In some sense, this observation eliminates the difficulty caused by primes (whose behaviour is rather irregular) to a certain extent, in determining the smoothness properties of F.

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The largest subset in $[1,n]$ whose integers have pairwise l.c.m. not exceeding n, II

by

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1. Let $g(n)$ denote the largest number of positive integers not exceeding n such that the l.c.m. (lowest common multiple) of any two of them does not exceed n. A conjecture of Erdös [2] states that the extremal sequence consists of the integers from 1 to $(n/2)^{1/2}$ and the even integers from $(n/2)^{1/2}$ to $(2n)^{1/2}$. Thus at any rate

$$g(n) > (3/2\sqrt{2})n^{1/2}-2 > (1.05)n^{1/2}-2.$$

In [1] it was established that

$$g(n) < (1+\lambda-\lambda^*)n^{1/2}+o(n^{1/2}), \tag{1}$$

where λ, λ^* are given by

$$\lambda = \sum_{j=1}^{\infty}\big((j+1)^{1/2}-j^{1/2}\big)(j+1)^{-1}, \tag{2}$$

$$\lambda^* = \sum_{j=2}^{\infty}\big(j^{-1/2}-(j+1)^{-1/2}\big)(j+1)^{-1}+\tfrac{9}{20}(1-2^{-1/2}). \tag{3}$$

In this paper we shall improve substantially upon the constant $1+\lambda-\lambda^*$ in (1) by a method which, while retaining certain features of the method in [1], is in some essential respects a different and considerably simpler one. We prove two theorems of which Theorem 1 gives the desired improvement over (1). We have included Theorem 2 because it is of related interest and is in any case essentially best possible.

Theorem 1. *We have*

$$n^{-1/2}g(n) \leqslant 1+\mu-\mu^*+o(1), \tag{4}$$

where μ and μ^* are given by

$$\mu = \sum_{j=1}^{\infty}a_j\big((j+1)^{1/2}-j^{1/2}\big)(j+1)^{-1}, \tag{5}$$

$$\mu^* = \sum_{j=1}^{\infty}b_j\big(j^{-1/2}-(j+1)^{-1/2}\big)(j+1)^{-1}, \tag{6}$$

with

(7) $$a_1 = 1, \quad a_2 = a_3 = 0, \quad a_j = 1 \ (j \geq 4)$$

and

(8) $$b_1 = 1, \quad b_2 = b_3 = 0, \quad b_4 = b_5 = 2, \quad b_j = 3 \ (j \geq 6).$$

Before stating Theorem 2, we need the following definition. For $i \geq 1$, we define $g_i(n)$ to be the largest number of positive integers in $(i^{-1/2}n^{1/2}, i^{1/2}n^{1/2}]$ with pairwise l.c.m. not exceeding n.

THEOREM 2. *We have*

(9) $$n^{-1/2}g_4(n) \leq \tfrac{1}{2}(2^{1/2}-2^{-1/2})+(2^{-1/2}-2^{-1})+o(1).$$

Furthermore, if $\mathcal{S}$ is a maximal set of integers in $(i^{-1/2}n^{1/2}, i^{1/2}n]$ with pairwise l.c.m. not exceeding n, then, with at most $o(n^{1/2})$ exceptions, $\mathcal{S}$ coincides with the integers in $(2^{-1}n^{1/2}, 2^{-1/2}n^{1/2}]$ and the even integers in $(2^{-1/2}n^{1/2}, 2^{1/2}n^{1/2}]$.

Direct computations reveal that $1+\lambda-\lambda^* = 1.63\ldots$, whereas $1+ +\mu-\mu^* = 1.43\ldots$

In connection with further possible extensions of Theorem 2, it would be interesting to determine if, for every k,

(9′) $$n^{-1/2}g_k(n) \leq \tfrac{1}{2}(2^{1/2}-2^{-1/2})+(2^{-1/2}-k^{-1/2})+o(1), \qquad n > n_0(k).$$

2. The following lemmas provide us with the essential tools for proving Theorems 1 and 2.

LEMMA 1. *Let $N \geq N_0(\varepsilon)$ and let N_1, N_2 satisfy $N \ll N_i \ll N$, $i = 1, 2$. Then there exist primes p_1, p_2 such that $p_i = N_i(1+o(1))$, $i = 1, 2$, and all the prime factors in p_1+2p_2 are $\geq N^\varepsilon$.*

