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On m times covering systems of congruences

by

ŠTEFAN PORUBSKÝ (Bratislava)

A system of residue classes

$$a_i(\operatorname{mod} n_i), \quad 0 \leqslant a_i < n_i,\ 1 < n_i,\ i = 1, 2, \ldots, k \tag{1}$$

is called to be *m times covering* if every integer belongs exactly to m of these classes, where m is a given positive integer.

Once covering system is usually called *exactly covering*. This topic is a subject of consideration of many papers, e.g. [1]–[9] except for [3]. It is evident that in any exactly covering system no residue class appears more than once, but such a restriction does not show to be useful for m times covering systems in general and therefore the repetition of residue classes will be allowed in (1).

In this paper we shall characterize the m times covering systems containing exactly one r-tuple of distinct residue classes with respect to the same modulus while the moduli of the remaining distinct classes are distinct for $r = 2, 3, 4$, and 5. These results extend those to be known for exactly covering systems and proved for $r = 2$ in [8], $r = 3$ in [9], $r = 4, 5$ in [7].

It is easy to check that m exactly covering systems form together an m times covering system. But on the other hand, there exist m times covering systems which are not of this form. The following one was constructed by Choi during the International Colloqium on Infinite and Finite Sets held in Keszthely (Hungary) in 1973.

The system

$$1(\operatorname{mod} 2), 0(\operatorname{mod} 3), 2(\operatorname{mod} 6), 0(\operatorname{mod} 10), 2(\operatorname{mod} 15);\ 1, 4, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 16, 18, 19, 22, 22, 23, 24, 25, 26, 28, 28, 29\ (\operatorname{mod} 30) \tag{2}$$

is twice covering. Suppose (2) to be a union of two exactly covering systems and let (1) be that of them which contains the class $1(\operatorname{mod} 2)$. Then (1) contains neither $0(\operatorname{mod} 3)$ nor $2(\operatorname{mod} 15)$ because of their non-empty intersection with class $1(\operatorname{mod} 2)$. Now the following two possibilities can occur:

(a) The residue class $2 \pmod 6$ belongs to (1). In this case $0 \pmod{10}$ cannot belong to (1), but then (1) does not cover the intersection of classes $0 \pmod{10}$ and $0 \pmod 3$ which contradicts the assumption that (1) is exactly covering.

(b) The class $2 \pmod 6$ does not belong to (1). Then (1) does not cover the common part of $2 \pmod 6$ and $2 \pmod{15}$, again a contradiction.

Before proving the main results we give some preliminaries. In what follows we shall suppose

$$n_1 \leqslant n_2 \leqslant \ldots \leqslant n_{k-1} \leqslant n_k$$

and that

$$n_1 = n_1^* < n_2^* < \ldots < n_v^* = n_k$$

are the all distinct moduli of (1). Moreover, we shall use the abbreviation

$$e(a/b) = \exp\left(\frac{2\pi i a}{b}\right).$$

The next lemma follows from Theorem 2 of [6] for m times covering systems.

Lemma 1. *The following statements are equivalent*:

A. *The system* (1) *is* m *times covering*.

B. $\sum_{\substack{i=1\\ n_j|sn_i}}^{k} n_i^{-1}\cdot e(sa_i/n_j) = m\cdot\delta_{s,n_j}$ *for every* $s = 1, \ldots, n_j$, $j = 1, \ldots, k$, *where* $\delta_{i,j}$ *is the delta of Kronecker and* $|$ *the sign of divisibility*.

C. $\sum_{i=1}^{k} n_i^{p-1}\cdot B_p(a_i/n_i) = m\cdot B_p$, $p = 0, 1, 2, \ldots$, *where* $B_p(x)$ *is the* p-*th Bernoulli polynomial and* $B_p = B_p(0)$ *the* p-*th Bernoulli number*.

Part B, resp. C was proved in [5], resp. [2] for exactly covering systems. In [6] these results are extended to general systems of residue classes.

For instance we get

$$\sum_{i=1}^{k}\frac{1}{n_i} = m$$

from C for $p = 0$; this yields the well-known result for exactly covering systems, that is for $m = 1$ (see [1]).

The following lemma justifies our further results.

