 of $(d-1)^{-1}$ times the last two terms inside the brackets on the right-hand side of (2.3) is

$$\tfrac{3}{2}\pi^2\sum_{\substack{a\geqslant 1\\ b,c\neq 0,0}}\sqrt{b^2+c^2}\,e^{-2\pi a\sqrt{b^2+c^2}}.$$

Combining these results completes the proof. ■

Corollary.

$$\zeta(3)=\tfrac{4}{3}\pi+4\pi^2\sum_{\substack{a\geqslant 1\\ b,c\neq 0,0}}\sqrt{b^2+c^2}\,e^{-2\pi a\sqrt{a^2+c^2}}-\frac{\pi^2}{3}-8\pi\sum_{\substack{a\geqslant 1\\ b\geqslant 1}}\frac{b}{a}K_1(2\pi ab).$$

Proof. This is a simple application of formula (1.5) and Proposition 4. ■

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