LEMMA 2. *Let $N \geq N_0(\varepsilon)$ and N_1, N_2, N_3 satisfy $N \ll N_i \ll N$, $i = 1, 2, 3$. Then there exist primes p_1, p_2, p_3 where $p_i = N_i(1+o(1))$, $i = 1, 2, 3$, such that all the prime factors in p_1+2p_2, $p_1+2p_2+4p_3$ are $\geq N^\varepsilon$ whereas $2p_2+4p_3 = 6q$, and all the prime factors in q are $\geq N^\varepsilon$.*

Proofs of Lemmas 1 and 2. The proofs involve only standard applications of sieve techniques. In the case of Lemma 1, we first choose p_1 to be a prime so that $p_1 = N_1(1+o(1))$. Then we sift out from the sequence p_1+2p_2, where p_2 runs through all the primes in $[(1-\varepsilon)N_2, (1+\varepsilon)N_2]$, all the multiples of primes $\leq N^\varepsilon$. In the case of Lemma 2 we first choose, as in the case of Lemma 1, p_1 and p_2 to be primes so that $p_1 = N_1(1+o(1))$ and $p_2 = N_2(1+o(1))$ such that all prime factors in p_1+2p_2 are $\geq N^\varepsilon$. Now we consider the sequence $(p_1+2p_2+4p_3)q$ where $6q = 2p_2+4p_3$ and p_3 runs through all the primes in $[N_3(1-\varepsilon), N_3(1+\varepsilon)]$, and sift out all multiples of primes $\leq N^\varepsilon$ from this sequence.

LEMMA 3. *Let $a_1 \leq a_2 \leq a_3$ be positive numbers such that $a_1a_2 > 1$ and $a_2a_3 > 2$. Let N be sufficiently large and for $i = 1, 2, 3$, let $\mathcal{X}_i$ be a set of $a_i(b_i-a_i)N$ integers in $[a_iN, b_iN]$. Suppose further that the integers in $\mathcal{X} = \mathcal{X}_1\cup\mathcal{X}_2\cup\mathcal{X}_3$ have pairwise l.c.m. not exceeding N^2. Then*

(10) $$a_1+a_2+a_3 \leq 1+o(1).$$

Proof. For $i = 1, 2, 3$, we can certainly find a subset $\mathcal{Y}_i$ of $\mathcal{X}_i$ consisting of at least $a_i(1+o(1))M$ integers in the interval $(c_iN, c_iN+M]$, where $M \gg N$, and $c_1 < c_2 < c_3$ (clearly $c_1 \geq a_1$, $c_2 \geq a_2$, $c_3 \geq a_3$). By Lemma 1 we can choose primes p_1 and p_2 where

$$p_1 = (c_2-c_1)N(1+o(1)) \quad \text{and} \quad 2p_2 = (c_3-c_2)N(1+o(1)),$$

such that p_1+2p_2 have all its prime factors $\geq N^\varepsilon$. To establish (10) it clearly suffices to show that for all but at most $o(M)$ triples of integers $a, a+p_1, a+p_1+2p_2$, where $a \in (c_1N, c_1N+M]$, any two integers from the same triple have l.c.m. $> N^2$. Now $(a+p_1, p_2) > 1$ or $(a, p_1p_2(p_1+2p_2)) > 1$ for only $o(M)$ integers a of $(c_1N, c_1N+M]$; and when $(a, p_1p_2(p_1+2p_2)) = 1$ and $(a+p_1, p_2) = 1$, then we have $(a, a+p_1) = 1$, $(a+p_1, a+p_1+2p_2) = 1$ or 2 and $(a, a+p_1+2p_2) = 1$ so that the integers a, $a+p_1$, $a+p_1+2p_2$ have indeed pairwise l.c.m. $> N^2$. (Note that $c_1c_2 > 1$ and $c_2c_3 > 2$.)

LEMMA 4. *Let $a_1 \leq a_2 \leq a_3 \leq a_4$ be positive numbers such that $a_1a_2 > 1$ and $a_2a_3 > 6$. Let N be sufficiently large and for $i = 1, 2, 3$, let $\mathcal{X}_i$ be a set of $a_i(b_i-a_i)N$ integers in $[a_iN, b_iN]$. Suppose further that the integers in $\mathcal{X} = \mathcal{X}_1\cup\mathcal{X}_2\cup\mathcal{X}_3\cup\mathcal{X}_4$ have pairwise l.c.m. $\leq N^2$. Then we have*

(11) $$a_1+a_2+a_3+a_4 \leq 1+o(1).$$

Proof. The proof proceeds along similar lines to those in the proof of Lemma 3. Here we use Lemma 2 instead of Lemma 1.

3. It will be seen that Theorems 1 and 2 are straightforward deductions of Lemma 5 below. All the estimates in Lemma 5 are proved by first appealing to Lemma 3 or 4 in § 2 and then using an argument similar to that employed in the proof of Lemma 2 in [1].

