On $m$ times covering systems of congruences

by

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A system of residue classes

\[(1) \quad a_i \pmod{n_i}, \quad 0 \leq a_i < n_i, \quad 1 < n_i, \quad i = 1, 2, \ldots, k\]

is called to be $m$ times covering if every integer belongs exactly to $m$ of these classes, where $m$ is a given positive integer.

Once covering system is usually called exactly covering. This topic is a subject of consideration of many papers, e.g. [1]-[9] except for [3]. It is evident that in any exactly covering system no residue class appears more than once, but such a restriction does not show to be useful for $m$ times covering systems in general and therefore the repetition of residue classes will be allowed in (1).

In this paper we shall characterize the $m$ times covering systems containing exactly one $r$-tuple of distinct residue classes with respect to the same modulus while the moduli of the remaining distinct classes are distinct for $r = 2, 3, 4, \text{ and } 5$. These results extend those to be known for exactly covering systems and proved for $r = 2$ in [3], $r = 3$ in [9], $r = 4, 5$ in [7].

It is easy to check that $m$ exactly covering systems form together an $m$ times covering system. But on the other hand, there exist $m$ times covering systems which are not of this form. The following one was constructed by Choi during the International Colloquium on Infinite and Finite Sets held in Köszthely (Hungary) in 1973.

The system

\[(2) \quad 1 \pmod{2}, 0 \pmod{3}, 2 \pmod{4}, 0 \pmod{6}, 2 \pmod{10}, 2 \pmod{15}; 1, 4, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 18, 19, 22, 23, 24, 25, 26, 28, 29 \pmod{30}\]

is twice covering. Suppose (2) to be a union of two exactly covering systems and let (1) be that of them which contains the class $1 \pmod{2}$. Then (1) contains neither $0 \pmod{3}$ nor $2 \pmod{15}$ because of their non-empty intersection with class $1 \pmod{2}$. Now the following two possibilities can occur:
(a) The residue class $2 \pmod{6}$ belongs to (1). In this case $0 \pmod{10}$ cannot belong to (1), but then (1) does not cover the intersection of classes $0 \pmod{10}$ and $0 \pmod{5}$ which contradicts the assumption that (1) is exactly covering.

(b) The class $2 \pmod{6}$ does not belong to (1). Then (1) does not cover the common part of $2 \pmod{6}$ and $2 \pmod{15}$, again a contradiction.

Before proving the main results we give some preliminaries. In what follows we shall suppose

$$n_1 \leq n_2 \leq \ldots \leq n_{k-1} < n_k$$

and that

$$n_1 = n_1^* < n_2^* < \ldots < n_s^* = n_h$$

are the all distinct moduli of (1). Moreover, we shall use the abbreviation

$$e(a/b) = \exp\left(\frac{2\pi ia}{b}\right).$$

The next lemma follows from Theorem 2 of [9] for $m$ times covering systems.

**Lemma 1.** The following statements are equivalent:

A. The system (1) is $m$ times covering.

B. \[ \sum_{n_i \neq n_j} n_i^{-1} \cdot e(a_i/n_i) = m \cdot \delta_{i,j} \] for every $s = 1, \ldots, n_j, j = 1, \ldots, k$,

where $\delta_{i,j}$ is the delta of Kronecker and $|i|$ the sign of divisibility.

C. \[ \sum_{i=1}^{k} n_i^{p-1} \cdot B_p(a_i/n_i) = m \cdot B_p, \quad p = 0, 1, 2, \ldots, \] where $B_p(x)$ is the $p$-th Bernoulli polynomial and $B_0 = B_0(0)$ the $0$-th Bernoulli number.

Part B, resp. C was proved in [5], resp. [2] for exactly covering systems. In [6] these results are extended to general systems of residue classes.

For instance we get

$$\sum_{i=1}^{k} \frac{1}{n_i} = m$$

from C for $p = 0$; this yields the well-known result for exactly covering systems, that is for $m = 1$ (see [1]).

The following lemma justifies our further results.

**Lemma 2.** Let (1) be an $m$ times covering system. Then (1) contains at least $p$ distinct residue classes modulo $n_h$ provided that $p$ is the least prime divisor of $n_k$.

