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On the distribution of arithmetic functions

by

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1. Introduction. In this paper, for each $\alpha \in (0,1)$, a density called α-density is defined for sets of positive integers. It is shown that if a set A, of natural numbers, has α-density for some α, then it has natural density. A set of positive integers with natural density one is constructed, which does not have α-density for any α. Some sufficient conditions are obtained for a real-valued additive arithmetic function to have a distribution in the sense of α-density. An example is given to show that this result is the best possible in some sense. In the last section of this paper some remarks on smoothness properties of the distributions of arithmetic functions are given.

2. Notations and definitions. Let A be a set of positive integers and let $0<\alpha<1$. Define

$$N(x, A) = \operatorname{card}\{m \in A:\ 1 \leqslant m \leqslant x\},$$
$$N(\alpha, x, A) = \operatorname{card}\{m \in A:\ x < m \leqslant x+x^{\alpha}\},$$
$$\underline{D}(\alpha, A) = \liminf_{x\to\infty} x^{-\alpha} N(\alpha, x, A),$$
$$\overline{D}(\alpha, A) = \limsup_{x\to\infty} x^{-\alpha} N(\alpha, x, A),$$
$$\underline{D}(A) = \liminf_{x\to\infty} x^{-1} N(x, A),$$
$$\overline{D}(A) = \limsup_{x\to\infty} x^{-1} N(x, A).$$

Clearly

$$0 \leqslant \underline{D}(\alpha, A) \leqslant \overline{D}(\alpha, A) \leqslant 1, \quad 0 \leqslant \underline{D}(A) \leqslant \overline{D}(A) \leqslant 1.$$

We say that A has *α-density* if $\underline{D}(\alpha, A) = \overline{D}(\alpha, A)$; in this case we denote the common value by $D(\alpha, A)$. A is said to have *natural density* if $\underline{D}(A) = \overline{D}(A)$ and the common value is denoted by $D(A)$.

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A complex-valued arithmetic function f is called *additive* if

$$f(mn) = f(m)+f(n)$$

whenever $(m, n) = 1$. f is said to be *strongly additive* if, in addition, it satisfies $f(p^k) = f(p)$ for all prime numbers p and positive integers k.

A complex-valued arithmetic function g is said to be *multiplicative* if $g(1) = 1$ and

$$g(mn) = g(m)g(n)$$

whenever $(m, n) = 1$.

A multiplicative function g is said to have an *a-mean value* if the limit of $x^{-a} \sum_{x<m\leqslant x+x^a} g(m)$ exists, as $x\to\infty$. The a-mean value of g is denoted by $M(a, g)$, whenever it exists.

A multiplicative function is said to have a *mean value* if the limit of $x^{-1} \sum_{1\leqslant m\leqslant x} g(m)$ exists, as $x\to\infty$, in which case the limit is denoted by $M(g)$.

We shall say that s-tuples $\{t_1(m), \ldots, t_s(m)\}$ of real-valued additive arithmetic functions have a *distribution* in the sense of a-density if there exists an s-dimensional distribution function $F(C_1, \ldots, C_s)$ such that

$$D(a, \{m\colon t_1(m) < C_1, \ldots, t_s(m) < C_s\})$$

exists and equals $F(C_1, \ldots, C_s)$ for all continuity points of it. Distribution in the sense of natural density can be defined in a similar way.

p, q with or without subscripts denote prime numbers and m, n denote positive integers. For any real number x, $[x]$ denotes the largest integer less than or equal to x.