In this section we shall adhere to the following notation. We let $\mathcal{A}$ denote any given set of integers in $[1, n]$ with pairwise l.c.m. not exceeding n. For $k = 1, 2, \ldots$, we let $\mathcal{A}_k$ be the subset of $\mathcal{A}$ in $(k^{1/2}n^{1/2}, (k+1)^{1/2}n^{1/2}]$ consisting of in all $a_k((k+1)^{1/2}-k^{1/2})n^{1/2}$ integers, and $\mathcal{A}_k^*$ be the subset of $\mathcal{A}$ in $((k+1)^{-1/2}n^{1/2}, k^{-1/2}n^{1/2}]$ consisting of in all $a_k^*(k^{-1/2}-(k+1)^{-1/2})n^{1/2}$ integers. We let $\mathcal{B}$, $\mathcal{B}^*$ be the subsets of $\mathcal{A}$ in $(2\cdot 3^{-1/2}n^{1/2}, 2^{1/2}n^{1/2}]$ and $(2^{-1/2}n^{1/2}, 3^{1/2}\cdot 2^{-1}n^{1/2}]$ consisting of in all $\beta(2^{1/2} - -2\cdot 3^{-1/2})n^{1/2}$ and $\beta^*(3^{1/2}\cdot 2^{-1}-2^{-1/2})n^{1/2}$ integers respectively. Similarly, let $\mathcal{C}$, $\mathcal{C}^*$ be the subsets of $\mathcal{A}$ in $(n^{1/2}, 2\cdot 3^{-1/2}n^{1/2}]$ and $(2^{-1}3^{1/2}n^{1/2}, n^{1/2}]$, and let the number of integers in these be $\gamma(2\cdot 3^{-1/2}-1)n^{1/2}$ and $\gamma^*(1- -2^{-1}3^{1/2})n^{1/2}$ respectively.

LEMMA 5. *We have*

(12) $$\alpha_1+\alpha_1^*\leqslant 1+o(1),$$

(13) $$2\alpha_k+\alpha_k^*\leqslant 1+o(1),\qquad k=2,3,4,5,$$

(14) $$3\alpha_k+\alpha_k^*\leqslant 1+o(1),\qquad k\geqslant 6,$$

(15) $$\alpha_2+\beta+\beta^*\leqslant 1+o(1),$$

(16) $$\alpha_3+\beta+\beta^*\leqslant 1+o(1),$$

(17) $$\gamma+\gamma^*\leqslant 1+o(1),$$

(18) $$2\beta+\alpha_2\leqslant 1+o(1),$$

and

(19) $$2\beta+\alpha_3\leqslant 1+o(1).$$

Proof. As the proofs of the estimates (12)–(19) are all very similar it suffices to give a detailed proof of one of them, say (15).

We divide the interval $(2^{1/2}n^{1/2}, 3^{1/2}n^{1/2}]$ into L subintervals each of length $(3^{1/2}-2^{1/2})n^{1/2}L^{-1}$. We let $\mathscr{A}_2^{(i)}$ be the subset of $\mathscr{A}_2$ in the ith subinterval, namely $(2^{1/2}n^{1/2}+(i-1)\Delta n^{1/2}, 2^{1/2}n^{1/2}+i\Delta n^{1/2}]$, where

(20) $$\Delta=(3^{1/2}-2^{1/2})L^{-1}.$$

Let $\alpha_2^{(i)}$ be defined by [1]

(21) $$|\mathscr{A}_2^{(i)}|=\alpha_2^{(i)}\Delta n^{1/2}$$

so that

(22) $$\sum_{i=1}^{L}\alpha_2^{(i)}=L\alpha_2.$$

Next we define $\mathscr{B}^{(i)}$ to be the subset of $\mathscr{B}$ in

$$\big(2(2^{1/2}+i\Delta)^{-1}n^{1/2}, 2(2^{1/2}+(i-1)\Delta)^{-1}n^{1/2}\big]$$

and $\mathscr{B}^{*(i)}$ to be the subset of $\mathscr{B}^*$ in

$$\big(2^{-1}(2^{1/2}+(i-1)\Delta)n^{1/2}, 2^{-1}(2^{1/2}+i\Delta)n^{1/2}\big].$$

Let $\beta^{(i)}, \beta^{*(i)}$ be defined by

(23) $$|\mathscr{B}^{(i)}|=\beta^{(i)}\left(\frac{2}{2^{1/2}+(i-1)\Delta}-\frac{2}{2^{1/2}+i\Delta}\right)n^{1/2},$$

(24) $$|\mathscr{B}^{*(i)}|=\beta^{*(i)}2^{-1}\Delta n^{1/2}.$$

We note that

(25) $$\sum_{i=1}^{L}\beta^{*(i)}=L\beta^*.$$

(1) For a set $\mathscr{X}$ of integers, $|\mathscr{X}|$ will denote the number of integers in it.