Lemma 2. *Let* (1) *be an* m *times covering system. Then* (1) *contains at least* p *distinct residue classes modulo* n_k *provided that* p *is the least prime divisor of* n_k.

Proof. We get

$$\sum_{n_i = n_k} e(a_i/n_k) = 0 \tag{3}$$

from B for $j = k$ and $s = 1$. Then

$$\sum_{j=1}^{q} c_j\cdot e(a_{t_j}/n_k) = 0 \tag{4}$$

grouping the equal terms of (3) where

$$a_{t_1} \pmod{n_k}, \ \ldots, \ a_{t_q} \pmod{n_k}$$

are the all distinct residue classes modulo n_k of (1) and c_j denotes the multiplicity of the appearence of class $a_{t_j} \pmod{n_k}$ in (1). Consequently, $c_j \neq 0$ for every $j = 1, \ldots, q$. It follows from Theorem 1 of [4] that $q \geqslant p$, the least prime divisor of n_k.

In case of exactly covering systems Lemma 2 yields the result conjectured by Znám in [9] and proved in [5] and independently by Newman in [4]. Znám's result presents a generalization of the well-known one due to Mirsky, D. Newman, Davenport and Rado (see [1]) asserting that every exactly covering system contains at least two residue classes with respect to the greatest modulus. Further generalization of these results is given in [6].

The following lemma will be important for our next considerations.

Lemma 3. *Let* c_i, b_i, m $(i = 1, \ldots, t)$ *be integers with* $t \leqslant 5$ *and* $0 \leqslant b_1 < b_2 < \ldots < b_t < m$ *and* $0 < c_i$ *for every* $i = 1, \ldots, t$. *Let the sum*

$$\sum_{j=1}^{t} c_j\cdot e(b_j/m) = 0 \tag{5}$$

have the following property: *if*

$$\sum_{j=1}^{t} d_j\cdot e(b_j/m) = 0 \quad \textit{with} \quad c_j \geqslant d_j \geqslant 0 \tag{5'}$$

then either $d_j = c_j$ *for every* j *or* $d_j = 0$ *for every* j. *Then* $t = 4$ *is impossible*; *if* $t = 2, 3, 5$ *then* $c_1 = c_2 = \ldots = c_t = 1$, t *divides* m *and residue class* $b_1 \pmod{m/t}$ *contains exactly those integers which belong to the system*

$$b_1 \pmod m, \ \ldots, \ b_t \pmod m.$$

Proof. Assigning the vectors to complex numbers $c_1\cdot e(b_1/m)$, $c_2\cdot e(b_2/m), \ldots, c_t\cdot e(b_t/m)$ in the usual way the relation (5) can be interpreted as a convex t-sided polygon. A polygon (5) with property (5′) is called minimal (after Mann [3]). From Theorem 6 of [3] it follows that the only t-sided minimal polygons with $t \leqslant 5$ are regular polygons with 3 or 5 sides. In case $t = 2$ it follows from Theorem 1 of [3] that the mentioned vectors are conversely oriented. Hence

$$\frac{2\pi}{m} b_{j+1} = \frac{2\pi}{m} b_j + \frac{2\pi}{t} \quad \text{for} \quad j = 1, \ldots, t-1$$

and

$$\frac{2\pi}{m} b_1 = \frac{2\pi}{m} b_t + \frac{2\pi}{t}$$

and our lemma immediately follows.

The following notation will be used to simplify the formulation of our next results. We say that system (1) has property $P(m, r)$ if:

(i) it is m times covering,

(ii) it contains exactly one r-tuple of distinct residue classes with respect to the greatest modulus while the moduli of the remaining distinct classes are distinct.

Owing to Lemma 2, $r \geqslant 2$ may be assumed in (ii).

The next lemma collects to this time known characterization of the above stated exactly covering systems ([7], [8], [9]).

LEMMA 4. *Let* (1) *be a system having property* $P(1, r)$ *with* $r = 2, 3, 4,$ *and* 5. *Then*

(a) $n_i = 2^i$ *for* $i = 1, \ldots, k-r$ *and* $n_{k-r+1} = \ldots = n_k = r \cdot 2^{k-r}$ *if* $r = 2, 3,$ *and* 5;

(b) $n = 2^i$, $i = 1, \ldots, k-4$; $n_{k-3} = \ldots = n_k = 2^{k-2}$ *or* $n_i = 2^i$, $i = 1, \ldots, k-5$; $n_{k-4} = 3 \cdot 2^{k-5}$, $n_{k-3} = \ldots = n_k = 3 \cdot 2^{k-4}$ *in case* $r = 4$.

