[5] E. Grosswald, *Sur une propriété des racines complexes des fonctions $L(s, \chi)$*, C. R. Acad. Sc. Paris 260 (1965), pp. 4299–4302.
[6] S. Knapowski and P. Turán, *Comparative prime number theory, I*, Acta Math. Acad. Sci. Hung. 8 (1962), pp. 299–314.
[7] Y. Motohashi, *On the distribution of the zeros of Dirichlet's L-functions*, Acta Arith. 22 (1972), pp. 107–112.
[8] A. Selberg, *Contributions to the theory of Dirichlet's L-functions*, Skr. Norske Vid. Akad. Oslo I, Mat.-Naturv. Klasse, 3 (1946), pp. 2–62.
[9] — *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid, 48 (1946), pp. 89–155.
[10] P. Turán, *On a problem concerning the zeros of Dirichlet's L-functions*, Publ. Ramanujan Institute 1 (1969), pp. 95–100.

INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey

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The half dimensional sieve

by

H. Iwaniec (Warszawa)

§ 1. Introduction. The purpose of this paper is to describe the half dimensional sieve in its general form (with conditions (Ω_1) and (Ω_2) of Halberstam and Richert type [3]) and to show that this sieve is powerful enough to establish the asymptotic formulae. In the paper a variant of Brun's method is used (without any weights or other combinatorial devices). Estimations of the 'sifting function' which are given in Theorem 1 cannot be essentially improved. Theorem 2 shows that in a special case the main term of the upper estimation coincides with the main term of the asymptotic formula for the sifting function. The fundamental result of the paper is included in Theorem 4. From this theorem we obtain the asymptotic formula for the number of quasi-primes of the form u^2+v^2+c lying in a short interval of consecutive terms of an arithmetic progression with the difference large in comparision to the length of the interval. It is the contents of Corollary 1. From Corollary 1 we obtain in particular the following Landau's theorem

$$B(x) := \sum_{\substack{n\leqslant x\\ n=a^2+b^2}} 1 \sim B\frac{x}{\sqrt{\log x}}$$

where

$$B = \frac{1}{\sqrt{2}} \prod_{p\equiv -1 (\mathrm{mod}\, 4)} \left(1-\frac{1}{p^2}\right)^{-1/2}.$$

Our proof is elementary and does not make use of the Prime Number Theorem (in contrast to the Levin and Fainleib's iterative method [11], p. 379). In 1953 K. Prachar [7] proved

$$(1.1) \qquad B(x, k, l) := \sum_{\substack{n\leqslant x\\ n=a^2+b^2\\ n\equiv l (\mathrm{mod}\, k)}} 1 \sim B_k \frac{x}{\sqrt{\log x}}$$

where

$$B_k = B\frac{(4,k)}{(2,k)k}\prod_{\substack{p\mid k\\ p\equiv -1 (\mathrm{mod}\, 4)}}\left(1+\frac{1}{p}\right).$$

The integers k, l are fixed and $(k,l)=1$, $l\equiv 1 \bmod (4,k)$. Next in 1965 G. J. Rieger [8] using Siegel's theorem proved that (1.1) is true uniformly for all $k<\log^c x$, where c is any fixed constant. Shortly afterwards in 1967 H. Bekić [1] extended the range of k up to $\exp c\sqrt{\log x}$. Quite recently G. S. Belozorov [9] moved k close to $\exp(\log x)^{2/3}$. Both Bekić and Belozorov also made use of the theory of Dirichlet's L-functions. From Corollary 1 we obtain

$$B(x,k,l) = B_k\frac{x}{\sqrt{\log x}}\left(1+O\left(\left(\frac{\log k}{\log x}\right)^{1/5}\right)\right).$$

The constant in symbol O is absolute. Our method does not allow us to replace exponent $1/5$ by a constant arbitrarily close to $1/2$.

A positive integer $n\leqslant N$ will be called quasi-prime in the sense of Hooley if it is not divisible by primes $<\min(n, N^{\log^{-2}\log N})$.

From Corollary 1 we obtain

$$\sum_{\substack{a\leqslant x\\ n=a^2+b^2+1\\ n\ \text{quasi-prime}}} 1 \sim \frac{e^{-C}}{\sqrt{2}}\prod_{p\equiv -1(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)^{-1/2}\left(1-\frac{1}{p(p-1)}\right)\frac{x\log^2\log x}{\log^{3/2}x},$$

where C is Euler's constant. From Theorem 1 we obtain

$$B(x+y)\leqslant B(x)+B(y)+O(y\log^{-0.7}y)$$

for all $x\geqslant y>1$. The constant in symbol O is absolute. Therefore $B(x)$ is "almost subadditive".

Theorem 2 can be extended without any difficulty to the case $\mathscr{P}=\{p;\ (D/p)=-1\}$, where D is an integer different from a perfect square and (D/p) denotes Kronecker's symbol. Combining our sieve result with some facts from the arithmetic of ideals of the field $Q(\sqrt{D})$ we can give an asymptotic formula for the number of positive integers $\leqslant x$ represented by a quadratic polynomial in two variables.

All the constants in symbols $\ll$, O will be absolute.

I conclude this introduction by expressing my thanks to Professor A. Schinzel for his interest in my paper.

§ 2. Statement of the results.

Let $\mathscr{A}$ be a finite set of positive integers and let $\mathscr{P}$ be a set of primes. For a real number $z>1$ let

$$P(z) := \prod_{p<z,\, p\in\mathscr{P}} p.$$

The sieve method is used to estimate the sifting function

$$S(\mathscr{A},z) := \left|\{a\in\mathscr{A};\ (a,P(z))=1\}\right|$$

where $|\{\ldots\}|$ denotes the cardinality of the set $\{\ldots\}$. Let $\omega(d)$ be a multiplicative arithmetic function such that

(Ω_1) $\qquad 0\leqslant\omega(p)<p$ and $\omega(p)=0$ for $p\notin\mathscr{P}$,

(Ω_2)
$$-L+\frac{1}{2}\log\frac{z}{w}\leqslant\sum_{w\leqslant p<z}\frac{\omega(p)}{p}\log p\leqslant\sum_{w\leqslant p<z}\frac{\omega(p)}{p-\omega(p)}\log p\leqslant K+\frac{1}{2}\log\frac{z}{w}$$

for any real numbers $z>w>1$, where K and L are constants >1. For a real number $z>1$ let

$$\Omega(z) := \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right)$$

and

$$\mathscr{A}_d := \{a\in\mathscr{A};\ a\equiv 0 (\mathrm{mod}\, d)\};\qquad R(\mathscr{A},d) := |\mathscr{A}_d|-\frac{\omega(d)}{d}X$$

for a certain real number $X_{\mathscr{A}}=X>1$.

THEOREM 1. *For $z\geqslant 2$ and $s\geqslant 1$ we have*

$$S(\mathscr{A},z)\geqslant\Omega(z)X\left\{1-\frac{f(s)}{\sqrt{s}}+O(e^{2K-s}\log^{-\gamma}z)\right\}-\sum_{\substack{d<z^s\\ d\mid P(z)}}|R(\mathscr{A},d)|,$$

$$S(\mathscr{A},z)\leqslant\Omega(z)X\left\{1+\frac{F(s)}{\sqrt{s}}+O(e^{2K-s}\log^{-\gamma}z)\right\}+\sum_{\substack{d<z^s\\ d\mid P(z)}}|R(\mathscr{A},d)|$$

where $\gamma=\frac{1}{2}-\frac{1}{(2-\log 2)e}=a-\frac{1}{2}$ and the functions $f(s)$ and $F(s)$ are defined in § 4. For $1\leqslant s\leqslant 2$ we have

$$1-\frac{f(s)}{\sqrt{s}}=\sqrt{\frac{e^C}{\pi s}}\int_1^s\frac{dt}{\sqrt{t(t-1)}};\qquad 1+\frac{F(s)}{\sqrt{s}}=2\sqrt{\frac{e^C}{\pi s}}.$$

Let a' denote the greatest odd divisor of a. We have

THEOREM 2. *Let us assume that*

$$a\in\mathscr{A}\Rightarrow a'\equiv 1(\mathrm{mod}\,4),\qquad \mathscr{P}\subset\{p\equiv -1(\mathrm{mod}\,4)\}.$$

Then, for $Q^6 < z^2 \leqslant A := \max_{a \in \mathscr{A}} a$ *we have*

$$S(\mathscr{A}, z) = \Omega(z) X \left\{1 + \frac{F(s)}{\sqrt{s}} + O\left(e^{4K-s}\left(\frac{L+\log Q}{\log A}\right)^{\gamma}\right)\right\} + \theta \sum_{\substack{d<A/Q \\ d|P(z)}} |R(\mathscr{A}, d)|$$

where $s = \log A/\log z$ *and* $|\theta| \leqslant 1$.