Finally we define c_i by

(26) $$c_i=\max_{j\geqslant i}(\alpha_2^{(j)}+\beta^{*(j)}),$$

so that c_i is monotone decreasing.

By Lemma 3 we have

$$\beta^{(i)}\leqslant\big(1+o(1)-c_{i+1}\big)$$

and this implies

$$|\mathscr{B}^{(i)}|\leqslant\big(1+o(1)-c_{i+1}\big)n^{1/2}\left\{\frac{2}{2^{1/2}+(i-1)\Delta}-\frac{2}{2^{1/2}+i\Delta}\right\}.$$

Thus

(27) $$n^{-1/2}\sum_{i=1}^{L}|\mathscr{B}^{(i)}|$$

$$\leqslant\big(1+o(1)\big)(2^{1/2}-2\cdot 3^{-1/2})-2\sum_{i=1}^{L-1}c_{i+1}\left\{\frac{2}{2^{1/2}+(i-1)\Delta}-\frac{1}{2^{1/2}+i\Delta}\right\}+O(\Delta).$$

Denoting the last sum by T, we have

(28) $$T\geqslant\Delta\sum_{i=2}^{L}c_i(2^{1/2}+i\Delta)^{-2}.$$

Since c_i and $(2^{1/2}+i\Delta)^{-2}$ are both monotone decreasing we have

(29) $$\sum_{i=2}^{L}c_i(2^{1/2}+i\Delta)^{-2}\geqslant\Big\{L^{-1}\sum_{i=2}^{L}c_i\Big\}\Big\{\sum_{i=2}^{L}(2^{1/2}+i\Delta)^{-2}\Big\}.$$

From (26) we clearly have

$$\sum_{i=2}^{L}c_i\geqslant\sum_{i=2}^{L}(\alpha_2^{(i)}+\beta^{*(i)})=L(\alpha_2+\beta^*)+O(1),$$

and it is easy to estimate $\sum_{i=2}^{L}(2^{1/2}+i\Delta)^{-2}$; in fact

$$\sum_{i=2}^{L}(2^{1/2}+i\Delta)^{-2}=\int_2^L(2^{1/2}+t\Delta)^{-2}dt+O(1)=\Delta^{-1}(2^{-1/2}-3^{-1/2})+O(1).$$

Using these estimates in (29) and (28) and recalling (20) we thus obtain

$$T\geqslant\{\alpha_2+\beta^*+O(L^{-1})\}\{2^{-1/2}-3^{-1/2}+O(L^{-1})\}$$
$$=(\alpha_2+\beta^*)(2^{-1/2}-3^{-1/2})+O(L^{-1}),$$

which, together with (27), yield

$$n^{-1/2}\sum_{i=1}^{L}|\mathscr{B}^{(i)}|\leqslant\big(1+o(1)-\alpha_2-\beta^*\big)(2^{1/2}-2\cdot 3^{-1/2}),$$

on choosing L sufficiently large. The above estimate clearly implies (15).

We are now in a position to prove Theorems 1 and 2. We shall find it convenient to prove Theorem 2 first.

Proof of Theorem 2. Let $\mathscr{S}$ be a maximal set of integers in $(4^{-1/2}n^{1/2}, 4^{1/2}n^{1/2}]$ with pairwise l.c.m. not exceeding n. On recalling the notation introduced before the statement of Lemma 5, we may regard $\mathscr{S}$ as a set $\mathscr{A}$ with $|\mathscr{A}_k| = |\mathscr{A}_k^*| = 0$ for $k \geqslant 4$, so that

$$\mathscr{S} = \mathscr{A}_3 \cup \mathscr{A}_3^* \cup \mathscr{A}_2 \cup \mathscr{A}_2^* \cup \mathscr{B}^* \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{C}^*$$

and

$$n^{-1/2}\,|\mathscr{S}| = a_3(4^{1/2}-3^{1/2}) + a_3^*(3^{-1/2}-4^{-1/2}) + a_2(3^{1/2}-2^{1/2}) + a_2^*(2^{-1/2}-3^{-1/2}) + \beta(2^{1/2}-2\cdot 3^{-1/2}) + \beta^*(3^{1/2}2^{-1}-2^{-1/2}) + \gamma(2\cdot 3^{-1/2}-1) + \gamma^*(1-2^{-1}\cdot 3^{1/2}).$$