THEOREM 1. *Every system of residue classes having property* $P(m, 2)$ *consists of* m *copies of a system having property* $P(1, 2)$.

Proof. Let (1) be a system having property $P(m, 2)$. Then (4) implies

$$c_1 \cdot e(a_{t_1}/n_k) + c_2 \cdot e(a_{t_2}/n_k) = 0, \tag{6}$$

where $a_{t_1} (\operatorname{mod} n_k)$ and $a_{t_2} (\operatorname{mod} n_k)$ are both distinct residue classes modulo n_k in (1). For the decomposition of (6) into minimal polygons can consist only from "twosided" minimal polygons, $c_1 = c_2$ $(= c)$ and

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) = 0.$$

Using Lemma 3 every pair of classes $a_{t_1} (\operatorname{mod} n_k)$, $a_{t_2} (\operatorname{mod} n_k)$ can be replaced by residue class $a_{t_1} (\operatorname{mod} n_k/2)$. Thus after the c-fold repetition of this step we obtain a system having property $P(m, 2)$ but with $k-c$ classes. The fact that this new system has property $P(m, 2)$ follows from Lemmas 2 and 3; the former according every m times covering system contains at least two distinct residue classes with respect to the greatest modulus and therefore $n_{v-1}^* = n_k/2$.

Now the proof follows by induction on k. Then this new system consists of m identical systems having property $P(1, 2)$. Since each of the classes modulo $n_k/2$ is one of two classes with respect to the greatest modulus in these systems having property $P(1, 2)$, it follows $c = m$. Replacing the classes modulo $n_k/2$ by the original ones $a_{t_1} (\operatorname{mod} n_k)$, $a_{t_2} (\operatorname{mod} n_k)$ we get that system (1) also consists of m copies of a system having property $P(1, 2)$.

It remains to prove that our theorem holds for systems having property $P(m, 2)$ with minimal possible k. In this case $n_k = 2$ (and hence the all n's of (1) are equal to 2), because in the opposite case using our reduction method we get a system having property $P(m, 2)$ but with less classes which contradicts our hypotheses. This system with minimal k is therefore evidently formed by m complete residue systems modulo 2 and our proof is complete.

Remark. Sometimes is convenient to consider the set of integers as a residue class modulo 1. In this case we may continue the reduction of the systems above and we get m copies of the set of all integers.

COROLLARY 1. *Let* (1) *be a system having property* $P(m, 2)$. *Then* $(v+1)m = k$ *and*

$$n_i^* = 2^i \quad \text{for} \quad i = 1, 2, \ldots, v.$$

Admitting no repetition of the residue classes in systems having property $P(m, 2)$ we get $v = k-1$ and consequently $m = 1$ is the only possible value of m in this case.

THEOREM 2. *Every system of residue classes having property* $P(m, 3)$ *consists of* m *copies of a system having property* $P(1, 3)$.

Proof. We get

$$\sum_{j=1}^{3} c_j \cdot e(a_{t_j}/n_k) = 0 \tag{7}$$

from (4) for our system (1) having property $P(m, 3)$. Decompose polygon (7) into minimal polygons. These minimal polygons can be of one of the following two forms, either

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) + e(a_{t_3}/n_k) = 0 \tag{8}$$

or

$$e(a_{t_j}/n_k) + e(a_{t_f}/n_k) = 0 \tag{8'}$$

with $1 \leqslant j < f \leqslant 3$. We now show that (8′) cannot occur. If there occur two distinct polygons of the form (8′) then we get

$$e(a_{t_i}/n_k) = e(a_{t_s}/n_k) \quad \text{with} \quad i < s$$

by their subtraction, which is impossible because $0 < a_{t_i} < a_{t_s} < n_k$.