**Proof.** We get

$$\sum_{j=1}^{k} e(a_j/n_k) = 0$$

from B for $j = k$ and $s = 1$. Then

$$\sum_{j=1}^{q} c_j \cdot e(a_j/n_k) = 0$$

grouping the equal terms of (3) where

$$a_{q_1} \pmod{n_k}, \ldots, a_{q_2} \pmod{n_k}$$

are the all distinct residue classes modulo $n_k$ of (1) and $c_j$ denotes the multiplicity of the appearance of class $a_j \pmod{n_k}$ in (1). Consequently, $c_j \neq 0$ for every $j = 1, \ldots, q$. It follows from Theorem 1 of [4] that $q \geq p$, the least prime divisor of $n_k$.

In case of exactly covering systems Lemma 2 yields the result conjectured by Znám in [9] and proved in [5] and independently by Newman in [4]. Znám's result presents a generalization of the well-known one due to Mirković, D. Newman, Davenport and Rado (see [1]) asserting that every exactly covering system contains at least two residue classes with respect to the greatest modulus. Further generalization of these results is given in [6].

The following lemma will be important for our next considerations.

**Lemma 3.** Let $a_i, b_i, m$ ($i = 1, \ldots, t$) be integers with $t \leq 5$ and $0 \leq b_i < b < \ldots < b_i < m$ and $0 < a_i$ for every $i = 1, \ldots, t$. Let the sum

$$\sum_{j=1}^{t} c_j \cdot e(b_j/m) = 0$$

have the following property: if

$$\sum_{j=1}^{t} d_j \cdot e(b_j/m) = 0 \quad \text{with} \quad c_j \geq d_j > 0$$

then either $d_j = c_j$ for every $j$ or $d_j = 0$ for every $j$. Then $t = 4$ is impossible; if $t = 2, 3, 5$ then $c_1 = c_2 = \ldots = c_t = 1$, $t$ divides $m$ and residue class $a_i \pmod{m}$ contains exactly those integers which belong to the system $b_1 \pmod{m}, \ldots, b_t \pmod{m}$.

**Proof.** Assigning the vectors to complex numbers $e(a_i/b_i)$, $e(b_i/m)$, $e(b_j/m)$, $e(D_j/m)$ in the usual way the relation (5) can be interpreted as a convex $t$-sided polygon. A polygon (5) with property (5') is called minimal (after Mann [3]). From Theorem 6 of [3] it follows that the only $t$-sided minimal polygons with $t \leq 5$ are regular polygons with $3$
or 5 sides. In case $i = 2$ it follows from Theorem 1 of [3] that the mentioned vectors are conversely oriented. Hence

\[
\frac{2\pi}{m} b_{j+1} = \frac{2\pi}{m} b_j + \frac{2\pi}{i} \quad \text{for} \quad j = 1, \ldots, i-1
\]

and

\[
\frac{2\pi}{m} b_i = \frac{2\pi}{m} b_1 + \frac{2\pi}{i}
\]

and our lemma immediately follows.

The following notation will be used to simplify the formulation of our next results. We say that system (1) has property $P(m, r)$ if:

(i) it is $m$ times covering,

(ii) it contains exactly one $r$-tuple of distinct residue classes with respect to the greatest modulus while the moduli of the remaining distinct classes are distinct.

Owing to Lemma 2, $r \geq 2$ may be assumed in (ii).

The next lemma collects to this time known characterization of the above stated exactly covering systems ([1], [8], [9]).

**Lemma 4.** Let (1) be a system having property $P(1, r)$ with $r = 2, 3, 4, \text{and } 5$. Then

(a) $n_i = 2^i$ for $i = 1, \ldots, k - r$ and $n_{k-r-1} = \ldots = n_r = r \cdot 2^{k-r}$ if $r = 2, 3, \text{and } 5$;

(b) $n_s = 2^i$, $i = 1, \ldots, k - 4$, $n_{k-3} = \ldots = n_r = 2^{k-s}$ or $n_s = 2^i$, $i = 1, \ldots, k - 5$, $n_{k-4} = 2 \cdot 2^{k-4}$ or $n_s = 2 \cdot 2^{k-4}$ in case $r = 4$.

**Theorem 1.** Every system of residue classes having property $P(m, 2)$ consists of $m$ copies of a system having property $P(1, 2)$.