Let

$$b(a) = \begin{cases} 1 & \text{if } \quad a = u^2+v^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$b^*(a) = \begin{cases} 1 & \text{if } \quad a = u^2+v^2,\ (u, v) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have

THEOREM 3. *Let us assume, that*

$$a \in \mathscr{A} \Rightarrow a \equiv 1, 2 \text{ or } 5 (\operatorname{mod} 8), \quad \mathscr{P} \subset \{p \equiv -1 (\operatorname{mod} 4)\}.$$

Then for $Q^6 < A$ *we have*

$$\sum_{a \in \mathscr{A}} b^*(a) = \sqrt{2} \prod_{p \equiv -1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} \left(1 - \frac{\omega(p)}{p}\right)\left(1 + \frac{1}{p}\right) X \log^{-1/2} A \times$$

$$\times \left\{1 + O\left(e^{6K}\left(\frac{L+\log Q}{\log A}\right)^{\gamma}\right)\right\} + \theta \sum_{\substack{d<A/Q \\ d|P(\sqrt{A})}} |R(\mathscr{A}, d)|.$$

THEOREM 4. *Let us assume, that*

$$a \in \mathscr{A} \Rightarrow a' \equiv 1 (\operatorname{mod} 4), \quad \mathscr{P} = \{p \equiv -1 (\operatorname{mod} 4)\},$$
$$\omega(p^a) = \omega(p) \quad \text{for} \quad a \geqslant 1.$$

Then for $Q^8 < A$ *we have*

$$\sum_{a \in \mathscr{A}} b(a) = \sqrt{2} \prod_{p \equiv -1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} \left(1 + \frac{1 - \omega(p)}{p}\right) X \log^{-1/2} A \times$$

$$\times \left\{1 + O\left(e^{6K}\left(\frac{L+\log Q}{\log A}\right)^{\gamma}\right)\right\} + O\left(\frac{A}{Q}\right) + \theta \sum_{\substack{d<A/Q \\ p|d \Rightarrow p \in \mathscr{P}}} |R(\mathscr{A}, d)|.$$

We shall deduce from this theorem the following

COROLLARY 1. *Let* $(Pl, 2k) = 1$, $l \equiv 1 (\operatorname{mod}(4, k))$, $\varepsilon > \dfrac{\log\log N}{\log N}$, $0 < |c| < kN + M < N^{1+\varepsilon}$, $p \,|\, P \Rightarrow p < N^{\varepsilon/\log\log N}$. *Then we have*

$$\sum_{\substack{M \leqslant n < M+N \\ n \equiv l (\operatorname{mod} k) \\ (n+c, P)=1}} b(n) = \frac{1}{\sqrt{2}} \prod_{p \equiv -1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} \prod_{p|P} \left(1 - \frac{1}{p}\right) \times$$

$$\times \prod_{\substack{p|P \\ p \nmid c \\ p \equiv -1 (\operatorname{mod} 4)}} \left(1 - \frac{1}{p(p-1)}\right) \frac{(4, k)}{(2, k) k} \prod_{\substack{p|(c,P)k \\ p \equiv -1 (\operatorname{mod} 4)}} \left(1 + \frac{1}{p}\right) \frac{N}{\sqrt{\log N}} \{1 + O(\varepsilon^{\gamma})\}.$$

Let

$$E(X, Q) = \sum_{q<Q} \max_{(l,q)=1} \left| \pi(X, q, l) - \frac{\operatorname{Li} X}{\varphi(q)} \right|,$$

$$B(X, Q) = \sum_{q<Q} \max_{(l,q)=1} \left| \sum_{\substack{n \leqslant X \\ n \equiv l (\operatorname{mod} q)}} b(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} b(n) \right|.$$

The following corollary can be deduced from Theorem 4 in the same way as Corollary 1. Therefore we shall not give the details.

COROLLARY 2. *Let* $\varepsilon > \log\log N/\log N$. *Then we have*

$$\sum_{p<N} b(p-1) = \frac{1}{\sqrt{2}} \prod_{p \equiv -1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} \left(1 - \frac{1}{p(p-1)}\right) \frac{N}{\log^{3/2} N} \times$$
$$\times \left(1 + O(\varepsilon^{\gamma})\right) + \theta_1 E(N, N^{1-\varepsilon}), \tag{2.1}$$

$$\sum_{p<N} b(N-p) = \frac{1}{\sqrt{2}} \prod_{p \equiv -1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} \left(1 - \frac{1}{p(p-1)}\right) \times$$
$$\times \prod_{\substack{p|N \\ p \equiv -1 (\operatorname{mod} 4)}} \left(1 + \frac{p}{p^2 - p - 1}\right) \frac{N}{\log^{3/2} N} \left(1 + O(\varepsilon^{\gamma})\right) + \theta_2 E(N, N^{1-\varepsilon}), \tag{2.2}$$

$$\sum_{n<N} b(n) b(n+1)$$
$$= \frac{3}{8} \frac{N}{\log N} \{1 + O(\varepsilon^{\gamma})\} + \theta_3 \{B(N, N^{1-\varepsilon}) + B(N/2, N^{1-\varepsilon})\}, \tag{2.3}$$

where $|\theta_i| \leqslant 1$ *for* $i = 1, 2, 3$.

(2.1) implies that Motohashi's conjecture [6] about the number of primes of the form $x^2 + y^2 + 1 \leqslant N$ follows in its corrected form (see [4],

p. 204) from Halberstam's well-known conjecture [5] that

$$E(N, N^{1-\varepsilon}) = O(N/\log^A N).$$

Similarly, (2.3) implies that Hooley's conjecture about $\sum_{n<N} b(n)b(n+1)$ would follow from the estimate $B(N, N^{1-\varepsilon}) = o(N/\log N)$.

§ 3. The fundamental identities of half dimensional sieve. The following identity

$$S(\mathscr{A}, z) = |\mathscr{A}| - \sum_{p|P(z)} S(\mathscr{A}_p, p) \tag{3.1}$$

provides a recursive formula for the sifting function. Using it many times we get

LEMMA 1. *For $R \geqslant 1$ we have*

$$\begin{aligned} S(\mathscr{A}, z) = |\mathscr{A}| &+ \sum_{r=1}^{R} (-1)^r \sum_{\substack{p_r < \ldots < p_1 < z \\ p_l < f_l \text{ for } l \leqslant r}} |\mathscr{A}_{p_1 \ldots p_r}| + \\ &+ \sum_{r=1}^{R} (-1)^r \sum_{\substack{f_r \leqslant p_r < \ldots < p_1 < z \\ p_l < f_l \text{ for } l < r}} S(\mathscr{A}_{p_1 \ldots p_r}, p_r) - \\ &- (-1)^R \sum_{\substack{p_{R+1} < \ldots < p_1 < z \\ p_l < f_l \text{ for } l \leqslant R}} S(\mathscr{A}_{p_1 \ldots p_{R+1}}, p_{R+1}) \end{aligned} \tag{3.2}$$

where $p_l \in \mathscr{P}$ and $f_l = f_l(p_1, \ldots, p_l)$ for $l \leqslant R+1$.

Let us put

$$S^-(\mathscr{A}, z) = |\mathscr{A}| + \sum_{1 \leqslant r \leqslant 2R+1} (-1)^r \sum_{\substack{p_r < \ldots < p_1 < z \\ p_{2l} < g_{2l} \text{ for } 2l \leqslant r}} |\mathscr{A}_{p_1 \ldots p_r}|,$$

$$S^+(\mathscr{A}, z) = |\mathscr{A}| + \sum_{1 \leqslant r \leqslant 2R} (-1)^r \sum_{\substack{p_r < \ldots < p_1 < z \\ p_{2l-1} < g_{2l-1} \text{ for } 2l-1 \leqslant r}} |\mathscr{A}_{p_1 \ldots p_r}|,$$

where $p_l \in \mathscr{P}$ and $g_l = g_l(p_1, \ldots, p_l)$ for $l = 2R$.

From Lemma 1 we get

LEMMA 2. *For $R \geqslant 1$ we have*

$$\begin{aligned} S(\mathscr{A}, z) = S^-(\mathscr{A}, z) &+ \sum_{r=1}^{R} \sum_{\substack{g_{2r} \leqslant p_{2r} < \ldots < p_1 < z \\ p_{2l} < g_{2l} \text{ for } l < r}} S(\mathscr{A}_{p_1 \ldots p_{2r}}, p_{2r}) + \\ &+ \sum_{\substack{p_{2R+2} < \ldots < p_1 < z \\ p_{2l} < g_{2l} \text{ for } l \leqslant R}} S(\mathscr{A}_{p_1 \ldots p_{2R+2}}, p_{2R+2}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} S(\mathscr{A}, z) = S^+(\mathscr{A}, z) &- \sum_{r=1}^{R} \sum_{\substack{g_{2r-1} \leqslant p_{2r-1} < \ldots < p_1 < z \\ p_{2l-1} < g_{2l-1} \text{ for } l < r}} S(\mathscr{A}_{p_1 \ldots p_{2r-1}}, p_{2r-1}) - \\ &- \sum_{\substack{p_{2R+1} < \ldots < p_1 < z \\ p_{2l-1} < g_{2l-1} \text{ for } l \leqslant R}} S(\mathscr{A}_{p_1 \ldots p_{2R+1}}, p_{2R+1}). \end{aligned} \tag{3.4}$$

Proof. We obtain (3.3) from (3.2) by substitution $f_{2l+1} = f_{2l} = g_{2l-1}$ and $R \to 2R$. Analogously we obtain (3.4) from (3.2) by substitution $f_{2l-1} = f_{2l} = g_{2l}$ and $R \to 2R+1$.

In particular, Lemma 2 gives us estimations of sifting function expressed only in terms of $|\mathscr{A}_d|$.

$$S^-(\mathscr{A}, z) \leqslant S(\mathscr{A}, z) \leqslant S^+(\mathscr{A}, z). \tag{3.5}$$

For $y > 1$ let us put $g_l = y/p_1 \cdots p_l$ and

$$\mathscr{D}_y^- = \{d = p_1 \cdots p_r;\ p_r < \ldots < p_1,\ p_{2l}^2 p_{2l-1} \cdots p_1 < y \text{ for } 2l \leqslant r \leqslant 2R+1\},$$
$$\mathscr{D}_y^+ = \{d = p_1 \cdots p_r;\ p_r < \ldots < p_1,\ p_{2l-1}^2 p_{2l-2} \cdots p_1 < y \text{ for } 2l-1 \leqslant r \leqslant 2R\}.$$

It is clear that

$$S^{\pm}(\mathscr{A}, z) = \sum_{\substack{d \in \mathscr{D}_y^{\pm} \\ d|P(z)}} \mu(d) |\mathscr{A}_d| = X \sum_{\substack{d \in \mathscr{D}_y^{\pm} \\ d|P(z)}} \mu(d) \frac{\omega(d)}{d} + \sum_{\substack{d \in \mathscr{D}_y^{\pm} \\ d|P(z)}} R(\mathscr{A}, d). \tag{3.6}$$