If there exist (8) and (8′) together in the decomposition of (7) then by subtraction we get $e(a_{t_u}/n_k) = 0$ for some $u = 1, 2$, or 3 which is again impossible. Since the all c's in (7) are positive, the only possibility how to decompose (7) into minimal polygons is to use polygons of the form (8). But now the remaining part of this proof is similar to that of previous theorem and therefore it will be omitted.

COROLLARY 2. *Let* (1) *be a system having property* $\mathrm{P}(m, 3)$. *Then* $(v+2)m = k$ *and*

$$n_i^* = 2^i \quad \textit{for} \quad i = 1, 2, \ldots, v-1 \quad \textit{and} \quad n_v^* = 3\cdot 2^{v-1}.$$

The next lemma is itself of some interest.

LEMMA 5. *No m times covering system* (1) *satisfies the following three conditions:*

(a) *system* (1) *contains exactly two distinct residue classes with respect to the greatest modulus* n_k;

(b) *there is a modulus $n_s \neq n_k$ among the moduli of* (1) *with $n_k \neq 2^w\cdot n_s$ for every integer $w \geqslant 0$ and* (1) *contains at most two distinct residue classes modulo* n_s;

(c) *system* (1) *does not contain a couple of distinct residue classes with respect to the same modulus which is distinct from n_k or n_s.*

Proof. Suppose the contrary. Let $a_{t_1} (\mathrm{mod}\, n_k)$ and $a_{t_2} (\mathrm{mod}\, n_k)$ be two distinct classes modulo n_k in (1). Following the first part of the proof of Theorem 1 we may replace every couple of classes $a_{t_1} (\mathrm{mod}\, n_k)$, $a_{t_2} (\mathrm{mod}\, n_k)$ by class $a_{t_1} (\mathrm{mod}\, n_k/2)$.

Let r be an integer with

$$\frac{n_k}{2^r} > n_s \geqslant \frac{n_k}{2^{r+1}}.$$

Since $n_k \neq 2^w\cdot n_s$ for every $w \geqslant 0$, the equality can be excluded. After the $(r+1)$-st reduction we get an m times covering system with greatest modulus n_s. (By the way, we see that (1) must contain two distinct residue classes modulo n_s.) Note that $n_k \neq 2^{r+1}$ because $n_s \geqslant 2$ and $n_k/2^r = 2$. Thus we get an m times covering system with less than k classes. Since $n_s < n_k$, also $n_s \neq \dfrac{n_k}{2^{r+1}}\cdot 2^t$ for every integer $t \geqslant 0$; and our lemma can be proved by induction on k. If (1) would be a system satisfying our conditions with minimal possible k, then $n_k = 2$. If $n_k > 2$ then our reduction leads to a system with less classes which contradicts the above minimality. But $n_k = 2$ implies $n_s = 2$ which is impossible too.

THEOREM 3. *Every system having property* $\mathrm{P}(m, 4)$ *consists of* m_1

copies of a system having property $\mathrm{P}(1, 2)$ *and of* $m - m_1$ *copies of a system having property* $\mathrm{P}(1, 4)$ *where* $0 \leqslant m_1 < m$.

Proof. Let (1) be a system having property $\mathrm{P}(m, 4)$. Let $a_{t_i} (\mathrm{mod}\, n_k)$, $i = 1, \ldots, 4$, be all distinct classes modulo n_k and let $a_\lambda (\mathrm{mod}\, n_{v-1}^*)$ be the class modulo n_{v-1}^* in (1). Furthermore, let (1) contain m_1 copies of class $a_\lambda (\mathrm{mod}\, n_{v-1}^*)$.

Again

$$\sum_{j=1}^{4} c_j\cdot e(a_{t_j}/n_k) = 0. \tag{9}$$

Reasoning as before we see that a decomposition of (9) into minimal polygons cannot contain two twosided polygons having a term in common. Now distinguish the following cases:

(a) The decomposition of (9) contains also triangles. If two distinct triangles appear in this decomposition then these must have two terms in common which yields the equality $e(a_{t_i}/n_k) = e(a_{t_j}/n_k)$ for some $i \neq j$. But this is impossible for $a_{t_i} \not\equiv a_{t_j} (\mathrm{mod}\, n_k)$. Similarly, a triangle and two distinct twosided polygons yield the contradictory equality $e(a_{t_i}/n_k) = 0$ for some $i = 1, 2, 3, 4$, because in this case the triangle must contain one of these 2-gons. Hence the only possibility how to decompose (9) into minimal polygons using a triangle is to use the triangle (after a proper renumbering if necessary)