3. Some properties of a-density. We shall, now, exhibit a set of positive integers with natural density one which does not have a-density for any a. If

$$B = \bigcup_{k=3}^{\infty}\left\{2^{k!}+\left[\frac{2^{k!}}{k!}\right]+1,\ 2^{k!}+\left[\frac{2^{k!}}{k!}\right]+2, \ldots, 2^{(k+1)!}\right\},$$

then, clearly, B has natural density one, but for every $a\in(0, 1)$

$$\underline{D}(a, B) = 0 \quad \text{and} \quad \overline{D}(a, B) = 1.$$

On the other hand, if a set A of positive integers has a-density for some $a\in(0, 1)$ then A has a natural density equal to $D(a, A)$. To show this, first note that for each x there exists ε_x such that $\varepsilon_x\to 0$, as $x\to\infty$ and

$$N(a, x, A) = \big(D(a, A)+\varepsilon_x\big)x^a.$$

Let

$$\delta_x = \max\big((\log x)^{-1}, \sup_{x^{1/2}\leqslant y\leqslant x} |\varepsilon_y|\big).$$

Clearly, $\delta_x > 0$ and $\delta_x\to 0$, as $x\to\infty$. Now we divide the interval $(x\delta_x, x]$ into n disjoint intervals $(a_i, b_i]$, where

$$a_1 = x\delta_x, \quad b_n = x, \quad a_n+a_n^a \geqslant x$$

and

$$a_{i+1} = b_i = a_i+a_i^a \quad \text{for} \quad i = 1, \ldots, n-1.$$

Note that

$$n \leqslant x^{1-a}\delta_x^{-a}.$$

Hence

$$\begin{aligned}|N(x, A)-xD(a, A)| &\leqslant x\delta_x+a_n^a+\sum_{i=1}^{n}|\varepsilon_{a_i}|a_i^a\\ &\leqslant x^a+x\delta_x+x^{1-a}\delta_x^{1-a}x^a\\ &= x(x^{a-1}+\delta_x+\delta_x^{1-a})\\ &= o(x) \quad \text{as} \quad x\to\infty.\end{aligned}$$

By a similar argument one can show that if a set has a-density then it has the same β-density provided $a<\beta$. Also if

$$A_\beta = \bigcup_{k=3}^{\infty}\{2^{k!}+[2^{k!\beta}]+1,\ 2^{k!}+[2^{k!\beta}]+2, \ldots, 2^{(k+1)!}\},$$

then it has a-density for every $a>\beta$, but $\underline{D}(a, A_\beta) = 0$ and $\overline{D}(a, A_\beta) = 1$ for every $a\leqslant\beta$.

Let g be a multiplicative function such that $|g(n)|\leqslant 1$, for all positive integers n. By modifying the above proof it is easy to deduce that $M(g)$ exists if $M(a, g)$ exists for some $a\in(0, 1)$.

4. Theorem. *Suppose $f_1, \ldots, f_s$ are real-valued strongly additive arithmetic functions satisfying the following conditions:*

(1) $$\sum_{|f_i(p)|\leqslant 1}\frac{1}{p}f_i(p) \text{ converges for } i = 1, \ldots, s,$$

(2) $$\sum_{|f_i(p)|\leqslant 1}\frac{1}{p}f_i^2(p) \text{ converges for } i = 1, \ldots, s,$$

and

(3) *for every* $\varepsilon > 0$,

$$\sum_{i=1}^{s}\sum_{\substack{p\leqslant x\\ |f_i(p)|\geqslant\varepsilon}} 1 = o(x^a).$$

Then the s-tuples $\{f_1(m), \ldots, f_s(m)\}$ have a distribution in the sense of a-density.

Proof. We need the following

Lemma ([1], p. 48). *Let the s-dimensional random vectors*

$$X_n = (X_{n_1}, \ldots, X_{n_s}) \quad and \quad X = (X_1, \ldots, X_s)$$

satisfy

$$\sum_{j=1}^{s} t_j X_{nj} \xrightarrow{D} \sum_{j=1}^{s} t_j X_j,$$

as $n \to \infty$, *for all s-tuples* $(t_1, \ldots, t_s)$ *of real numbers. Then* $X_n \xrightarrow{D} X$, *where* $\xrightarrow{D}$ *denotes convergence in distribution.*

In view of this lemma, it is sufficient to prove the theorem, when $s = 1$. So we drop the suffix and write f instead of f_1. First we note that (3) implies