**Proof.** Let (1) be a system having property $P(m, 2)$. Then (4) implies

\[
e_1 \cdot e(a_1/n_k) + e(a_2/n_k) = 0,
\]

where $a_1 \pmod{n_k}$ and $a_2 \pmod{n_k}$ are both distinct residue classes modulo $n_k$ in (1). For the decomposition of (6) into minimal polygons can consist only from "two-sided" minimal polygons, $c_1 = c_2 = 0$ and

\[
e(a_1/n_k) + e(a_2/n_k) = 0.
\]

Using Lemma 3 every pair of classes $a_1 \pmod{n_k}$, $a_2 \pmod{n_k}$ can be replaced by residue class $a_2 \pmod{n_k/2}$. Thus after the $\sigma$-fold repetition of this step we obtain a system having property $P(m, 2)$ but with $k - \sigma$ classes. The fact that this new system has property $P(m, 2)$ follows from Lemma 2 and 3; the former according every $m$ times covering system contains at least two distinct residue classes with respect to the greatest modulus and therefore $n_{k-1} = n_k/2$.

Now the proof follows by induction on $k$. Then this new system consists of $m$ identical systems having property $P(1, 2)$. Since each of the classes modulo $n_k/2$ is one of two classes with respect to the greatest modulus in these systems having property $P(1, 2)$, it follows $c = m$. Replacing the classes modulo $n_k/2$ by the original ones $a_1 \pmod{n_k}$, $a_2 \pmod{n_k}$ we get that system (1) also consists of $m$ copies of a system having property $P(1, 2)$.

It remains to prove that our theorem holds for systems having property $P(m, 2)$ with minimal possible $k$. In this case $n_k = 2$ (and hence all $n_s$ of (1) are equal to 2), because in the opposite case using our reduction method we get a system having property $P(m, 2)$ but with less classes which contradicts our hypotheses. This system with minimal $k$ is therefore evidently formed by $m$ complete residue systems modulo 2 and our proof is complete.

**Remark.** Sometimes is convenient to consider the set of integers as a residue class modulo 1. In this case we may continue the reduction of the systems above and we get $m$ copies of the set of all integers.

**Corollary 1.** Let (1) be a system having property $P(m, 2)$. Then

\[
1 + m = k \quad \text{and} \quad n_i^2 = 2^i \quad \text{for} \quad i = 1, 2, \ldots, m.
\]

Admitting no repetition of the residue classes in systems having property $P(m, 2)$ we get $v = k - 1$ and consequently $m = 1$ is the only possible value of $m$ in this case.

**Theorem 2.** Every system of residue classes having property $P(m, 3)$ consists of $m$ copies of a system having property $P(1, 3)$.

**Proof.** We get

\[
\sum_{f=1}^{3} c_f \cdot e(a_f/n_k) = 0
\]

from (4) for our system (1) having property $P(m, 3)$. Decompose polygon (7) into minimal polygons. These minimal polygons can be of the following two forms, either

\[
e(a_1/n_k) + e(a_2/n_k) = 0
\]

or

\[
e(a_1/n_k) + e(a_2/n_k) = 0
\]

with $1 \leq f < f \leq 3$. We now show that (8) cannot occur. If there occur two distinct polygons of the form (8) then we get

\[
e(a_i/n_k) = e(a_j/n_k) \quad \text{with} \quad i < j
\]

by their subtraction, which is impossible because $0 < a_i < a_j < n_k$. 
If there exist (8) and (8') together in the decomposition of (7) then by subtraction we get $e(a_u / n_b) = 0$ for some $u = 1, 2, 3$ which is again impossible. Since the all $e_i$'s in (7) are positive, the only possibility how to decompose (7) into minimal polygons is to use polygons of the form (8). But now the remaining part of this proof is similar to that of previous theorem and therefore it will be omitted.

**Corollary 2.** Let (1) be a system having property $P(m, 3)$. Then $(v+2)m = k$ and

$$n_i^* = 2^i \quad \text{for} \quad i = 1, 2, \ldots, v-1 \quad \text{and} \quad n_v^* = 3 \cdot 2^{v-1}.$$  

The next lemma is itself of some interest.

**Lemma 5.** No $m$ times covering system (1) satisfies the following three conditions:

(a) system (1) contains exactly two distinct residue classes with respect to the greatest modulus $n_b$;

(b) there is a modulus $n_i \neq n_b$ among the moduli of (1) with $n_i \neq 2^o n_b$ for every integer $o \geq 0$ and (1) contains at most two distinct residue classes modulo $n_i$;

(c) system (1) does not contain a couple of distinct residue classes with respect to the same modulus which is distinct from $n_b$ or $n_k$.

Proof. Suppose the contrary. Let $a_i \mod(n_b)$ and $a_k \mod(n_b)$ be two distinct classes modulo $n_b$ in (1). Following the first part of the proof of Theorem 1 we may replace every couple of classes $a_i \mod(n_b)$, $a_k \mod(n_b)$ by class $a_i \mod(n_b/2)$.