For $y > 1$, $s \geqslant 1$ and $r \geqslant 1$ let us put

$$d_{2r,y}(s) = \sum_{\substack{g_{2r} \leqslant p_{2r} < \ldots < p_1 < y^{1/s} \\ p_{2l} < g_{2l} \text{ for } l < r}} \frac{\omega(p_1 \cdots p_{2r})}{p_1 \cdots p_{2r}} \Omega(p_{2r});$$

$$g_{2r,y}(s) = \sum_{\substack{g_{2r} \leqslant p_{2r} < \ldots < p_1 < y^{1/s} \\ p_{2l} < g_{2l} \text{ for } l < r}} \frac{\omega(p_1 \cdots p_{2r})}{p_1 \cdots p_{2r}} \log^{-1/2} p_{2r};$$

$$h_{2r-1,y}(s) = \sum_{\substack{\sqrt{g_{2r-1}} \leqslant p_{2r-1} < \ldots < p_1 < y^{1/s} \\ p_{2l} < g_{2l} \text{ for } l < r}} \frac{\omega(p_1 \cdots p_{2r-1})}{p_1 \cdots p_{2r-1}} \log^{-a} 3 g_{2r-1}.$$

For $y > 1$, $s > 0$ and $r \geqslant 1$ let us put

$$d_{2r-1,y}(s) = \sum_{\substack{g_{2r-1} \leqslant p_{2r-1} < \ldots < p_1 < y^{1/s} \\ p_{2l-1} < g_{2l-1} \text{ for } l < r}} \frac{\omega(p_1 \cdots p_{2r-1})}{p_1 \cdots p_{2r-1}} \Omega(p_{2r-1});$$

$$g_{2r-1,y}(s) = \sum_{\substack{g_{2r-1} \leqslant p_{2r-1} < \ldots < p_1 < y^{1/s} \\ p_{2l-1} < g_{2l-1} \text{ for } l < r}} \frac{\omega(p_1 \cdots p_{2r-1})}{p_1 \cdots p_{2r-1}} \log^{-1/2} p_{2r-1};$$

$$h_{2r,y}(s) = \sum_{\substack{\sqrt{g_{2r}} \leqslant p_{2r} < \ldots < p_1 < y^{1/s} \\ p_{2l} < g_{2l} \text{ for } l<r}} \frac{\omega(p_1 \ldots p_{2r})}{p_1 \ldots p_{2r}} \log^{-a} 3g_{2r}.$$

It is easy to see that

$$\sum_{\substack{d \in \mathscr{D}_y^+, \mu(d)=1 \\ d|P(y^{1/s}), d^* \geqslant \sqrt{y/d}}} \frac{\omega(d)}{d} \log^{-a}(3y/d) = \sum_{r=1}^{\infty} h_{2r,y}(s) \quad \text{for} \quad s \geqslant 1, \tag{3.7}$$

where d^* is the least prime divisor of d (for $d = 1$, d^* means 1).

The following lemma is obvious.

Lemma 3. *We have*

$$h_{i+1,y}(s) = \sum_{y^{1/(i+3)} \leqslant p < y^{1/s}} \frac{\omega(p)}{p} h_{i,y/p}\left(\frac{\log(y/p)}{\log p}\right) \quad \textit{for} \quad i \geqslant 1,\ s \geqslant \frac{3-(-1)^i}{2},$$

(3.8)

$$h_{2n,y}(s) = h_{2n,y}(2) \quad \textit{for} \quad n \geqslant 1,\ 0 \leqslant s \leqslant 2;$$

$$g_{i+1,y}(s) = \sum_{y^{1/(i+2)} \leqslant p < y^{1/s}} \frac{\omega(p)}{p} g_{i,y/p}\left(\frac{\log(y/p)}{\log p}\right) \quad \textit{for} \quad i \geqslant 1,\ s \geqslant \frac{3+(-1)^i}{2},$$

(3.9)

$$g_{2n+1,y}(s) = g_{2n+1,y}(2) \quad \textit{for} \quad n \geqslant 1,\ 0 \leqslant s \leqslant 2.$$

The $\Omega(z)$ satisfies the recursive formula

$$\Omega(z) = 1 - \sum_{p<z} \frac{\omega(p)}{p} \Omega(p)$$

very similar to (3.1). Hence having replaced in (3.3) and (3.4) the sifting functions $S(\mathscr{A}_{p_1 \ldots p_i}, p_i)$ by $\dfrac{\omega(p_1 \ldots p_i)}{p_1 \ldots p_i} \Omega(p_i)$, we get

Lemma 4. *For $R \geqslant 1$ we have*

$$\sum_{\substack{d \in \mathscr{D}_y^- \\ d|P(y^{1/s})}} \mu(d) \frac{\omega(d)}{d} = \Omega(y^{1/s}) - \sum_{r=1}^{R} d_{2r,y}(s) - \sum_{\substack{p_{2R+2} < \ldots < p_1 < y^{1/s} \\ p_{2l} < g_{2l} \text{ for } l \leqslant R}} \frac{\omega(p_1 \ldots p_{2R+2})}{p_1 \ldots p_{2R+2}} \Omega(p_{2R+2}) \tag{3.10}$$

for $s \geqslant 1$ and

$$\sum_{\substack{d \in \mathscr{D}_y^+ \\ d|P(y^{1/s})}} \mu(d) \frac{\omega(d)}{d} = \Omega(y^{1/s}) + \sum_{r=1}^{R} d_{2r-1,y}(s) + \sum_{\substack{p_{2R+1} < \ldots < p_1 < y^{1/s} \\ p_{2l-1} < g_{2l-1} \text{ for } l \leqslant R}} \frac{\omega(p_1 \ldots p_{2R+1})}{p_1 \ldots p_{2R+1}} \Omega(p_{2R+1}) \tag{3.11}$$

for $s > 0$.

§ 4. The functions $G_a(s)$, $f(s)$ and $F(s)$. For $0 < a < 1 \leqslant s$ let us put

$$G_a(s) = \frac{1}{2} \int_s^{\infty} t^{-1} \left(1 - \frac{1}{t}\right)^{-a} G(t-1)\, dt$$

where

$$G(s) = \begin{cases} e^{-s} & \text{if} \quad s \geqslant 1, \\ e^{-2} & \text{if} \quad 0 \leqslant s < 1. \end{cases}$$

Lemma 5. *For $a = 1 - 1/(2-\log 2)e$ we have*

$$\beta := \sup e^s G_a(s) < 1. \tag{4.1}$$

Proof. First, let us note that for $s \geqslant 1$ we have

$$\int_s^{\infty} t^{-1} e^{-t}\, dt = -e^{-s}\log s + \int_s^{\infty} e^{-t} \log t\, dt$$

$$< -e^{-s}\log s + \int_s^{\infty} e^{-t}\left(\log(t+1) - \frac{1}{t+1}\right) dt = e^{-s} \log\left(1 + \frac{1}{s}\right).$$

Hence for $s \geqslant 2$ we get

$$e^s G_a(s) < \frac{1}{2} \int_{s-1}^{\infty} t^{-1} e^{s-t}\, dt < \frac{e}{2} \log 2 < 1$$

and for $1 \leqslant s \leqslant 2$ we have

$$R(s) := e^{s+1}\log 2 + e^s \int_s^2 t^{-1}\left(1 - \frac{1}{t}\right)^{-a} dt$$

$$> 2e^{s+2} G_a(2) + e^s \int_s^2 t^{-1} \left(1 - \frac{1}{t}\right)^{-a} dt = 2e^{s+2} G_a(s).$$

To prove the lemma it is enough to show that $R(s) < 2e^2$ for $1 \leqslant s \leqslant 2$. Let us put

$$r(s) = s^{-1}\left(1-\frac{1}{s}\right)^{-\alpha} e^s.$$

Hence

$$\frac{r'}{r}(s) = \frac{(s-1)^2-\alpha}{s(s-1)} < 0 \quad \text{for} \quad 1 < s < 1+\sqrt{\alpha},$$

that is $r(s)$ is decreasing in $1 < s < 1+\sqrt{\alpha}$. In the interval $1+\sqrt{\alpha} \leqslant s \leqslant 2$ we have $r(s) < e^s/(s-1) < 2e^2$ and hence

$$s_0 := \sup\{1 \leqslant s \leqslant 2;\ r(s) = 2e^2\} < 1+\sqrt{\alpha}.$$

We shall consider the intervals

$$I_1 = \{1 \leqslant s \leqslant s_0\} \quad \text{and} \quad I_2 = \{s_0 \leqslant s \leqslant 2\}$$

separately. In the interior of I_1 we have

$$\begin{aligned} e^{-s}R'(s) &= e\log 2 + \int_s^2 t^{-1}\left(1-\frac{1}{t}\right)^{-\alpha} dt - s^{-1}\left(1-\frac{1}{s}\right)^{-\alpha} \\ &< e\log 2 + \int_s^2 (t-1)^{-\alpha}dt - s^{-1}\left(1-\frac{1}{s}\right)^{-\alpha} \\ &= 2e - s^{-1}\left(1-\frac{1}{s}\right)^{-\alpha}\left\{1+\frac{(s-1)s^{1-\alpha}}{1-\alpha}\right\} \\ &\leqslant 2e - s^{-1}\left(1-\frac{1}{s}\right)^{-\alpha}\{1+3(s-1)\} \leqslant 2e - \frac{r(s)}{e} \leqslant 2e - \frac{r(s_0)}{e} = 0. \end{aligned}$$

Thus, $R(s)$ is decreasing and

$$R(s) \leqslant R(1) < e^2\log 2 + e\int_1^2 (t-1)^{-\alpha}dt = 2e^2 \quad \text{for} \quad s \in I_1.$$

On the other hand, $R(2) = e^3\log 2 < 2e^2$. Hence, if for some $s \in I_2$ the inequality $R(s) \geqslant 2e^2$ held, the function $R(s)$ would have the maximum in the interior of I_2, say at the point $s_1 \in I_2$. Then

$$0 = R'(s_1) = R(s_1) - r(s_1) > 2e^2 - 2e^2 = 0.$$

The contradiction completes the proof of $R(s) < 2e^2$.

