In some sense, this observation eliminates the difficulty caused by primes (whose behaviour is rather irregular) to a certain extent, in determining the smoothness properties of $F$.

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References


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(475)

The largest subset in $[1,n]$ whose integers have pairwise l.c.m. not exceeding $n$, II

by

S. L. G. Choi (Vancouver, Canada)

1. Let $g(n)$ denote the largest number of positive integers not exceeding $n$ such that the l.c.m. (lowest common multiple) of any two of them does not exceed $n$. A conjecture of Erdős [2] states that the extremal sequence consists of the integers from 1 to $(n/2)^{1/3}$ and the even integers from $(n/2)^{1/3}$ to $(2n)^{1/3}$. Thus at any rate

$$g(n) > (3/2)^{1/3} n^{1/3} - 2 > (1.05) n^{1/3} - 2.$$ 

In [1] it was established that

$$(1) \quad g(n) < (1 + \lambda - \lambda^*) n^{1/3} + o(n^{1/3}),$$

where $\lambda, \lambda^*$ are given by

$$(2) \quad \lambda = \sum_{j=1}^{\infty} [(j+1)^{1/3} - j^{1/3}](j+1)^{-1},$$

$$\lambda^* = \sum_{j=2}^{\infty} [(j^{-1/3} - (j+1)^{-1/3})(j+1)^{-1} + \frac{3}{5}(1-2^{-1/3}).$$

In this paper we shall improve substantially upon the constant $1 + \lambda - \lambda^*$ in (1) by a method which, while retaining certain features of the method in [1], is in some essential respects a different and considerably simpler one. We prove two theorems of which Theorem 1 gives the desired improvement over (1). We have included Theorem 2 because it is of related interest and is in any case essentially best possible.

**Theorem 1.** We have

$$(4) \quad n^{-1/3} g(n) \leq 1 + \mu - \mu^* + o(1),$$

where $\mu$ and $\mu^*$ are given by

$$(5) \quad \mu = \sum_{j=1}^{\infty} a_j [(j+1)^{1/3} - j^{1/3}](j+1)^{-1},$$

$$(6) \quad \mu^* = \sum_{j=2}^{\infty} b_j [(j^{-1/3} - (j+1)^{-1/3})(j+1)^{-1},$$

where $a_j, b_j$ are given by

$$(7) \quad a_j = \frac{1}{3} \left[ (j+1)^{1/3} - j^{1/3} \right],$$

$$\frac{1}{3} \left[ j^{-1/3} - (j+1)^{-1/3} \right].$$

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The largest subset in \([1, n]\)

Lemma 3. Let \(a_1 \leq a_2 \leq a_3\) be positive numbers such that \(a_2 a_3 > 1\) and \(a_2 a_3 > 2\). Let \(N\) be sufficiently large and for \(i = 1, 2, 3\), let \(X_i\) be a set of \(a_i(b_i - a_i)N\) integers in \([a_i N, b_i N]\). Suppose further that the integers in \(X = X_1 \cup X_2 \cup X_3\) have pairwise i.e.m. not exceeding \(N^3\). Then

\[ a_1 + a_2 + a_3 \leq 1 + o(1). \]

Proof. For \(i = 1, 2, 3\), we can certainly find a subset \(Y_i\) of \(X_i\) consisting of at least \(a_i(1 + o(1))M\) integers in the interval \((c_i N, c_i N + M)\), where \(M \gg N\), and \(a_1 < a_2 < a_3\) (clearly \(c_1 \geq a_1, c_2 \geq a_2, c_3 \geq a_3\)). By Lemma 1 we can choose primes \(p_1\) and \(p_2\) where

\[ p_1 = (c_1 - e_1)N(1 + o(1)) \quad \text{and} \quad p_2 = (c_2 - e_2)N(1 + o(1)), \]

such that \(p_1 + p_2\) have all its prime factors \(\gg N\). To establish (10) it clearly suffices to show that for all but at most \(o(M)\) triples of integers \(a_1 - p_1, a_1 - p_2, a_1 - p_3\), where \(a_1 \geq c_1, c_2, c_3\), any two integers from the same triple have i.e.m. \(\gg N^3\). Now \(a_1(1 + p_1 + p_2 + p_3) > 1\) for only \(o(M)\) integers \(a_1 \geq c_1, c_2, c_3\); and when \(a_1(1 + p_1 + p_2 + p_3) = 1\) and \((a_1 - p_1, a_1 - p_2) = 1\) then we have \(a_2 = a_3 = 1\) and \((a_2 - p_1, a_2 - p_2) = 1\) and \((a_3 - p_1, a_3 - p_2) = 1\) so that the integers \(a_1 - p_1, a_1 - p_2, a_1 - p_3\) have indeed pairwise i.e.m. \(\ll N^3\). (Note that \(c_1 > c_2 > c_3\).