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) + e(a_{t_3}/n_k) = 0$$

together with

$$e(a_{t_j}/n_k) + e(a_{t_4}/n_k) = 0$$

for some $j = 1, 2$, or 3. After the reduction we get a system possessing only the m times covering property (but not necessarily property (ii)) in which $n_{v-1}^* = n_k/2$ (Lemma 2) and one of the moduli of which is $n_k/3$. Lemma 5 shows that this is impossible.

(b) The only minimal polygons appearing in the decomposition of (9) are twosided polygons having no term in common, e.g.

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) = e(a_{t_3}/n_k) + e(a_{t_4}/n_k) = 0.$$

Now we may replace the classes modulo n_k of (1) by say, m_2 classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and m_3 classes $a_{t_3} (\mathrm{mod}\, n_k/2)$. The classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and $a_{t_3} (\mathrm{mod}\, n_k/2)$ are distinct. Suppose $0 \leqslant a_{t_1} < a_{t_3} < n_k$. If these two classes coincide then $a_{t_3} = a_{t_1} + n_k/2$. On the other hand, Lemma 3 implies $a_{t_2} = a_{t_1} + n_k/2$ or $a_{t_1} = a_{t_2} + n_k/2$, that is $a_{t_2} = a_{t_3}$ or $a_{t_3} = a_{t_2} + n_k$ which is a contradiction, because $a_{t_i} (\mathrm{mod}\, n_k)$ are distinct classes.

(ba) Let $n_{v-1}^* = n_k/2$. Distinguish:

(baa) If $a_\lambda \equiv a_{t_1} (\mathrm{mod}\, n_k/2)$ then we get a system having property

$P(m, 2)$. By Theorem 1, this new system consists of m copies of a once covering system, and hence $m_1 + m_2 = m_3 = m$. Moreover, in these once covering systems the classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and $a_{t_3} (\mathrm{mod}\, n_k/2)$ are the only distinct residue classes with respect to the greatest modulus. If now, in m_1 copies of the mentioned once covering system we replace only class $a_{t_3} (\mathrm{mod}\, n_k/2)$ by classes $a_{t_3} (\mathrm{mod}\, n_k)$ and $a_{t_4} (\mathrm{mod}\, n_k)$, and if we similarly split both classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and $a_{t_3} (\mathrm{mod}\, n_k/2)$ in the remaining m_2 copies of the above stated once covering system, then we get the original system having property $P(m, 4)$.

(bab) If $a_\lambda \not\equiv a_{t_i} (\mathrm{mod}\, n_k/2)$ for $i = 1, 3$ then we get a system having property $P(m, 3)$ which all distinct residue classes with respect to the greatest modulus are $a_\lambda (\mathrm{mod}\, n^*_{v-1})$, $a_{t_1} (\mathrm{mod}\, n_k/2)$, $a_{t_3} (\mathrm{mod}\, n_k/2)$. Thus $m_1 = m_2 = m_3 = m$. By arguments similar to the above ones we see that the given system (1) consists of m copies of a system having property $P(1, 4)$.

(bb) $n^*_{v-1} \neq n_k/2$. Evidently $n_k/2 > n^*_{v-1}$. Now we get a system having property $P(m, 2)$ in which the residue classes with respect to the greatest modulus are $a_{t_1} (\mathrm{mod}\, n_k/2)$, $a_{t_3} (\mathrm{mod}\, n_k/2)$. In this case the given system (1) consists of m copies of a system having property $P(1, 4)$.