Let $r$ be an integer with

$$\frac{n_2}{2^r} > n_b > \frac{n_2}{2^{r+1}}.$$  

Since $n_b \neq 2^o n_2$ for every $o \geq 0$, the equality can be excluded. After the $(r+1)$-st reduction we get an $m$ times covering system with greatest modulus $n_b$. (By the way, we see that (1) must contain two distinct residue classes modulo $n_b$.) Note that $n_2 \neq 2^{r+1}$ because $n_b \geq 2$ and $n_2 < 2^r$. Thus we get an $m$ times covering system with less than $k$ classes.

Since $n_b < n_2$, also $n_2 \neq \frac{n_2}{2^{r+1}} \cdot 2^i$ for every integer $i \geq 0$; and our lemma can be proved by induction on $k$. If (1) would be a system satisfying our conditions with minimal possible $k$, then $n_b = 2$. If $n_b > 2$ then our reduction leads to a system with less classes which contradicts the above minimality. But $n_b = 2$ implies $n_b = 2$ which is impossible too.

**Theorem 3.** Every system having property $P(m, 4)$ consists of $m$ copies of a system having property $P(1, 2)$ and of $m - m_1$ copies of a system having property $P(1, 4)$ where $0 \leq m_1 < m$.

Proof. Let (1) be a system having property $P(m, 4)$. Let $a_i \mod(n_b)$, $i = 1, \ldots, 4$, be all distinct classes modulo $n_b$ and let $a_i \mod(n_b^{i-1})$ be the class modulo $n_b^{i-1}$ in (1). Furthermore, let (1) contain $m_1$ copies of class $a_i \mod(n_b^{i-1})$.

Again

$$\sum_{j=1}^4 c_j e(a_j / n_b) = 0.$$  

Reasoning as before we see that a decomposition of (9) into minimal polygons cannot contain two twosided polygons having a term in common. Now distinguish the following cases:

(a) The decomposition of (9) contains also triangles. If two distinct triangles appear in this decomposition then these must have two terms in common which yields the equality $e(a_j / n_b) = e(a_j / n_b)$ for some $i \neq j$. But this is impossible for $a_i \neq a_j \mod(n_b)$. Similarly, a triangle and two distinct twosided polygons yield the contradictory equality $e(a_i / n_b) = 0$ for some $i = 1, 2, 3, 4$ because in this case the triangle must contain one of these 2-gons. Hence the only possibility how to decompose (9) into minimal polygons using a triangle is to use the triangle (after a proper renumbering if necessary)

$$e(a_i / n_b) + e(a_j / n_b) + e(a_k / n_b) = 0,$$

together with

$$e(a_i / n_b) + e(a_k / n_b) = 0$$

for some $j = 1, 2, 3, 4$. After the reduction we get a system possessing only the $m$ times covering property (but not necessarily property (ii)) in which $n_b^{i-1} = n_b/2$ (Lemma 2) and one of the moduli of which is $n_b/3$. Lemma 5 shows that this is impossible.

(b) The only minimal polygons appearing in the decomposition of (9) are twosided polygons having no term in common, e.g.

$$e(a_i / n_b) + e(a_j / n_b) = e(a_i / n_b) + e(a_k / n_b) = 0.$$  

Now we may replace the classes modulo $n_b$ of (1) by say, $m_1$ classes $a_i \mod(n_b/2)$ and $m_2$ classes $a_k \mod(n_b/2)$. The classes $a_i \mod(n_b/2)$ and $a_k \mod(n_b/2)$ are distinct. Suppose $0 < a_1 < a_2 < n_b$. If these two classes coincide then $a_2 = a_1 + n_b/2$. On the other hand, Lemma 3 implies $a_2 = a_1 + n_b/2$ or $a_2 = a_1 + n_b/2$, that is $a_2 = a_1$ or $a_2 = a_1 + n_b$ which is a contradiction, because $a_i \mod(n_b/2)$ are distinct classes.

(ba) Let $n_b^{i-1} = n_b/2$. Distinguish:

(baa) If $a_i = a_k \mod(n_b/2)$ then we get a system having property...
P(m, 2). By Theorem 1, this new system consists of m copies of a once covering system, and hence \( m_1 + m_2 = m_2 = m \). Moreover, in these once covering systems the classes \( a_{i_1} \mod n_{i_2} \) and \( a_{i_2} \mod n_{i_3} \) are the only distinct residue classes with respect to the greatest modulus. If now, in \( m_1 \) copies of the mentioned once covering system we replace only class \( a_{i_1} \mod n_{i_2} \) by classes \( a_{i_1} \mod n_{i_3} \) and \( a_{i_2} \mod n_{i_4} \), and if we similarly split both classes \( a_{i_1} \mod n_{i_2} \) and \( a_{i_2} \mod n_{i_3} \) in the remaining \( m_2 \) copies of the above stated once covering system, then we get the original system having property \( P(m, 4) \).