Let us introduce the following functions

$$g_1(s) = \begin{cases} \sqrt{2}-\sqrt{s} & \text{if} \quad 0 \leqslant s \leqslant 2, \\ 0 & \text{if} \quad s \geqslant 2, \end{cases}$$

$$g_{i+1}(s) = \frac{1}{2}\int_s^\infty \frac{g_i(t-1)}{\sqrt{t(t-1)}}dt \quad \text{for} \quad i \geqslant 1,\ s \geqslant \frac{3+(-1)^i}{2},$$

$$g_{2n+1}(s) = g_{2n+1}(2) \quad \text{for} \quad n \geqslant 1,\ 0 \leqslant s \leqslant 2.$$

The functions $g_i(s)$ are clearly of class C^{i-1} on the half-lines $\left[\frac{3-(-1)^i}{2}, \infty\right)$ and have supports $\left[\frac{1+(-1)^i}{2}, i+1\right]$.

We shall prove the following

LEMMA 6. *For $i \geqslant 1$ and $s \geqslant \frac{1+(-1)^i}{2}$ we have*

$$g_i(s) \leqslant \sqrt{2}\beta^{i-1}e^{-s}. \tag{4.2}$$

Proof. For $i = 1$ (4.2) is obvious. Let us assume that (4.2) is true for some $i \geqslant 1$. It follows from (4.1) and the inductive assumption that for $s \geqslant 2$

$$g_{i+1}(s) \leqslant \sqrt{2}\beta^{i-1}G_{1/2}(s) < \sqrt{2}\beta^{i-1}\, G_{\hat{\alpha}}(s) < \sqrt{2}\beta^i e^{-s}.$$

In particular, for $0 \leqslant s \leqslant 2$ and $i \equiv 0 \pmod 2$, we have

$$g_{i+1}(s) = g_{i+1}(2) < \sqrt{2}\beta^i e^{-2} \leqslant \sqrt{2}\beta^i e^{-s}.$$

It follows from (4.1) that for $1 \leqslant s \leqslant 2$ and $i \equiv 1 \pmod 2$

$$g_{i+1}(s) = \frac{1}{2}\int_s^\infty \frac{g_i(t-1)}{\sqrt{t(t-1)}}dt + \frac{g_i(2)}{2}\int_s^2 \frac{dt}{t(t-1)} < \sqrt{2}\beta^{i-1}G_{1/2}(s) < \sqrt{2}\beta^i e^{-s}.$$

The proof is complete.

Owing to Lemma 6 we can introduce the functions $f(s)$ and $F(s)$ in the following way:

$$f(s) := \sum_{i=1}^\infty g_{2i}(s) \quad \text{for} \quad s \geqslant 1,$$

$$F(s) := \sum_{i=1}^\infty g_{2i-1}(s) \quad \text{for} \quad s \geqslant 0.$$

It follows from Lemma 6 that $f(s) \ll e^{-s}$ and $F(s) \ll e^{-s}$. The following integral equations hold

$$f(s) = \frac{1}{2}\int_s^\infty \frac{F(t-1)}{\sqrt{t(t-1)}}\,dt \quad \text{for} \quad s \geqslant 1, \tag{4.3}$$

$$F(s) = \frac{1}{2}\int_s^\infty \frac{f(t-1)}{\sqrt{t(t-1)}}\,dt \quad \text{for} \quad s \geqslant 2. \tag{4.4}$$

We shall show

LEMMA 7. *We have*

$$f(s) = \sqrt{s} - \sqrt{\frac{e^C}{\pi}}\int_1^s \frac{dt}{\sqrt{t(t-1)}} \quad \textit{for} \quad 1 \leqslant s \leqslant 3,$$

$$F(s) = 2\sqrt{\frac{e^C}{\pi}} - \sqrt{s} \quad \textit{for} \quad 0 \leqslant s \leqslant 2.$$

This lemma and equations (4.3), (4.4) determine the $f(s)$ and $F(s)$ completely. In virtue of Lemma 7 we can compute $f(s)$ and $F(s)$ step by step.

Proof of Lemma 7. It is enough to show

$$f(1) = 1; \quad F(0) = 2\sqrt{\frac{e^C}{\pi}},$$

because it follows from (4.3) and (4.4) that

$$F(s) = F(0) - \sqrt{s} \quad \text{for} \quad 0 \leqslant s \leqslant 2,$$

$$f(s) = f(1) - 1 + \sqrt{s} - \frac{F(0)}{2}\int_1^s \frac{dt}{\sqrt{t(t-1)}} \quad \text{for} \quad 1 \leqslant s \leqslant 3.$$

From (4.3) and (4.4) for $s > 2$ we have

$$\frac{d}{ds}2\sqrt{s}\big(f(s)+F(s)\big) = \frac{f(s)+F(s)}{\sqrt{s}} - \frac{f(s-1)+F(s-1)}{\sqrt{s-1}}.$$

Hence by $f(s) \ll e^{-s}$, $F(s) \ll e^{-s}$ we get

$$2\sqrt{s}\big(f(s)+F(s)\big) = \int_{s-1}^s \frac{f(x)+F(x)}{\sqrt{x}}\,dx \quad \text{for} \quad s \geqslant 2.$$

In particular, we have

$$2\sqrt{2}\big(f(2)+F(2)\big) = \int_1^2 \frac{f(x)+F(x)}{\sqrt{x}}\,dx$$

that is $f(1) = 1$. Hence we have also

$$\omega_1(s) := 1 + \frac{F(s)-f(s)}{2\sqrt{s}} = \frac{F(0)}{2}\left(1 + \frac{1}{2}\int_1^s \frac{dt}{\sqrt{t(t-1)}}\right) \quad \text{for} \quad 1 \leqslant s \leqslant 2$$

and

$$2s\omega_1'(s) = \omega_1(s-1) - \omega_1(s) \quad \text{for} \quad s > 2.$$

Let $\omega(s)$ be the continuous function satisfying the following differential equation with shifted argument:

$$\omega(s) = 1/\sqrt{s} \qquad (0 < s \leqslant 1),$$
$$2s\omega'(s) = \omega(s-1) - \omega(s) \qquad (s > 2).$$

It is easy to see that $\omega_1(s) = \dfrac{F(0)}{2}\omega(s)$. From Lemma 1.3.2 of [10] we get

$$1 = \omega_1(\infty) = \frac{1}{2}\sqrt{\frac{\pi}{e^C}}F(0)$$

which proves the lemma.

§ 5. Preliminary lemmata. By (Ω_2) using partial summation we have

$$-\frac{L}{\log w} + \frac{1}{2}\log\frac{\log z}{\log w} \tag{Ω_3}$$
$$\leqslant \sum_{w \leqslant p < z}\frac{\omega(p)}{p} \leqslant \sum_{w \leqslant p < z}\frac{\omega(p)}{p-\omega(p)} \leqslant \frac{K}{\log w} + \frac{1}{2}\log\frac{\log z}{\log w}.$$

Hence we get

$$\frac{\Omega(w)}{\Omega(z)} = \prod_{w \leqslant p < z}\left(1 - \frac{\omega(p)}{p}\right)^{-1} \geqslant \exp\sum_{w \leqslant p < z}\frac{\omega(p)}{p} \geqslant \sqrt{\frac{\log z}{\log w}}\,e^{-L/\log w},$$

$$\frac{\Omega(w)}{\Omega(z)} = \prod_{w \leqslant p < z}\left(1 + \frac{\omega(p)}{p-\omega(p)}\right) \leqslant \exp\sum_{w \leqslant p < z}\frac{\omega(p)}{p-\omega(p)} \leqslant \sqrt{\frac{\log z}{\log w}}\,e^{K/\log w}.$$

It follows from the above inequality that there exists the limit

$$\Omega = \lim_{z\to\infty}\Omega(z)\sqrt{\log z}$$

and

$$e^{-L/\log w} \leqslant \Omega^{-1}\Omega(w)\sqrt{\log w} \leqslant e^{K/\log w}. \tag{Ω_4}$$

From Mertens's formula

$$\prod_{p<z}\left(1-\frac{1}{p}\right) \sim e^{-C}\log^{-1}z,$$

we have

$$\Omega = e^{-C/2}\prod_{p}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1/2}.$$

By (Ω_3) using partial summation we obtain

LEMMA 8. *Let $B(x)$ be a positive, continuous and monotone function in the interval $w \leqslant x < z$. Then we have*

$$-LB\log^{-1}w \leqslant \sum_{w\leqslant p<z}\frac{\omega(p)}{p}B(p)-\frac{1}{2}\int_{w}^{z}\frac{B(t)}{t\log t}\,dt \leqslant KB\log^{-1}w,$$

where $B = \max(B(z), B(w))$.