(4) $$\sum_{|f(p)| \geqslant 1} \frac{1}{p} < \infty.$$

In view of (1), (2) and (4), adopting the usual probabilistic methods (see the proof of the sufficiency part of Erdös–Wintner Theorem in [5], pp. 79–81) and using a slight modification of Turán–Kubilius inequality (see [5], Lemma 3.1, p. 31), it is easy to show that there exists a distribution function F such that

$$x^{-a} N\big(a, x, \{m\colon f_{x^a}(m) < C\}\big) \to F(C)$$

as $x \to \infty$, at each continuity point C of F. Here we used the notation

$$f_t(m) = \sum_{\substack{p \leqslant t \\ p|m}} f(p).$$

So to complete the proof it suffices to show that for every $\delta > 0$

$$N(a, x, E) = o(x^a),$$

where

$$E = \{m \in (x, x+x^a]\colon h(m, x) > \delta\}$$

and

$$h(m, x) = |f(m) - f_{x^a}(m)|.$$

For any $\varepsilon > 0$, let

$$A(\varepsilon) = \{p \in (x^a, x+x^a]\colon p \text{ prime and } |f(p)| \geqslant \varepsilon\}$$

and

$$B(\varepsilon) = \{m \in (x, x+x^a]\colon p \nmid m \text{ for every } p \in A(\varepsilon)\}.$$

Note that, for any fixed $\delta > 0$ and for every $\varepsilon > 0$,

$$\begin{aligned} E &\subset \{m \in (x, x+x^a]\colon p \mid m \text{ for some } p \in A(\varepsilon\delta)\} \\ &\quad \cup \big(\{m \in (x, x+x^a]\colon h(m, x) > \delta\} \cap B(\varepsilon\delta)\big) \\ &= E_1 \cup E_2 \quad \text{(say)}. \end{aligned}$$

Clearly by (3)

$$\operatorname{card} E_1 \leqslant \sum_{p \in A(\varepsilon\delta)} \left(\left[\frac{x+x^a}{p}\right] - \left[\frac{x}{p}\right]\right) = O\Big(\sum_{\substack{p \leqslant x+x^a \\ p \in A(\varepsilon\delta)}} 1\Big)$$
$$< \varepsilon x^a \quad \text{for every } x > x(\varepsilon\delta).$$

Since each $m \in (x, x+x^a]$ is divisible by at most $([1/a]+2)$ primes $q > x^a$ and since, as a result, for each $m \in B(\varepsilon\delta)$,

$$h(m, x) < \varepsilon\delta([1/a]+2),$$

we have

$$\operatorname{card} E_2 \leqslant \delta^{-1} \sum_{m \in B(\varepsilon\delta)} h(m, x) < \delta^{-1} \varepsilon\delta([1/a]+2) x^a.$$

So, for every $x > x(\varepsilon\delta)$,

$$N(a, x, E) < \left(\varepsilon + \varepsilon\left(\frac{1}{a} + 2\right)\right) x^a.$$

Since ε is arbitrary it follows, for every $\delta > 0$, that

(5) $$x^{-a} N(a, x, E) \to 0 \quad \text{as} \quad x \to \infty.$$

This completes the proof of the theorem.

Remarks. We can, of course, replace (3) by a weaker, but clumsy looking condition

(6) for every $\varepsilon > 0$

$$\sum_{m \leqslant x^{1-a}} Q\left(\varepsilon, \frac{x}{m}, \frac{x+x^a}{m}\right) = o(x^a),$$

where

$$Q(\varepsilon, r, y) = \operatorname{card}\{p \in (r, y]\colon |f_i(p)| \geqslant \varepsilon, \text{ for some } i\}.$$

For, in the proof of the theorem, $m \in E_1$ implies $m = ap$, where $a \leqslant x^{1-a}+1$, $p > x^a$, $|f(p)| \geqslant \varepsilon\delta$. Hence

$$\operatorname{card} E_1 \leqslant \sum_{a \leqslant x^{1-a}+1} Q\left(\varepsilon, \frac{x}{a}, \frac{x+x^a}{a}\right) = o(x^a).$$