Lemma 4. Let \(a_1 \leq a_2 \leq a_3 \leq a_4\) be positive numbers such that \(a_2 a_3 > 1\) and \(a_2 a_3 > 2\). Let \(N\) be sufficiently large and for \(i = 1, 2, 3, 4\), let \(X_i\) be a set of \(a_i(b_i - a_i)N\) integers in \([a_i N, b_i N]\). Suppose further that the integers in \(X = X_1 \cup X_2 \cup X_3 \cup X_4\) have pairwise i.e.m. \(\ll N^4\). Then we have

\[ a_1 + a_2 + a_3 + a_4 \leq 1 + o(1). \]

Proof. The proof proceeds along similar lines to those in the proof of Lemma 3. Here we use Lemma 2 instead of Lemma 1.

3. It will be seen that Theorems 1 and 2 are straightforward deductions of Lemma 5 below. All the estimates in Lemma 5 are proved by first appealing to Lemma 3 or 4 in § 2 and then using an argument similar to that employed in the proof of Lemma 3 in [1].

In this section we shall adhere to the following notation. We denote any given set of integers in \([1, n]\) with pairwise i.e.m. not exceeding \(n\). For \(k = 1, 2, \ldots, \) we let \(\mathcal{S}_k\) be the subset of \(\mathcal{S}\) in \((2^k n^{1/2}, (k+1) n^{1/2})\) consisting of all \(a_0(2^k n^{1/2}, (k+1) n^{1/2})\) integers, and \(\mathcal{S}_k\) be the subset of \(\mathcal{S}\) in \((k+1) n^{1/2}, (k+2) n^{1/2})\) consisting of all \(a_0(k+1) n^{1/2}, (k+2) n^{1/2})\) integers. We let \(\mathcal{S}_0\) be the subsets of \(\mathcal{S}\) in \((2^3 \cdot n^{1/2}, 2^{3/2} n^{3/2})\) and \((2^3 \cdot n^{1/2}, 3^{1/2} - 2^{1/2} n^{3/2})\) consisting of all \(\beta(3^{1/2} - 2^{1/2} n^{3/2})\) and \(\beta(3^{1/2} - 2^{1/2} n^{3/2})\) integers respectively. Similarly, let \(\mathcal{S}_1\) be the subsets of \(\mathcal{S}\) in \((n^{1/2}, 2^{3/2} n^{3/2})\) and \((3^{1/2} - 2^{1/2} n^{3/2})\), and let the number of integers in these be \(\gamma(3^{1/2} - 2^{1/2} n^{3/2})\) and \(\gamma(1 - 2^{1/2} n^{3/2})\) respectively.
Lemma 5. We have

\begin{align}
(12) & \quad a_1 + a^* \leq 1 + o(1), \\
(13) & \quad 2a_k + a^* \leq 1 + o(1), \quad k = 2, 3, 4, 5, \\
(14) & \quad 3a_5 + a^* \leq 1 + o(1), \quad k \geq 6, \\
(15) & \quad a_1 + \beta + \beta^* \leq 1 + o(1), \\
(16) & \quad a_1 + \beta + \beta^* \leq 1 + o(1), \\
(17) & \quad \gamma + \gamma^* \leq 1 + o(1), \\
(18) & \quad 2\beta + a_2 \leq 1 + o(1), \\
(19) & \quad 2\beta + a_5 \leq 1 + o(1). 
\end{align}

Proof. As the proofs of the estimates (12)–(19) are all very similar it suffices to give a detailed proof of one of them, say (15).

We divide the interval \((2^{1/2}n^{1/2}, 3^{1/2}n^{1/2})\) into \(L\) subintervals each of length \((3^{1/2} - 2^{1/2})n^{1/2}L^{-1}\). We let \(\mathcal{A}^{(0)}\) be the subset of \(\mathcal{A}^*\) in the \(i\)th subinterval, namely \([2^{1/2}n^{1/2} + (i-1)A_n^{1/2}, 2^{1/2}n^{1/2} + iA_n^{1/2}]\), where

\[A = (3^{1/2} - 2^{1/2})L^{-1}.
\]

Let \(a_i^{(0)}\) be defined by

\[|\mathcal{A}^{(0)}| = a_0^{(0)} A_n^{1/2},
\]

so that

\[\sum_{i=1}^{L} a_i^{(0)} = A_n.
\]