We shall prove

LEMMA 9. *For $y>1$, $i\geqslant 1$ and $\frac{3-(-1)^i}{2}\leqslant s\leqslant i+2$ we have*

$$\sum_{y^{1/(i+3)}\leqslant p<y^{1/s}}\frac{\omega(p)}{p}G\left(\frac{\log(y/p)}{\log p}\right)\log^{-\alpha}(3y/p) \leqslant G_{\alpha}(s)\log^{-\alpha}3y+4iKG(s-1)\log^{-1}3y, \tag{5.1}$$

$$\sum_{y^{1/(i+2)}\leqslant p<y^{1/s}}\frac{\omega(p)}{p}g_i\left(\frac{\log(y/p)}{\log p}\right)\log^{-1/2}(3y/p) \leqslant g_{i+1}(s)\log^{-1/2}3y+4iKg_i(s-1)\log^{-1}3y, \tag{5.2}$$

$$\sum_{y^{1/(i+2)}\leqslant p<y^{1/s}}\frac{\omega(p)}{p}g_i\left(\frac{\log(y/p)}{\log p}\right)\log^{-1/2}(3y/p) \geqslant g_{i+1}(s)\log^{-1/2}3y-6iL^{\gamma}\beta^{i}e^{-s}\log^{-\alpha}3y. \tag{5.3}$$

Proof. It is enough to prove (5.3) because the lower estimations (5.1) and (5.2) are easy consequences of Lemma 8. For $L\geqslant\log 3y$ the right hand side of (5.3) is negative, so (5.3) is trivial. Let $L<\log 3y$ and $s_1/s = 1+(L/\log 3y)^{2\gamma}$. Then we have

$$g_{i+1}(s)-g_{i+1}(s_1) = \frac{1}{2}\int_{s}^{s_1}\frac{g_i(t-1)}{\sqrt{t(t-1)}}\,dt$$

$$\leqslant \frac{g_i(s-1)}{\sqrt{s}}\left(\sqrt{s_1-1}-\sqrt{s-1}\right)\leqslant g_i(s-1)\sqrt{\frac{s_1}{s}-1} = g_i(s-1)\left(\frac{L}{\log 3y}\right)^{\gamma},$$

$$\left(1-\frac{1}{s_1}\right)\log^3 3y \geqslant \frac{1}{2}\left(\frac{L}{\log 3y}\right)^{2\gamma}\log^3 3y \geqslant \frac{1}{2}L^{2(1-\gamma)}\log^{2\alpha}3y.$$

It follows from Lemma 8 that

$$\sum_{y^{1/(i+2)}\leqslant p<y^{1/s}}\frac{\omega(p)}{p}g_i\left(\frac{\log(y/p)}{\log p}\right)\log^{-1/2}(3y/p)$$

$$\geqslant \sum_{(3y)^{1/(i+2)}\leqslant p<(3y)^{1/s_1}}\frac{\omega(p)}{p}g_i\left(\frac{\log(3y/p)}{\log p}\right)\log^{-1/2}(3y/p)$$

$$\geqslant g_{i+1}(s_1)\log^{-1/2}3y - L\frac{i+2}{\log 3y}\,\frac{g_i(s_1-1)}{\sqrt{\left(1-\frac{1}{s_1}\right)\log 3y}}$$

$$\geqslant g_{i+1}(s)\log^{-1/2}3y - g_i(s-1)L^{\gamma}(\sqrt{2}i+2\sqrt{2}+1)\log^{-\alpha}3y$$

which completes the proof.

LEMMA 10. *For $y>1$, $i\geqslant 1$ and $\frac{1-(-1)^i}{2}\leqslant s\leqslant i+2$ we have*

$$h_{i,y}(s) \ll \beta_1^i e^{K/2-s}\log^{-\alpha}3y,$$

where $2\beta_1 = 1+\beta$.

Proof. From (5.1) we get

$$h_{1,y}(s)\leqslant e^2\sum_{y^{1/3}\leqslant p<y^{1/s}}\frac{\omega(p)}{p}G\left(\frac{\log(y/p)}{\log p}\right)\log^{-\alpha}(3y/p) \ll K\log^{-\alpha}3y$$

which proves the lemma for $i=1$. Let us assume that for some $i\geqslant 1$ we have

$$h_{i,y}(s) < C_1\beta_1^i e^{K/2-s}\log^{-\alpha}3y. \tag{5.4}$$

If C_1 is a sufficiently large positive constant we shall prove

$$h_{i+1,y}(s) < C_1\beta_1^{i+1}e^{K/2-s}\log^{-\alpha}3y. \tag{5.5}$$

For $s\leqslant i+3$ we have

$$h_{i+1,y}(s)\leqslant\frac{1}{(i+1)!}\left(\sum_{p<y}\frac{\omega(p)}{p}\right)^{i+1}$$

$$< 4^{-i-1}\exp\left(4\sum_{p<y}\frac{\omega(p)}{p}\right) < \left(\frac{e}{4}\right)^{i+1}e^{K/4+4e^{16}+2-s}\log^2 y.$$

Hence (5.5) is trivial if

$$\log^{2+\alpha}3y < C_1\left(\frac{4\beta_1}{e}\right)^{i+1}e^{K/4-4e^{16}-2}. \tag{5.6}$$

Let us assume, that (5.6) is false. Therefore for sufficiently large C_1 we have

$$\log^{1-\alpha}3y > 8e(1-\beta)^{-i}iK.$$

Now, it follows from (5.1), (5.4)

$$h_{i+1,y}(s) \leqslant C_1\beta_1^i e^{K/2} \sum_{y^{1/(i+3)}\leqslant p<y^{1/s}} \frac{\omega(p)}{p} G\left(\frac{\log(y/p)}{\log p}\right)\log^{-a}(3y/p)$$
$$\leqslant C_1\beta_1^i e^{K/2-s}\left\{e^s G_a(s)+\frac{4iKe}{\log^{1-a}3y}\right\}\log^{-a}3y$$
$$\leqslant C_1\beta_1^{i+1}e^{K/2-s}\log^{-a}3y, \quad \text{for } s\geqslant \frac{3+(-1)^i}{2}.$$

For $\frac{1+(-1)^i}{2}\leqslant s<\frac{3+(-1)^i}{2}$ we have $2\nmid i$ and $h_{i+1,y}(s) = h_{i+1,y}(2)$. This completes the proof.

From Lemma 10 we obtain

LEMMA 11. *For $y\geqslant z>1$ we have*

$$\sum_{\substack{d\in\mathscr{D}_y^+,\, d|P(z)\\ \mu(d)=1,\, d^*>\sqrt{y/d}}} \frac{\omega(d)}{d}\log^{-a}(3y/d) \ll e^{K/2-s}\log^{-a}3y,$$

where $s = \log y/\log z$.

LEMMA 12. *For $y>1$, $i\geqslant 1$ and $\frac{1+(-1)^i}{2}\leqslant s\leqslant i+1$ we have*

$$g_{i,y}(s)\leqslant g_i(s)\log^{-1/2}3y+O(\beta_1^i e^{K/2-s}\log^{-a}3y).$$

Proof. From Lemma 8 we obtain

$$g_{1,y}(s)\leqslant \sum_p \frac{\omega(p)}{p}\log^{-1/2}p \ll K$$

for $y>1$ and

$$g_{1,y}(s) = \sum_{y^{1/2}\leqslant p<y^{1/s}} \frac{\omega(p)}{p}\log^{-1/2}p \leqslant g_1(s)\log^{-1/2}y+2\sqrt{2}K\log^{-3/2}y$$
$$\leqslant g_1(s)\log^{-1/2}3y+O(K\log^{-3/2}3y)$$

for $y\geqslant 3$. The above two estimations prove the lemma for $i=1$.

Let us assume

$$g_{i,y}(s) < g_i(s)\log^{-1/2}3y+C_2\beta_1^i e^{K/2-s}\log^{-a}3y \tag{5.7}$$

for some $i\geqslant 1$. If C_2 is a sufficiently large positive constant we shall prove

$$g_{i+1,y}(s) < g_{i+1}(s)\log^{-1/2}3y+C_2\beta_1^{i+1}e^{K/2-s}\log^{-a}3y. \tag{5.8}$$

For $s\leqslant i+2$ we have

$$g_{i+1,y}(s)\leqslant \frac{1}{(i+1)!}\left(\sum_{p<y}\frac{\omega(p)}{p}\right)^{i+1}$$
$$< 4^{-i-1}\exp\left(4\sum_{p<y}\frac{\omega(p)}{p}\right) < \left(\frac{e}{4}\right)^{i+1} e^{K/4+4c^{16}+1-s}\log^2 y.$$

Hence (5.8) is trivial if

$$\log^{2+a}3y < C_2\left(\frac{4\beta_1}{e}\right)^{i+1} e^{K/4-4c^{16}-1}. \tag{5.9}$$

Let us assume that (5.9) is false. Therefore for sufficiently large C_2 we have

$$\log^{1-a}3y > 16eiK(1-\beta)^{-1}.$$

Now, it follows from (5.1), (5.2) and (5.7)

$$g_{i+1,y}(s)\leqslant \sum_{y^{1/(i+2)}\leqslant p<y^{1/s}} \frac{\omega(p)}{p}\left\{g_i\left(\frac{\log(y/p)}{\log p}\right)\log^{-1/2}(3y/p)+\right.$$
$$\left.+C_2\beta_1^i e^{K/2}G\left(\frac{\log(y/p)}{\log p}\right)\log^{-a}(3y/p)\right\}$$
$$\leqslant g_{i+1}(s)\log^{-1/2}3y+\frac{4iKg_i(s-1)}{\log 3y}+$$
$$+C_2\beta_1^i e^{K/2-s}\log^{-a}3y\left\{e^s G_a(s)+\frac{4eiK}{\log^{1-a}3y}\right\}$$
$$\leqslant g_{i+1}(s)\log^{-1/2}3y+C_2\beta_1^i e^{K/2-s}\log^{-a}3y\left\{e^s G_a(s)+\frac{8eiK}{\log^{1-a}3y}\right\}$$
$$\leqslant g_{i+1}(s)\log^{-1/2}3y+C_2\beta_1^{i+1}e^{K/2-s}\log^{-a}3y$$

for $s\geqslant \frac{3-(-1)^i}{2}$. For $\frac{1-(-1)^i}{2}\leqslant s<\frac{3-(-1)^i}{2}$ we have $2\nmid i$, $g_{i+1,y}(s) = g_{i+1,y}(2)$ and $g_{i+1}(s) = g_{i+1}(2)$. This completes the proof.