Of course, (3) clearly, implies (6). Now we shall give an example (essentially due to Erdös) to show that, in general, (6) does not imply (3). Let $a \in (\frac{2}{3}, 1)$ and let $x_1 = 3, x_2 = x_1^{2/a}, \ldots, x_{k+1} = x_k^{2/a}, \ldots$ Note that $x_k \to \infty$ as $k \to \infty$. Define a strongly additive arithmetic function f by

$$f(p) = \begin{cases} 1 & \text{if} \quad x_k < p \leqslant x_k + x_k^a \log x_k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $a > 7/12$, we have the number of primes in the interval $(x_k, x_k + x_k^a \log x_k]$ to be (see [3])(1)

$$\big(1+o(1)\big)x_k^a \geqslant \tfrac{1}{2}(x_k + x_k^a \log x_k)^a \quad \text{for all sufficiently large } k.$$

Hence (3) is not satisfied. Now we shall show that (6) is satisfied.

Suppose $x \in [x_k, x_k \log x_k]$, then for any $\varepsilon > 0$, the left hand side of (6) is equal to

$$\sum_{m \leqslant (\log x)^2} Q\Big(\varepsilon, \frac{x}{m}, \frac{x+x^a}{m}\Big) = O\Big(\sum_{m \leqslant (\log x)^2} x^a (m \log x)^{-1}\Big)$$
$$= O(x^a \log\log x/\log x) = o(x^a),$$

since $f(p) = 0$ for p in $(x_{k-1} + x_{k-1}^a \log x_{k-1}, x_k)$ and

$$Q\Big(\varepsilon, \frac{x}{m}, \frac{x+x^a}{m}\Big) = 0 \quad \text{if} \quad m \in \big((\log x)^2, x^{1-a}\big),$$

as for any such m,

$$\frac{x}{m} \geqslant x^a \geqslant x_k^a = x_{k-1}^2 > x_{k-1}.$$

If $x \in (x_k \log x_k, x_{k+1})$, then the left hand side of (6) is less than the number of primes $p \leqslant x$, with $f(p) = 1$. Which in turn is less than

$$2x_{k-1} + \pi(x_k + x_k^a \log x_k) - \pi(x_k) = O(x_k^a) = o(x^a).$$

Here $\pi(x)$ denotes the number of primes not exceeding x. Hence (6) holds.

Now we shall give an example to show that our theorem is the best possible, in the sense that (6) can not be dropped.

Let a be a positive real number less than 1 and very close to 1. Let $\{y_k\}$ be an increasing sequence of positive real numbers tending to infinity sufficiently fast. Let $a_1 < a_2 < \ldots$ be the set of primes in the intervals

$$\Big(\frac{y_k}{t}, \frac{y_k + y_k^a}{t}\Big), \quad k = 1, 2, \ldots, \; 1 \leqslant t \leqslant y_k^{1-a}.$$

(1) In fact, if $x > h > x^{7/12+\varepsilon}$, it follows from (28.27), (28.32), (28.33) (see p. 121), that $\psi(x+h) - \psi(x) \sim h$. Hence $\pi(x+h) - \pi(x) = (1+o(1))h/\log x$, where $\pi(x)$ denotes the number of primes $p \leqslant x$.

Clearly, $\sum a_i^{-1} < \infty$. Let b be the natural density of the set of positive integers not divisible by any a_i (which, clearly, exists). Erdös ([2], see p. 179) has shown that the number of integers $m \in (y_k, y_k + y_k^a)$ which are not divisible by any a_i is less than $(b-\eta)y_k^a$ for a fixed $\eta > 0$. So from the results of Section 3, it follows that the set of positive integers not divisible by any a_i does not have a-density. If we define a strongly additive arithmetic function f by

$$f(p) = \begin{cases} 1 & \text{if} \quad p = a_i \text{ for some } i, \\ 0 & \text{otherwise,} \end{cases}$$

then clearly (6) is violated and f does not have distribution in the sense of a-density, eventhough (1) and (2) are satisfied. On the other hand a more detailed analysis tend to show if a is very close to 1, then (6) is also necessary for the existence of the distribution of an additive arithmetic function f in the sense of a-density. But we are unable to prove it rigorously.