COROLLARY 3. *The moduli of each system having property* $P(m, 4)$ *satisfy one of the following three relations*:

(a) $$n^*_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, v;$$

(b) $$n^*_i = 2^i \quad \text{for} \quad i = 1, \ldots, v-2; \quad n^*_{v-1} = 3 \cdot 2^{v-2}, \quad n^*_v = 3 \cdot 2^{v-1};$$

(c) $$n^*_i = 2^i \quad \text{for} \quad i = 1, \ldots, v-1; \quad n^*_v = 2^{v+1}.$$

THEOREM 4. *Let* (1) *be a system having property* $P(m, 5)$. *Then either* (1) *consists of* m_1 *copies of a system having property* $P(1, 4)$ *and of* m_2 *copies of a system having property* $P(1, 3)$, *where* m_1, m_2 *are positive integers with* $m_1 + m_2 = m$, *or* (1) *consists of* m *copies of a system having property* $P(1, 5)$.

Proof. Let $a_{t_i} (\mathrm{mod}\, n_k)$, $i = 1, \ldots, 5$ be all distinct residue classes with respect to the greatest modulus. Let $a_{\lambda_i} (\mathrm{mod}\, n^*_{v-i})$ be classes modulo n^*_{v-i} for $i = 1, 2$, and 3. Moreover, let m_i denotes the number of classes modulo n^*_{v-i}, $i = 1, 2$, in (1). Then we have

$$\sum_{j=1}^{5} c_j \cdot e(a_{t_j}/n_k) = 0. \tag{10}$$

First we may exclude all decompositions of (10) into minimal polygons containing three distinct twosided polygons. In the opposite case we should have two 2-gons having one term in common which is impossible.

Assume a decomposition of (10) containing two distinct triangles.

Since they cannot have two terms in common, they have exactly one common term, e.g.

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) + e(a_{t_3}/n_k) = e(a_{t_3}/n_k) + e(a_{t_4}/n_k) + e(a_{t_5}/n_k) = 0.$$

Then

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) - e(a_{t_4}/n_k) - e(a_{t_5}/n_k) = 0$$

or

$$e(2a_{t_1}/2n_k) + e(2a_{t_2}/2n_k) + e\big((2a_{t_4} + n_k)/2n_k\big) + e\big((2a_{t_5} + n_k)/2n_k\big) = 0.$$

According to Lemma 3 we can decompose this tetragon by means of twosided polygons only. The use of

$$e(2a_{t_1}/2n_k) + e(2a_{t_2}/2n_k) = 0$$

implies $e(a_{t_3}/n_k) = 0$ which is impossible. Let

$$e(2a_{t_1}/2n_k) + e\big((2a_{t_4} + n_k)/2n_k\big) = 0.$$

Then Lemma 3 gives: $2a_{t_1} = 2a_{t_4} + n_k + 2n_k/2$ if $2a_{t_1} > 2a_{t_4} + n_k$ or $2a_{t_4} + n_k = 2a_{t_1} + 2n_k/2$ if $2a_{t_1} < 2a_{t_4} + n_k$, i.e. $a_{t_1} = a_{t_4} + n_k$ or $a_{t_1} = a_{t_4}$ but this is impossible for $a_{t_1} (\mathrm{mod}\, n_k)$ and $a_{t_4} (\mathrm{mod}\, n_k)$ are different classes. Therefore a decomposition of (10) into minimal polygons can contain at most one triangle.

A decomposition of the form

$$e(a_{t_1}/n_k) + e(a_{t_2}/n_k) + e(a_{t_3}/n_k) = e(a_{t_4}/n_k) + e(a_{t_5}/n_k) = 0$$

can be ruled out after the reductions using Lemma 5.

Now it can be shown that the only possibility for using a triangle in a decomposition of (10) is of the following type:

$$\begin{aligned} e(a_{t_1}/n_k) + e(a_{t_2}/n_k) + e(a_{t_3}/n_k) &= 0, \\ e(a_{t_1}/n_k) + e(a_{t_4}/n_k) &= 0, \\ e(a_{t_2}/n_k) + e(a_{t_5}/n_k) &= 0. \end{aligned} \tag{11}$$

Let this decomposition contain m_5 such triangles, m_3 and m_4 twosided polygons of the first and second form respectively. After reduction we get m_5 classes $a_{t_1} (\mathrm{mod}\, n_k/3)$, m_3 classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and m_4 classes $a_{t_2} (\mathrm{mod}\, n_k/2)$. The classes $a_{t_1} (\mathrm{mod}\, n_k/2)$ and $a_{t_2} (\mathrm{mod}\, n_k/2)$ are again distinct.