LEMMA 13. *For $y>1$, $i\geqslant 1$ and $\frac{1+(-1)^i}{2}\leqslant s<i+1$ we have*

$$g_{i,y}(s)\geqslant g_i(s)\log^{-1/2}3y+O(L^\gamma\beta_1^{i-1}e^{K/2-s}\log^{-a}3y).$$

Proof. For $L\geqslant \log 3y$ we have

$$g_{1,y}(s)\geqslant 0\geqslant g_1(s)\log^{-1/2}3y-\sqrt{2}(L\log^{-1}3y)^\gamma\log^{-1/2}3y.$$

For $L < \log 3y$ from Lemma 8 we obtain

$$g_{1,y}(s) \geqslant g_1(s)\log^{-1/2} y - 2\sqrt{2}L\log^{-3/2} y \geqslant g_1(s)\log^{-1/2}3y + O(L^\gamma \log^{-\alpha}3y).$$

The above two estimations prove the lemma for $i = 1$. Let us assume

$$g_{i,y}(s) \geqslant g_i(s)\log^{-1/2}3y - C_3 L^\gamma \beta_1^{i-1} e^{K/2-s}\log^{-\alpha}3y, \tag{5.10}$$

for some $i \geqslant 1$. If C_3 is a sufficiently large positive constant we shall prove

$$g_{i+1,y}(s) \geqslant g_{i+1}(s)\log^{-1/2}3y - C_3 L^\gamma \beta_1^{i} e^{K/2-s}\log^{-\alpha}3y. \tag{5.11}$$

For

$$\log^{1/2+\alpha}3y < \frac{\sqrt{2}}{2}C_3 L^\gamma \left(\frac{\beta_1}{\beta}\right)^i e^{K/2} \tag{5.12}$$

the right hand side of (5.11) is negative, so (5.11) is trivial. Let us assume that (5.12) is false. Therefore for sufficiently large C_3 we have

$$\log^{1-\alpha}3y > 16eiK(1-\beta)^{-1} \quad \text{and} \quad C_3 > 24i\left(\frac{\beta}{\beta_1}\right)^{i-1}.$$

Now, it follows from (5.1), (5.3) and (5.10)

$$g_{i+1,y}(s) \geqslant \sum_{y^{1/(i+2)} \leqslant p < y^{1/s}} \frac{\omega(p)}{p}\left\{g_i\left(\frac{\log(y/p)}{\log p}\right)\log^{-1/2}(3y/p) - \right.$$

$$\left. - C_3 L^\gamma \beta_1^{i-1} e^{K/2} G\left(\frac{\log(y/p)}{\log p}\right)\log^{-\alpha}(3y/p)\right\}$$

$$\geqslant g_{i+1}(s)\log^{-1/2}3y - 6iL^\gamma \beta^i e^{-s}\log^{-\alpha}3y -$$

$$- C_3 L^\gamma \beta_1^{i-1} e^{K/2-s}\log^{-\alpha}3y\left\{e^s G_\alpha(s) + \frac{4eiK}{\log^{1-\alpha}3y}\right\}$$

$$\geqslant g_{i+1}(s)\log^{-1/2}3y - C_3 L^\gamma \beta_1^{i-1} e^{K/2-s}\log^{-\alpha}3y \times$$

$$\times \left\{e^s G_\alpha(s) + 6i\left(\frac{\beta_1}{\beta}\right)^{i-1} C_3^{-1} + \frac{4eiK}{\log^{1-\alpha}3y}\right\}$$

$$\geqslant g_{i+1}(s)\log^{-1/2}3y - C_3 L^\gamma \beta_1^{i} e^{K/2-s}\log^{-\alpha}3y \quad \text{for} \quad s \geqslant \frac{3-(-1)^i}{2}.$$

For $\frac{1-(-1)^i}{2} \leqslant s < \frac{3-(-1)^i}{2}$ we have $2 \nmid i$, $g_{i+1,y}(s) = g_{i+1,y}(2)$ and $g_{i+1}(s) = g_{i+1}(2)$. This completes the proof.

LEMMA 14. *For $y > 1$, $i \geqslant 1$ and $\frac{1+(-1)^i}{2} \leqslant s \leqslant i+1$ we have*

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s) \leqslant g_i(s) + O(\beta_1^i e^{2K-s}\log^{-\gamma}3y).$$

Proof. For $1 \geqslant \vartheta \geqslant 0$, $u \geqslant 0$ we have

$$e^{u\vartheta} - 1 = \sum_{n=1}^{\infty}\frac{(u\vartheta)^n}{n!} < \vartheta\sum_{n=1}^{\infty}\frac{u^n}{n!} = \vartheta(e^u - 1).$$

Hence for $p < y^{1/s}$ we get

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,\Omega(p) \leqslant \sqrt{\frac{\log y}{\log p}}\,e^{K/\log p} \leqslant \sqrt{\frac{\log 3y}{\log p}}\left(1 + \frac{2e^{K/\log 2}}{\log p}\right).$$

Hence by definition $d_{i,y}(s)$ we obtain

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s)\log^{-1/2}3y$$

$$\leqslant \sum_{\substack{q_i \leqslant p_i < \ldots < p_i < y^{1/s} \\ \ldots\ldots\ldots\ldots}} \frac{\omega(p_1\ldots p_i)}{p_1\ldots p_i}\left(\log^{-1/2}p_i + 2e^{K/\log 2}\log^{-\alpha}p_i\right).$$

Since

$$\sum_{p \geqslant x}\frac{\omega(p)}{p}\log^{-\alpha}p \leqslant 5K\log^{-\alpha}3x \tag{5.13}$$

it follows

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s)\log^{-1/2}3y \leqslant \begin{cases} g_{i,y}(s) + 30Ke^{K/\log 2}\log^{-\alpha}3y & \text{if} \quad i = 1, \\ g_{i,y}(s) + 10Ke^{K/\log 2}h_{i-1,y}(s) & \text{if} \quad i > 1. \end{cases}$$

Hence by Lemmata 10 and 12 we obtain Lemma 14.

LEMMA 15. *For $y > 1$, $i \geqslant 1$ and $\frac{1+(-1)^i}{2} \leqslant s \leqslant i+1$ we have*

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s) \geqslant g_i(s) + O\left(\left(\frac{L}{\log 3y}\right)^\gamma \beta_1^i e^{K-s}\right).$$

Proof. For $p < y^{1/s}$ we have

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,\Omega(p)$$

$$\geqslant \sqrt{\frac{\log y}{\log p}}\,e^{-L/\log p} \geqslant \sqrt{\frac{\log y}{\log p}}\max\left(0, 1 - \frac{L}{\log p}\right) \geqslant \sqrt{\frac{\log y}{\log p}}\left(1 - \left(\frac{L}{\log p}\right)^\gamma\right).$$

Hence by definition $d_{i,y}(s)$ we obtain

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s)\log^{-1/2}y$$

$$\geqslant \sum_{\substack{q_i \leqslant p_i < \ldots < p_1 < y^{1/s} \\ \ldots\ldots\ldots\ldots}} \frac{\omega(p_1\ldots p_i)}{p_1\ldots p_i}\left(\log^{-1/2}p_i - L^\gamma\log^{-\alpha}p_i\right).$$

It follows from (5.13)

$$\sqrt{s}\,\Omega^{-1}(y^{1/s})\,d_{i,y}(s)\log^{-1/2}y \geqslant \begin{cases} g_{i,y}(s)-5KL^{\gamma}h_{i-1}(s-1) & \text{if} \quad i>1,\\ g_{i,y}(s)-15KL^{\gamma}\log^{-a}3y & \text{if} \quad i=1.\end{cases}$$

Hence by Lemmata 10 and 13 we obtain Lemma 15 for $y \geqslant 3$. For $1 \leqslant y \leqslant 3$ lemma is trivial.

§ 6. Proof of Theorem 1. From Lemmata 14 and 15 we have

LEMMA 16. *For $y>1$ and $s=\log y/\log z>0$ we have*

$$S^{+}(\mathscr{A},z) \leqslant \Omega(z)X\left\{1+\frac{F(s)}{\sqrt{s}}+O(e^{2K-s}\log^{-\gamma}3y)\right\}+\sum_{\substack{d\in\mathscr{D}_y^{+}\\ d\mid P(z)}} R(\mathscr{A},z),$$

$$S^{+}(\mathscr{A},z) \geqslant \Omega(z)X\left\{1+\frac{F(s)}{\sqrt{s}}+O\left(e^{K-s}\left(\frac{L}{\log 3y}\right)^{\gamma}\right)\right\}+\sum_{\substack{d\in\mathscr{D}_y^{+}\\ d\mid P(z)}} R(\mathscr{A},z).$$

LEMMA 17. *For $y>1$ and $s=\log y/\log z\geqslant 1$ we have*

$$S^{-}(\mathscr{A},z) \leqslant \Omega(z)X\left\{1-\frac{f(s)}{\sqrt{s}}+O\left(e^{K-s}\left(\frac{L}{\log 3y}\right)^{\gamma}\right)\right\}+\sum_{\substack{d\in\mathscr{D}_y^{-}\\ d\mid P(z)}} R(\mathscr{A},z),$$

$$S^{-}(\mathscr{A},z) \geqslant \Omega(z)X\left\{1-\frac{f(s)}{\sqrt{s}}+O(e^{2K-s}\log^{-\gamma}3y)\right\}+\sum_{\substack{d\in\mathscr{D}_y^{-}\\ d\mid P(z)}} R(\mathscr{A},z).$$

It is easy to see that for $z \leqslant y$, $d\mid P(z)$, $d\in\mathscr{D}_y^{-}\cup\mathscr{D}_y^{+}$ we have $d<y$. Therefore Theorem 1 follows from (3.5) and Lemmata 16 and 17.