(bab) If \( a_{i_1} \neq a_{i_2} \mod n_{i_3} \) for \( i = 1, 3 \) then we get a system having property \( P(m, 3) \) which all distinct residue classes with respect to the greatest modulus are \( a_{i_1} \mod n_{i_3} \), \( a_{i_2} \mod n_{i_4} \), \( a_{i_3} \mod n_{i_5} \). Thus \( m_1 = m_2 = m = m \). By arguments similar to the above ones we see that the given system (1) consists of \( m \) copies of a system having property \( P(1, 4) \).

(bb) \( n_{i_3} = n_{i_4} = 2 \). Evidently \( n_{i_3} = 2 \). Now we get a system having property \( P(m, 2) \) in which the residue classes with respect to the greatest modulus are \( a_{i_1} \mod n_{i_3} \), \( a_{i_2} \mod n_{i_4} \). In this case the given system (1) consists of \( m \) copies of a system having property \( P(1, 4) \).

Corollary 3. The modulus of each system having property \( P(m, 4) \) satisfy one of the following three relations:

(a) \( n_{i_3} = 2^i \) for \( i = 1, 2, \ldots, v \); 
(b) \( n_{i_3} = 2^i \) for \( i = 1, \ldots, v - 1 \), \( n_{v-1} = 3 \cdot 2^{v-2} \), \( n_{v} = 3 \cdot 2^{v-1} \); 
(c) \( n_{i_3} = 2^i \) for \( i = 1, \ldots, v - 1 \), \( n_{v} = 2^{v+1} \).

Theorem 4. Let (1) be a system having property \( P(m, 5) \). Then either (1) consists of \( m \) copies of a system having property \( P(1, 4) \) and of \( m \) copies of a system having property \( P(1, 3) \), where \( m_1, m_2 \) are positive integers with \( m_1 + m_2 = m \), or (1) consists of \( m \) copies of a system having property \( P(1, 5) \).

Proof. Let \( a_{i_1} \mod n_{i_2} \), \( i = 1, \ldots, 5 \) be all distinct residue classes with respect to the greatest modulus. Let \( a_{i_3} \mod n_{i_4} \) be classes modulo \( n_{i_4} \) for \( i = 1, 2, \) and 3. Moreover, let \( m_i \) denote the number of classes modulo \( n_{i_4} \), \( i = 1, 2 \), in (1). Then we have

\[
\sum_{i=1}^{5} c_i \cdot e(a_{i_3} \mod n_{i_4}) = 0.
\]

First we may exclude all decompositions of (10) into minimal polygons containing three distinct two-sided polygons. In the opposite case we should have two 2-gons having one term in common which is impossible.

Assume a decomposition of (10) containing two distinct triangles.

Since they cannot have two terms in common, they have exactly one common term, e.g.

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) + e(a_{i_5} \mod n_{i_6}) = e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) + e(a_{i_5} \mod n_{i_6}) = 0.
\]

Then

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) + e(a_{i_5} \mod n_{i_6}) = 0
\]

or

\[
e(2a_{i_1} \mod n_{i_2}) + e(2a_{i_3} \mod n_{i_4}) + e(2a_{i_5} \mod n_{i_6}) + e(2a_{i_7} \mod n_{i_8}) + e(2a_{i_9} \mod n_{i_10}) + e(2a_{i_11} \mod n_{i_12}) = 0.
\]

According to Lemma 3 we can decompose this tetragon by means of twosided polygons only. The use of

\[
e(2a_{i_1} \mod n_{i_2}) + e(2a_{i_3} \mod n_{i_4}) = 0
\]

implies \( e(a_{i_5} \mod n_{i_6}) = 0 \) which is impossible. Let

\[
e(2a_{i_1} \mod n_{i_2}) + e(2a_{i_3} \mod n_{i_4}) = 0.
\]

Then Lemma 3 gives:

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) = e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) = 0
\]

can be ruled out after the reductions using Lemma 5.