Next we define \(\mathcal{B}^{(0)}\) to be the subset of \(\mathcal{B}\) in

\[\{(2^{1/2} + iA)^{-1} n^{1/2}, 2^{1/2} + (i-1) A^{-1} n^{1/2}\},
\]

and \(\mathcal{B}^{(0)}\) to be the subset of \(\mathcal{B}^*\) in

\[\left[2^{-1}(2^{1/2} + (i-1) A)n^{1/2}, 2^{-1}(2^{1/2} + iA)n^{1/2}\right].
\]

Let \(\beta^{(0)}, \beta^{(0)}\) be defined by

\[|\mathcal{B}^{(0)}| = \beta^{(0)} \left(2^{1/2} +(i-1)A - \frac{2}{2^{1/2} + iA}\right)n^{1/2},
\]

\[|\mathcal{B}^{(0)}| = \beta^{(0)} 2^{-1} A_n^{1/2}.
\]

We note that

\[\sum_{i=1}^{L} \beta^{(0)} = L \beta^*.
\]

Finally we define \(c_i\) by

\[c_i = \max \{a_i^{(0)} + \beta^{(0)}\}
\]

so that \(c_i\) is monotone decreasing.

By Lemma 3 we have

\[\beta^{(0)} \leq (1 + o(1) - c_i^{(0)}).
\]

and this implies

\[|\mathcal{B}^{(0)}| \leq (1 + o(1) - c_i^{(0)}) n^{1/2} \left(\frac{2}{2^{1/2} +(i-1)A} - \frac{2}{2^{1/2} + iA}\right).
\]

Thus

\[n^{-1/2} \sum_{i=1}^{L} |\mathcal{B}^{(0)}|
\]

\[\leq (1 + o(1))(2^{1/2} - 2 \cdot 3^{-1/2}) - 2 \sum_{i=1}^{L} c_i^{(0)} \left(\frac{2}{2^{1/2} +(i-1)A} - \frac{1}{2^{1/2} + iA}\right) + O(1).
\]

Denoting the last sum by \(T\) we have

\[T \geq A \sum_{i=1}^{L} c_i^{(0)} (2^{1/2} + iA)^{-2}.
\]

Since \(c_i\) and \(2^{1/2} + iA)^{-2}\) are both monotone decreasing we have

\[\sum_{i=1}^{L} c_i^{(0)} (2^{1/2} + iA)^{-2} \geq \left[L^{-1} \sum_{i=1}^{L} c_i^{(0)} \left(2^{1/2} + iA)^{-2}\right)\right].
\]

From (26) we clearly have

\[\sum_{i=1}^{L} c_i \geq \sum_{i=1}^{L} (a_i^{(0)} + \beta^{(0)}) = L(a_1 + \beta^*) + O(1),
\]

and it is easy to estimate \(\sum_{i=1}^{L} (2^{1/2} + iA)^{-2}\); in fact

\[\sum_{i=1}^{L} (2^{1/2} + iA)^{-2} = \int_{L}^{2^{1/2} + L} (t - 2^{1/2} + iA)^{-2} dt = O(1) = A^{-1} (2^{1/2} - 3^{-1/2}) + O(1).
\]

Using these estimates in (29) and (28) and recalling (20) we thus obtain

\[T \geq (a_1 + \beta^* + O(1)) \left[2^{-1/2} - 3^{-1/2} + O(L^{-1})\right]
\]

\[= (a_1 + \beta^*) (2^{1/2} - 3^{-1/2}) + O(L^{-1}),
\]

which, together with (27), yield

\[n^{-1/2} \sum_{i=1}^{L} |\mathcal{B}^{(0)}| \leq (1 + o(1) - a_1 - \beta^*) (2^{1/2} - 2 \cdot 3^{-1/2}),
\]

on choosing \(L\) sufficiently large. The above estimate clearly implies (15).
We are now in a position to prove Theorems 1 and 2. We shall find
it convenient to prove Theorem 2 first.

**Proof of Theorem 2.** Let $\mathcal{S}$ be a maximal set of integers in $(4^{-1/12}n^{1/2},
4^{-1/3}n^{1/2})$ with pairwise l.c.m. not exceeding $n$. On recalling the notation introduced before the statement of Lemma 5, we may regard $\mathcal{S}$ as a set $\mathcal{S}'$
with $|\mathcal{S}'| = |\mathcal{S}_1|$ for $k \geq 4$, so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8 \cup \mathcal{S}_9$$

and

$$n^{-1/2} |\mathcal{S}| = a_0(4^{-1/2} - 3^{-1/2}) + a_0(3^{-1/2} - 4^{-1/2}) + a_0(3^{-1/2} - 2^{-1/2}) + \alpha$$

$$+ \alpha(2^{-1/2} - 3^{-1/2}) + \alpha(2^{-1/2} - 2^{-1/2}) + \alpha(3^{-1/2} - 2^{-1/2}) + \gamma(2^{-1/2} - 1) + \eta.$$