§ 7. Proof of Theorem 2. Let $z\leqslant y<A$, $a\in\mathscr{A}\Rightarrow a'\equiv 1 \pmod 4$ and $p\in\mathscr{P}\Rightarrow p\equiv -1 \pmod 4$. From (3.4) we get

$$S^{+}(\mathscr{A},z)-S(\mathscr{A},z) \leqslant \sum_{\substack{d\in\mathscr{D}_y^{+},\,d\mid P(z)\\ \mu(d)=1,\,d^{*}>\sqrt{y/d}}}\ \sum_{\substack{\sqrt{y/d}\leqslant p<\sqrt{A/d}\\ p<d^{*},\,p\in\mathscr{P}}} S(\mathscr{A}_{pd},p) \tag{7.1}$$

and from Lemma 16 we have

$$S(\mathscr{A}_{pd},p) \leqslant S^{+}(\mathscr{A}_{pd},p^{1/3}) \tag{7.2}$$
$$\leqslant O\left(e^{2K}\frac{\omega(pd)}{pd}\,\Omega(p^{1/3})X\right)+\sum_{\substack{d_1\in\mathscr{D}^{+}_{p^{1/3}}\\ d_1\mid P(p^{1/3})}} |R(\mathscr{A}_{pd},d_1)|.$$

It is obvious that $(d_1d, pd_1, pd)=1$ and taking $X_{\mathscr{A}_{pd}}=X\dfrac{\omega(pd)}{pd}$ we find $R(\mathscr{A}_{pd},d_1)=R(\mathscr{A},pdd_1)$. Moreover

$$pdd_1<p^{4/3}d<\left(\frac{A}{d}\right)^{2/3}d<\left(\frac{A}{y}\right)^{2/3}y.$$

From Lemma 8 we have

$$\sum \frac{\omega(p)}{p}\log^{-1/2}p<2K+1$$

and

$$\sum_{\sqrt{y/d}\leqslant p<\sqrt{A/d}} \frac{\omega(p)}{p}\log^{-1/2}p$$
$$\leqslant \sqrt{\frac{2}{\log(y/d)}}-\sqrt{\frac{2}{\log(A/d)}}+\frac{2\sqrt{2}K}{\log^{3/2}(y/d)}$$
$$\leqslant \frac{\sqrt{2}\log(A/y)}{\sqrt{\log(y/d)\log(A/d)}}+\frac{2\sqrt{2}K}{\log^{3/2}(y/d)} \leqslant \frac{\sqrt{2}}{\sqrt{\log(y/d)}}\left(\frac{\log(A/y)}{\log(A/d)}\right)^{\gamma}+\frac{2\sqrt{2}K}{\log^{3/2}(y/d)}$$
$$\leqslant \sqrt{2}\,\frac{\log^{\gamma}(A/y)}{\log^{a}(y/d)}+\frac{2\sqrt{2}K}{\log^{3/2}(y/d)}.$$

Hence

$$\sum_{\sqrt{y/d}\leqslant p<\sqrt{A/d}} \frac{\omega(p)}{p}\log^{-1/2}p \leqslant 5K\,\frac{\log^{\gamma}(3A/y)}{\log^{a}(3y/d)}. \tag{7.3}$$

Since for $z\geqslant p$

$$\Omega(p^{1/3})=\Omega(\max(2,p^{1/3}))\leqslant \Omega(z)\sqrt{\frac{3\log z}{\log p}}\;e^{K/\log 2}$$

it follows from (7.1)–(7.3) and Lemma 11

$$S^{+}(\mathscr{A},z)-S(\mathscr{A},z)$$
$$\leqslant O\left(\Omega(z)\sqrt{\log z}\,Xe^{(2+\log^{-1}2)K}K\log^{\gamma}(3A/y)\right)\sum_{\substack{d\in\mathscr{D}_y^{+},\,d\mid P(z)\\ \mu(d)=1,\,d^{*}>\sqrt{y/d}}} \frac{\omega(d)}{d}\log^{-a}(3y/d)+$$
$$+\sum_{\substack{d<(yA^2)^{1/3}\\ d\mid P(z)}} |R(\mathscr{A},d)|$$
$$\leqslant O\left(\Omega(z)Xe^{4K-s_1}\left(\frac{\log(3A/y)}{\log 3y}\right)^{\gamma}\right)+\sum_{\substack{d<(yA^2)^{1/3}\\ d\mid P(z)}} |R(\mathscr{A},d)|.$$

Hence by Lemma 16 we get

$$S(\mathscr{A},z) \tag{7.4}$$
$$=\Omega(z)X\left\{1+\frac{F(s_1)}{\sqrt{s_1}}+O\left(\left(\frac{L+\log(A/y)}{\log 3y}\right)^{\gamma}e^{4K-s_1}\right)\right\}+\theta\sum_{\substack{d<(yA^2)^{1/3}\\ d\mid P(z)}} |R(\mathscr{A},d)|,$$

where $1<z\leqslant y<A$, $s_1=\log y/\log z\geqslant 1$.

Let $Q^3 < z$, $y = AQ^{-3}$, $s = \log A/\log z$. Then

$$s/s_1 = \log A/\log y = 1+3\log Q/\log y < 1+1/s_1,$$
$$\left|\frac{F(s)}{\sqrt{s}} - \frac{F(s_1)}{\sqrt{s_1}}\right| \leqslant \left(\frac{s}{s_1}-1\right) f(s_1-1) \leqslant e^{-s}\frac{\log Q}{\log y}.$$

Hence and from (7.4) we obtain Theorem 2.

§ 8. Proof of Theorem 3. From Theorem 2 we obtain

$$\sum_{a\in\mathscr{A}} b^*(a) = S(\mathscr{A},\sqrt{A}) = \Omega(\sqrt{A})X\left\{1+\frac{F(2)}{\sqrt{2}}+O\left(e^{4K}\left(\frac{L+\log Q}{\log A}\right)^{\gamma}\right)\right\} + \\ + \theta\sum_{\substack{d<A/Q\\ d|P(\sqrt{A})}} |R(\mathscr{A},d)|.$$

Since

$$\frac{\pi}{4} = L(1,\chi_4) = \prod_p\left(1-\frac{\chi_4(p)}{p}\right)^{-1}$$

it follows

$$(\sqrt{2}+F(2))\Omega = \frac{2}{\sqrt{\pi}}\prod_p\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1/2}$$
$$= \sqrt{2}\prod_{p\equiv -1 (\mathrm{mod}\, 4)}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1/2}\left(1+\frac{1}{p}\right).$$

On the other hand from (Ω_4) we have

$$\left(1+\frac{F(2)}{\sqrt{2}}\right)\Omega(\sqrt{A}) \leqslant (\sqrt{2}+F(2))\,\Omega\log^{-1/2}A e^{2K/\log A}$$
$$\leqslant (\sqrt{2}+F(2))\,\Omega\log^{-1/2}A\left(1+\frac{e^{2K}}{\log A}\right),$$
$$\left(1+\frac{F(2)}{\sqrt{2}}\right)\Omega(\sqrt{A}) \geqslant (\sqrt{2}+F(2))\,\Omega\log^{-1/2}A e^{-2L/\log A}$$
$$\geqslant (\sqrt{2}+F(2))\,\Omega\log^{-1/2}A\left(1-\left(\frac{2L}{\log A}\right)^{\gamma}\right)$$

which completes the proof.

§ 9. Proof of Theorem 4. Let r be the square of an integer with all prime divisors $\equiv -1 (\mathrm{mod}\, 4)$ and let us put

$$\omega'(p^a) = \begin{cases}\omega(p) & \text{if } p\in\mathscr{P},\ p\nmid r,\\ 1 & \text{if } p\in\mathscr{P},\ p\mid r, \qquad (a\geqslant 1),\\ 0 & \text{if } p\notin\mathscr{P},\end{cases}$$
$$\mathscr{A}' = \{a/r;\ a\in\mathscr{A}_r\}; \qquad A' := \max_{a'\in\mathscr{A}'} a'.$$

Therefore

$$\sum_{w\leqslant p<z}\frac{\omega'(p)}{p-\omega'(p)}\log p \leqslant \sum_{w\leqslant p<z}\frac{\omega(p)}{p-\omega(p)}\log p + \sum_{p|r}\frac{\log p}{(p-2)(1+\omega(p))}$$
$$\leqslant K'+\frac{1}{2}\log\frac{z}{w},$$

$$\sum_{w\leqslant p<z}\frac{\omega'(p)}{p}\log p \geqslant \sum_{w\leqslant p<z}\frac{\omega(p)}{p}\log p - \log r \geqslant -L'+\frac{1}{2}\log\frac{z}{w}.$$

For $X_{\mathscr{A}'} = X' = \dfrac{\omega(r)}{r}X$ we have

$$R(\mathscr{A}',d) = |\mathscr{A}'_d| - \frac{\omega'(d)}{d}X' = |\mathscr{A}_{rd}| - \frac{\omega'(d)\,\omega(r)}{dr}X = R(\mathscr{A},rd).$$

We shall prove

$$\text{(9.1)}\qquad S(\mathscr{A}',\sqrt{A'}) = \frac{\omega(r)}{r}X(\sqrt{2}+F(2))\,\Omega'\log^{-1/2}A\times$$
$$\times\left\{1+O\left(e^{6K'}\left(\frac{L+\log Q}{\log A}\right)^{\gamma}\right)\right\} + 2\theta A/Q + r + \theta\sum_{\substack{rd<A/Q\\ d|P(\sqrt{A})}}|R(\mathscr{A},rd)|$$

where

$$\Omega' = e^{-C/2}\prod_p\left(1-\frac{\omega'(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1/2}.$$

Let us consider two cases

I. $r\geqslant Q^2$ or $A'\leqslant 2A/(Q+r)$. We have

$$S(\mathscr{A}',\sqrt{A'}) \leqslant |\mathscr{A}'| = |\mathscr{A}_r| \leqslant \min(A', A/r) \leqslant 2A/(Q+r).$$

Moreover

$$\frac{\omega(r)}{r}X(\sqrt{2}+F(2))\,\Omega'\log^{-1/2}A\left\{1-e^{6K'}\left(\frac{\log Q}{\log A}\right)^{\gamma}\right\} \leqslant 0 \leqslant S(\mathscr{A}',\sqrt{A'})$$

if $e^{6K'}\left(\dfrac{\log Q}{\log A}\right)^{\gamma} \geqslant 1$ and

$$\frac{\omega(r)}{r}X(\sqrt{2}+F(2))\,\Omega'\log^{-1/2}A - |R(\mathscr{A},r)| - 2A/(Q+r)$$
$$\leqslant 4e^{K'/\log 2}\log^{-1/2}A\,\frac{\omega(r)}{r}X - \frac{\omega(r)}{r}X + |\mathscr{A}_r| - \frac{2A}{Q+r} \leqslant 0 \leqslant S(\mathscr{A}',\sqrt{A'})$$

if $e^{6K'}\left(\dfrac{\log Q}{\log A}\right)^{\gamma} < 1$. This shows that in the case I (9.1) is trivial.