Now it can be shown that the only possibility for using a triangle in a decomposition of (10) is of the following type:

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) + e(a_{i_5} \mod n_{i_6}) = 0,
\]

(11)

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) + e(a_{i_5} \mod n_{i_6}) = 0,
\]

\[
e(a_{i_1} \mod n_{i_2}) + e(a_{i_3} \mod n_{i_4}) = 0.
\]

Let this decomposition contain \( m \) such triangles, \( m \) and \( m \) twosided polygons of the first and second form respectively. After reduction we get \( m \) classes \( a_{i_1} \mod n_{i_2} \), \( m \) classes \( a_{i_3} \mod n_{i_4} \) and \( m \) classes \( a_{i_5} \mod n_{i_6} \). The classes \( a_{i_1} \mod n_{i_2} \) and \( a_{i_3} \mod n_{i_4} \) are again distinct.

The case \( n_{v-1} = n_{v-2} \) (i.e. \( n_{v-1} < n_{v-2} \)) is impossible according to Lemma 5.
(a) \( w_{i,j}^n = n_i/2 \). The case \( a_{i,j} = a_{i,j}^+(\text{mod } n_i/2) \) may be similarly verified to be impossible. Hence \( a_{i,j} \not\equiv a_{i,j}^+(\text{mod } n_i/2) \), \( i = 1, 2 \). Now we get an \( m \) times covering system having three distinct classes with respect to the greatest modulus, namely \( a_{i,j}^+(\text{mod } n_i/2) \), \( a_{i,j}^+(\text{mod } n_i/2) \) and \( a_{i,j}^+(\text{mod } n_i/2) \); thus \( m_1 = n_i = m_1 \). This new system contains also \( m_2 \) classes \( a_{i,j}^+(\text{mod } n_i/3) \). Since 2 and 3 divide \( n_1, n_2, n_3 > n_2/3 > n_2/6 \geq 1 \), and we may continue our reduction. The \( m_1 \)-fold using of Lemma 3 yields \( m_1 \) classes \( a_{i,j}^+(\text{mod } n_i/6) \). Now \( n_1/3 > n_1/6 \), and therefore \( a_{i,j}^+(\text{mod } n_i/3) \) and \( a_{i,j}^+(\text{mod } n_i/6) \) are all distinct residue classes with respect to the greatest modulus \( n_i = n_i/6 \), and also \( m_2 = m_1 \). These classes may replace by \( m_2 \) classes \( a_{i,j}^+(\text{mod } n_i/6) \). Thus we get \( m_1 + m_2 \) classes \( a_{i,j}^+(\text{mod } n_i/6) \).

(b) If \( n_2/6 = 1 \) then we get \( m \) copies of the set of all integers. The reverse procedure shows that the initial system (1) consists of \( m_1 \) copies of a system having property \( P(1, 4) \) (by the way, with classes either modulo 3 or modulo 6) and \( m_2 \) copies of a system having property \( P(1, 3) \) (with moduli 2 and 6) where \( m_1 + m_2 = m \).

(a) The case \( n_2 = n_1/6 \) can be excluded, because then we should have an \( m \) times covering system with one "distinct class" with respect to the greatest modulus which contradicts Lemma 2.

(b) \( n_2 = n_1/6 \). Reasoning as before we get \( a_{i,j} = a_{i,j}^+(\text{mod } n_i/6) \). Now we have a system having property \( P(m, 2) \) which consists of \( m \) copies of a system having property \( P(1, 2) \). By arguments as in the previous paragraph we get that (1) consists of \( m_1 \) copies of a system having property \( P(1, 4) \) and \( m_2 \) copies of a system having property \( P(1, 3) \) with \( m_1 + m_2 = m \).

The last possibility how to decompose (10) into minimal polygons consists in using pentagons. A pentagon together with a \( p \)-sided polygon with \( p < 5 \) leads to a contradiction with the definition of minimal polygons. Following the proof of Theorem 1 we may show that the system (1) having property \( P(m, 5) \) consists of \( m \) copies of a system having property \( P(1, 5) \).

**Corollary 4.** Let (1) be a system having property \( P(m, 5) \). Then one of the following alternatives holds:

(a) \( w_i^* = 2^i \) for \( i = 1, \ldots, v-1 \) and \( w_v^* = 5 \cdot 2^v-1 \),

(b) \( w_i^* = 2^i \) for \( i = 1, \ldots, v-2 \), \( w_{v-1}^* = 3 \cdot 3^{v-3} \), \( w_v^* = 3 \cdot 2^{v-2} \).

Our theorems suggest the conjecture that every system having property \( P(m, r) \) consists of exactly covering systems, but this is not true in general as it is shown by example due to Choi.

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References