5. In this section we shall remark on the smoothness properties of distributions of additive arithmetic functions. As has been already pointed out, if an additive arithmetic function has a distribution in the sense of a-density, then it has a distribution in the sense of natural density and both the distributions coincide. Let f be a real-valued additive arithmetic function having a distribution F. It is shown in [4] (see, in particular, the proof of Theorem 1 of [4]), that F is absolutely continuous (a.c.) if and only if the distribution corresponding to the characteristic function (c.f.)

$$\varphi_1(t) = \exp\Big(\sum_{n=3}^{\infty} p_n^{-1}(e^{ita_n} - 1 - ita_n)\Big)$$

is a.c., where

$$a_n = \begin{cases} f(p_n) & \text{if} \quad |f(p_n)| < 1, \text{ and } n \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

Here p_n denotes the nth prime. By using an argument similar to that in [4] and observing

$$\sum_{n=3}^{\infty} \Big(\frac{1}{p_n} - \frac{1}{n \log n}\Big) < \infty$$

we can easily deduce that F is a.c. if and only if the distribution corresponding to the c.f.

$$\varphi_2(t) = \exp\Big(\sum_{n=3}^{\infty} \big((e^{ita_n} - 1 - ita_n)/n \log n\big)\Big)$$

is a.c.

In some sense, this observation eliminates the difficulty caused by primes (whose behaviour is rather irregular) to a certain extent, in determining the smoothness properties of F.

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References

[1] P. Billingsley, *Convergence of Probability Measures*, J. Wiley, New York 1968.
[2] P. Erdös, *On the difference of consecutive terms of sequences defined by divisibility properties*, Acta. Arith. 12 (1966), pp. 175–182.
[3] M. N. Huxley, *The Distribution of Prime Numbers*, Oxford University Press, London 1972.
[4] G. Jogesh Babu, *Some results on the distribution of additive arithmetic functions III*, Acta. Arith. 25 (1973), pp. 39–49.
[5] J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Transl. Math. Mono., Amer. Math. Soc. 11 (1964).

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The largest subset in $[1,n]$ whose integers have pairwise l.c.m. not exceeding n, II

by

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1. Let $g(n)$ denote the largest number of positive integers not exceeding n such that the l.c.m. (lowest common multiple) of any two of them does not exceed n. A conjecture of Erdös [2] states that the extremal sequence consists of the integers from 1 to $(n/2)^{1/2}$ and the even integers from $(n/2)^{1/2}$ to $(2n)^{1/2}$. Thus at any rate

$$g(n) > (3/2\sqrt{2})n^{1/2}-2 > (1.05)n^{1/2}-2.$$

In [1] it was established that

$$g(n) < (1+\lambda-\lambda^*)n^{1/2}+o(n^{1/2}), \tag{1}$$

where λ, λ^* are given by

$$\lambda = \sum_{j=1}^{\infty}\big((j+1)^{1/2}-j^{1/2}\big)(j+1)^{-1}, \tag{2}$$

$$\lambda^* = \sum_{j=2}^{\infty}\big(j^{-1/2}-(j+1)^{-1/2}\big)(j+1)^{-1}+\tfrac{9}{20}(1-2^{-1/2}). \tag{3}$$

In this paper we shall improve substantially upon the constant $1+\lambda-\lambda^*$ in (1) by a method which, while retaining certain features of the method in [1], is in some essential respects a different and considerably simpler one. We prove two theorems of which Theorem 1 gives the desired improvement over (1). We have included Theorem 2 because it is of related interest and is in any case essentially best possible.

THEOREM 1. *We have*

$$n^{-1/2}g(n) \leqslant 1+\mu-\mu^*+o(1), \tag{4}$$

where μ and μ^* are given by

$$\mu = \sum_{j=1}^{\infty}a_j\big((j+1)^{1/2}-j^{1/2}\big)(j+1)^{-1}, \tag{5}$$

$$\mu^* = \sum_{j=1}^{\infty}b_j\big(j^{-1/2}-(j+1)^{-1/2}\big)(j+1)^{-1}, \tag{6}$$