The case $n^*_{v-1} \neq n_k/2$ (i.e. $n^*_{v-1} < n_k/2$) is impossible according to Lemma 5.

(a) $n_{v-1}^* = n_k/2$. The case $a_{\lambda_1} \equiv a_{t_1} (\mathrm{mod}\, n_k/2)$ may be similarly verified to be impossible. Hence $a_{\lambda_1} \not\equiv a_{t_i} (\mathrm{mod}\, n_k/2)$, $i = 1, 2$. Now we get an m times covering system having three distinct classes with respect to the greatest modulus, namely $a_{t_1}(\mathrm{mod}\, n_k/2)$, $a_{t_2}(\mathrm{mod}\, n_k/2)$ and $a_{\lambda_1}(\mathrm{mod}\, n_{v-1}^*)$; thus $m_3 = m_4 = m_1$. This new system contains also m_5 classes $a_{t_1}(\mathrm{mod}\, n_k/3)$. Since 2, and 3 divide n_k, $n_k/2 > n_k/3 > n_k/6 \geqslant 1$, and we may continue our reduction. The m_1-fold using of Lemma 3 yields m_1 classes $a_{t_1}(\mathrm{mod}\, n_k/6)$. Now $n_k/3 > n_k/6$, and therefore $a_{t_1}(\mathrm{mod}\, n_k/3)$ and $a_{\lambda_2}(\mathrm{mod}\, n_{v-2}^*)$ are all distinct residue classes with respect to the greatest modulus $n_{v-2}^* = n_k/3$, and also $m_5 = m_2$. These classes we may replace by m_2 classes $a_{t_1}(\mathrm{mod}\, n_k/6)$. Thus we get $m_1 + m_2$ classes $a_{t_1}(\mathrm{mod}\, n_k/6)$.

(aa) If $n_k/6 = 1$ then we get m copies of the set of all integers. The reverse procedure shows that the initial system (1) consists of m_1 copies of a system having property $\mathrm{P}(1, 4)$ (by the way, with classes either modulo 3 or modulo 6) and m_2 copies of a system having property $\mathrm{P}(1, 3)$ (with moduli 2 and 6) where $m_1 + m_2 = m$.

(aba) The case $n_{v-3}^* \neq n_k/6$ can be excluded, because then we should have an m times covering system with one "distinct class" with respect to the greatest modulus which contradicts Lemma 2.

(abb) $n_{v-3}^* = n_k/6$. Reasoning as before we get $a_{t_1} \not\equiv a_{\lambda_3}(\mathrm{mod}\, n_k/6)$. Now we have a system having property $\mathrm{P}(m, 2)$ which consists of m copies of a system having property $\mathrm{P}(1, 2)$. By arguments as in the previous paragraph we get that (1) consists of m_1 copies of a system having property $\mathrm{P}(1, 4)$ and m_2 copies of a system having property $\mathrm{P}(1, 3)$ with $m_1 + m_2 = m$.

The last possibility how to decompose (10) into minimal polygons consists in using pentagons. A pentagon together with a p-sided polygon with $p < 5$ leads to a contradiction with the definition of minimal polygons. Following the proof of Theorem 1 we may show that merely pentagonal decomposition of (10) into minimal polygons gives that the initial system (1) having property $\mathrm{P}(m, 5)$ consists of m copies of a system having property $\mathrm{P}(1, 5)$.

COROLLARY 4. *Let* (1) *be a system having property* $\mathrm{P}(m, 5)$. *Then one of the following alternatives holds*:

(a) $n_i^* = 2^i$ *for* $i = 1, \ldots, v-1$ *and* $n_v^* = 5 \cdot 2^{v-1}$,

(b) $n_i^* = 2^i$ *for* $i = 1, \ldots, v-2$, $n_{v-1}^* = 3 \cdot 2^{v-3}$, $n_v^* = 3 \cdot 2^{v-2}$.

Our theorems suggest the conjecture that every system having property $\mathrm{P}(m, r)$ consists of exactly covering systems, but this is not true in general as it is shown by example due to Choi.

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MATHEMATICAL INSTITUTE
OF THE SLOVAK ACADEMY OF SCIENCES
Bratislava, Czechoslovakia

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