On using (13) with $k = 2, 3$, (15), (16) and (17) and denoting η by

(30) $$\eta = \max(a_2, a_3),$$

we obtain

(31) $$n^{-1/2}|\mathscr{S}| \leqslant \eta E_1 + \beta E_2 + \gamma E_3 + (1-4^{-1/2})(1+o(1)),$$

where E_1, E_2, E_3 are given by

(32) $$E_1 = (4^{1/2}-3^{1/2}) - 2(3^{-1/2}-4^{-1/2}) + (3^{1/2}-2^{1/2}) - 2(2^{-1/2}-3^{-1/2}) - (3^{1/2}2^{-1}-2^{-1/2}),$$

(33) $$E_2 = (2^{1/2}-2\cdot 3^{-1/2}) - (3^{1/2}\cdot 2^{-1}-2^{-1/2}),$$

and

(34) $$E_3 = (2\cdot 3^{-1/2}-1) - (1-2^{-1}3^{1/2}).$$

To prove the theorem it suffices to show that if $\eta > 0$, then

$$n^{-1/2}|\mathscr{S}| - \{(1-4^{-1/2}) + \tfrac{1}{2}(2^{1/2}-1-(1-2^{-1/2}))\} < 0;$$

and in view of (31), it suffices to prove

(35) $$\eta E_1 + (\beta-\tfrac{1}{2})E_2 + (\gamma-\tfrac{1}{2})E_3 < 0.$$

Clearly

(36) $$(\gamma-\tfrac{1}{2})E_3 \leqslant 0,$$

since $\gamma \leqslant \frac{1}{2}$.

Furthermore direct computations give

(37) $$E_1 \leqslant .014$$

whereas, on using (18), (19) we have

(38) $$(\beta-\tfrac{1}{2})E_2 \leqslant -\eta(.105).$$

Obviously (36), (37), (38) yield (35), as desired.

Proof of Theorem 1. Let $\mathscr{A}$ be a maximal set of integers in $[1, n]$ with pairwise l.c.m. $\leqslant n$. By Theorem 2 we have the estimation

$$\sum_{k=1}^{3}\{|\mathscr{A}_k| + |\mathscr{A}_k^*|\} \leqslant |\mathscr{S}| \leqslant \tfrac{1}{2}(2^{1/2}-2^{-1/2}) + (2^{-1/2}-2^{-1}) + o(1);$$

and estimates for $|\mathscr{A}_k|$, $|\mathscr{A}_k^*|$, $k \geqslant 4$ are given by (13) and (14). To complete the proof of the estimate (4) we use the simple fact that $a_k \leqslant \dfrac{1}{k+1}$, which is obtained on observing that any two integers in $\mathscr{A}_k$ have l.c.m. $> n$ unless their g.c.d. is $> k$, i.e. unless they are separated by a distance $\geqslant k+1$.

Concluding remarks. It seems interesting to pose the following question. Let $\mathscr{A}_k$ be a set of $a_k((k+1)^{1/2}-k^{1/2})n^{1/2}$ integers in $(k^{1/2}n^{1/2}, (k+1)^{1/2}n^{1/2}]$ with pairwise l.c.m. $\leqslant n$. We define $\mathscr{A}_k^*$ to be a largest subset in $((k+1)^{-1/2}n^{1/2}, k^{-1/2}n^{1/2}]$ so that the integers in $\mathscr{A}_k \cup \mathscr{A}_k^*$ have a pairwise l.c.m. $\leqslant n$. Let the number of integers in $\mathscr{A}_k^*$ be $a_k^*(k^{-1/2}-(k+1)^{-1/2})n^{1/2}$. Determine the largest number m_k so that for all choices of $\mathscr{A}_k$, we have

$$m_k a_k + a_k^* \leqslant 1 + o(1).$$

Lemma 5 gives $m_1 \geqslant 1$, $m_k \geqslant 2$, $k = 2, 3, 4, 5$, and $m_k \geqslant 3$, $k \geqslant 6$. It is clear from the proof of Theorem 1 that it remains valid with b_k replaced by m_k. Finally we remark that it can be shown that $m_1 = 1$, $m_2 = m_3 = m_4 = m_5 = 2$, $m_6 = m_7 = m_8 = m_9 = m_{10} = m_{11} = 4$, and that (9′) holds with $k = 9$. We have refrained however from giving the proofs of these results as these would require a substantial elaboration of our present method resulting in a great many complications in details.

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