II. $r < Q^2$ and $A' > 2A/(Q+r)$. We have $Q^6 < A'$ and

$$\log^{-1/2} A' = \log^{-1/2} A \left(1 + O\left(\frac{\log Q}{\log A}\right)\right).$$

Since the function ω' fulfils (Ω_1) and (Ω_2) with constants L' and K' and the set $\mathscr{A}'$ fulfils the assumptions of Theorem 2 it follows that

$$\begin{aligned} S(\mathscr{A}', \sqrt{A'}) &= \Omega'(\sqrt{A'})\, X' \left\{1 + \frac{F(2)}{\sqrt{2}} + O\left(e^{4K'}\left(\frac{L' + \log Q}{\log A}\right)^{\gamma}\right)\right\} + \theta \sum_{\substack{rd < A/Q \\ d \mid P(\sqrt{A})}} |R(\mathscr{A}, rd)| \\ &= \big(\sqrt{2} + F(2)\big)\, \Omega' \frac{\omega(r)}{r} X \log^{-1/2} A \left\{1 + O\left(e^{6K'}\left(\frac{L + \log Q}{\log A}\right)^{\gamma}\right)\right\} + \\ &\qquad + \theta \sum_{\substack{rd < A/Q \\ d \mid P(\sqrt{A})}} |R(\mathscr{A}, rd)| \end{aligned}$$

which completes the proof of (9.1).

Like in the proof of Theorem 3 we have

$$\big(\sqrt{2} + F(2)\big)\, \Omega' \frac{\omega(r)}{r} = \sqrt{2} \prod_{p \equiv -1 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2} \left(1 - \frac{\omega(p)}{p}\right)\left(1 + \frac{1}{p}\right) \lambda(r)$$

where

$$\lambda(r) = \frac{\omega(r)}{r} \prod_{\substack{p \mid r \\ p \equiv -1 \pmod 4}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{\omega(p)}{p}\right)^{-1}.$$

We have also

$$\sum_r \lambda(r) = \prod_{p \equiv -1 \pmod 4} \left(1 + \frac{1 - \omega(p)}{p}\right)\left(1 - \frac{\omega(p)}{p}\right)^{-1}\left(1 + \frac{1}{p}\right)^{-1}$$

and

$$\sum_r \lambda(r) e^{6K'} = e^{6K} \sum_r \lambda(r) \prod_{p \mid r} p^{\frac{6}{(p-2)(1+\omega(p))}} \ll e^{6K} \sum_r \lambda(r).$$

Since

$$\sum_{a \in \mathscr{A}} b(a) = \sum_r \sum_{a \in \mathscr{A}_r} b^*\left(\frac{a}{r}\right) = \sum_r S(\mathscr{A}', \sqrt{A'})$$

the proof of Theorem 4 is thus complete.

§ 10. Proof of Corollary 1. Let $a = 0$ if $2 \mid k$ and $a \geqslant 0$ if $2 \nmid k$. Let us put

$$\mathscr{A} = \{M \leqslant n < M + N;\ (n + c, P) = 1,\ n \equiv l \pmod k,\ n \equiv 2^a \pmod{2^{a+2}}\}.$$

It is obvious that $a \in \mathscr{A} \Rightarrow a' \equiv 1 \pmod 4$ and $(a, kc, kP) = 1$. For $(d, 2kc, 2kP) = 1$ we have

$$\mathscr{A}_d = \{M \leqslant n < M + N;\ (n + c, P) = 1,\ n \equiv l \pmod k,\ n \equiv 2^a \pmod{2^{a+2}},\ n \equiv 0 \pmod d\}.$$

We shall estimate the cardinality of $\mathscr{A}_d$ using the linear sieve. Let us put

$$\mathscr{M} = \{M + c \leqslant m < M + N + c,\ m \equiv c + l \pmod k,\ m \equiv c + 2^a \pmod{2^{a+2}},\ m \equiv c \pmod d\}.$$

It is obvious that $m \in \mathscr{M} \Rightarrow (m, d, P) = 1$. Therefore for $P_1 = \prod_{p \mid P,\, p \nmid d} p$ we have

$$|\mathscr{A}_d| = |\{m \in \mathscr{M};\ (m, P_1) = 1\}|.$$

For $b \mid P_1$ we have

$$\mathscr{M}_b = \{M + c \leqslant m < M + N + c;\ m \equiv c + l \pmod k,\ m \equiv c + 2^a \pmod{2^{a+2}},\ m \equiv c \pmod d,\ m \equiv 0 \pmod b\}.$$

The system of four congruences involved in the definition of $\mathscr{M}_b$ has exactly one solution mod $[2^{a+2}, k]bd$. Hence

$$\left| |\mathscr{M}_b| - \frac{N}{[2^{a+2}, k]bd} \right| \leqslant 1 \quad \text{for} \quad b \mid P_1.$$

Using the fundamental lemma from [2] we get

$$|\mathscr{A}_d| = \frac{N}{[2^{a+2}, k]d} \prod_{\substack{p \mid P \\ p \nmid d}} \left(1 - \frac{1}{p}\right) \{1 + O(\log^{-2} N)\}$$

$$\text{for} \quad d < N^{1-5\varepsilon},\ 2^{a+2} < N^{2\varepsilon}.$$

In particular, we have

$$N^{1+\varepsilon} \geqslant M + N \geqslant A \geqslant |\mathscr{A}| \gg \frac{N}{2^a k \log N} \geqslant N^{1-4\varepsilon}.$$

Let us substitute in Theorem 4

$$Q = N^{6\varepsilon}, \qquad X = \frac{N}{[2^{a+2}, k]} \prod_{p \mid P} \left(1 - \frac{1}{p}\right),$$

$$\omega(p^\alpha) = \begin{cases} \dfrac{p}{p-1} & \text{if} \quad p \equiv -1 \pmod 4,\ p \mid P,\ p \nmid c, \\ 1 & \text{if} \quad p \equiv -1 \pmod 4,\ p \nmid kP, \qquad (\alpha > 0) \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\frac{A}{Q} \ll N^{1-5\varepsilon} \ll \prod_{p|P}\left(1-\frac{1}{p}\right)\frac{N}{[2^{a+2}, k]\log N}, \qquad |R(\mathscr{A}, d)| \ll \frac{\prod_{p|P}\left(1-\frac{1}{p}\right)N}{2^{a}k\varphi(d)\log^2 N}.$$

Since the function ω fulfils (Ω_1) and (Ω_2) with constants

$$K \ll 1, \qquad L \ll \log\log(c, P)k \ll \log\log N$$

it follows from Theorem 4 that

$$\sum_{a\in\mathscr{A}} b(a) = \sqrt{2}\prod_{p\equiv -1 (\mathrm{mod}\, 4)}\left(1-\frac{1}{p^2}\right)^{-1/2}\prod_{p|P}\left(1-\frac{1}{p}\right)\prod_{\substack{p|(P,c)\\ p\equiv -1(\mathrm{mod}\, 4)}}\left(1+\frac{1}{p}\right)\times$$

$$\times\prod_{\substack{p|P, p\nmid c\\ p\equiv -1(\mathrm{mod}\, 4)}}\left(1-\frac{1}{p(p-1)}\right)\prod_{\substack{p|k\\ p\equiv -1(\mathrm{mod}\, 4)}}\left(1+\frac{1}{p}\right)\frac{N}{[2^{a+2}, k]\sqrt{\log N}}\left(1+O(\varepsilon^{\gamma})\right)$$

for $2^{a+2} < N^{2\varepsilon}$. For other a we have the following trivial estimations

$$\sum_{a\in\mathscr{A}} b(a) \leqslant 1+N/[2^{a+2}, k] \quad \text{if} \quad N^{2\varepsilon} \leqslant 2^{a+2} \leqslant 4(M+N),$$

$$\sum_{a\in\mathscr{A}} b(a) = 0 \quad \text{if} \quad 4(M+N) < 2^{a+2}.$$

On the other hand, we have

$$\sum_{\substack{M\leqslant n<M+N\\ n\equiv l(\mathrm{mod}\, k)\\ (n+c, P)=1}} b(n) = \sum_{a}\sum_{a\in\mathscr{A}} b(a)$$

which completes the proof.

References

[1] H. Bekić, *Über die Anzahl der Zahlen a^2+b^2 in einer arithmetischen Reihe grosser Differenz*, Journ. Reine Angew. Math. 226 (1967), pp. 120–131.

[2] H. Halberstam and H.-E. Richert, *Brun's method and the Fundamental Lemma*, Acta Arith. 24 (1973), pp. 113–133.

[3] — — *Mean value theorems for a class of arithmetic functions*, Acta Arith. 18 (1971), pp. 243–256.

[4] H. Iwaniec, *Primes of the type $\varphi(x, y)+A$, where φ is a quadratic form*, Acta Arith. 21 (1972), pp. 203–234.

[5] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Berlin 1971.

[6] Y. Motohashi, *On the distribution of prime numbers which are of the form x^2+y^2+1*, Acta Arith. 16 (1970), pp. 351–363.

[7] K. Prachar, *Über Zahlen der Form a^2+b^2 in einer arithmetischen Progression*, Math. Nachr. 10 (1953), pp. 51–54.

[8] G. J. Rieger, *Über die Anzahl der als Summe von zwei Quadraten darstellbaren und in einer primen Restklasse gelegenen Zahlen unterhalb einer positiven Schranke. II*, Journ. Reine Angew. Math. 217 (1965), pp. 200–216.

[9] Г.С. Белозёров, *О числах вида $n = u^K+dv^K$ в арифметической прогрессии*, Тезисы докладов всесоюзной конференции „Проблемы аналитической теории чисел и её применений'', Вильнюс 1974, pp. 26–27.

[10] Б. В. Левин и А. С. Файнлейб, *Применение некоторых интегральных уравнений к вопросам теории чисел*, УМН 22, вып. 3 (135) (1967), pp. 119–197.

[11] А. Г. Постников, *Введение в аналитическую теорию чисел*, Москва 1971.

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