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On the zeros of Dirichlet L-functions (VI)

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§ 1. Here we will see a q-analogue of the author's previous work [4]. We will quote this by (V).

Let $L(s,\chi)$ be a Dirichlet L-function with a character χ to modulus q. We write a nontrivial zero of $L(s,\chi)$ by $\varrho(\chi)=\beta(\chi)+i\gamma(\chi)$. As before for given two Dirichlet L-functions $L(s,\chi_1)$ and $L(s,\chi_2)$, we call ϱ a coincident zero of $L(s,\chi_1)$ and $L(s,\chi_2)$ if $L(\varrho,\chi_1)=L(\varrho,\chi_2)=0$ with the same multiplicity. We call ϱ a noncoincident zero of $L(s,\chi_1)$ and $L(s,\chi_2)$ if ϱ is not a coincident zero. We assume the order is given in the set of ordinates of zeros of $L(s,\chi)$ by $0\leqslant\gamma_n(\chi)\leqslant\gamma_{n+1}(\chi)$. Also in the set $\{\gamma_n(\chi_1),\gamma_m(\chi_2): n=1,2,\ldots, m=1,2,\ldots\}$ the order is given by

$$\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2),\quad \text{if}\quad \gamma_n(\chi_1)<\gamma_m(\chi_2)$$

and

$$\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2)\leqslant\gamma_{n+1}(\chi_1)\leqslant\gamma_{m+1}(\chi_2)\leqslant\ldots$$

if

$$\gamma_n(\chi_1)=\gamma_{n+1}(\chi_1)=\ldots=\gamma_m(\chi_2)=\gamma_{m+1}(\chi_2)=\ldots$$

Now we are concerned with the following problems, which are similar to problems (i), (ii) and (iii) in (V).

(i) *Have different primitive L-functions $L(s,\chi_1)$ and $L(s,\chi_2)$ a coincident zero?*

(ii) *For given positive real numbers t_1 and t_2, and for almost all pairs of primitive characters (χ_1,χ_2) does there exist a zero of $L(s,\chi_2)$ in*

$$\gamma_n(\chi_1)\leqslant \operatorname{Im} s\leqslant\gamma_{n+1}(\chi_1)$$

for each $\gamma_n(\chi_1)$ in $t_1\leqslant\gamma_n(\chi_1)\leqslant t_2$?

(iii) *For some $\gamma_n(\chi_1)$, does it happen that $\gamma_n(\chi)\leqslant\gamma_n(\chi_1)\leqslant\gamma_{n+1}(\chi)$ for almost all primitive characters χ?*

Our answers to these are the following theorems.

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THEOREM 1. *Let* $0 < t_1 < t_2 < q^{1/4-a}$, *where* $a > 0$. *Assume that*

$$(t_2-t_1)\log q = C > C_0,$$

where C_0 *is a suitable positive absolute constant. Then for positive proportion of pairs* (χ_1, χ_2) *of primitive characters to modulus* q, *there are at least* $C_1(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$ *number of* $\gamma_n(\chi_1)$ *in* $t_1 < \gamma_n(\chi_1) < t_2$ *such that there is no* $\gamma(\chi_2)$ *in*

$$\gamma_n(\chi_1) \leqslant t \leqslant \gamma_{n+1}(\chi_1),$$

where ε *is a positive small number and* C_1 *is some positive absolute constant.*

Similarly

THEOREM 1′. *Under the same hypothesis to* t_1 *and* t_2 *as above, for each primitive character* χ *to modulus* q, $L(s,\chi)$ *and* $L(s,\chi_1)$ *have at least* $C_1(\log C)^{1/2}\times$ $\times(\log\log C)^{1/2+\varepsilon}$ *number of noncoincident zeros in* $t_1 < \operatorname{Im} s < t_2$ *for positive proportion of primitive characters* χ_1 *to modulus* q, *where the number of noncoincident zeros is counted with multiplicities and* C_1 *and* ε *are the same as in Theorem* 1.

For longer intervals we have

THEOREM 2. *Let* $0 < t_1 < t_2 < q^{1/4-a}$, *where* $a > 0$. *Assume that* $(t_2-t_1)\log q$ *tends to* ∞ *as* q *tends to* ∞. *Let* χ *be an arbitrarily given primitive character to modulus* q *and* $\Phi(q)$ *be an arbitrarily given positive increasing function which goes to* ∞ *as* q *tends to* ∞. *Then for almost all primitive characters* χ_1 *to modulus* q, *either*

(i) *for at least* $\dfrac{\sqrt{4\log\big((t_2-t_1)\log q\big)}}{\Phi(q)}$ *number of* $\gamma_n(\chi)$'*s in* $t_1 < \gamma_n(\chi) < t_2$, *there is no zero of* $L(s,\chi_1)$ *in* $\gamma_n(\chi) \leqslant \operatorname{Im} s \leqslant \gamma_{n+1}(\chi)$, *or*

(ii) *for at least* $\dfrac{\sqrt{4\log\big((t_2-t_1)\log q\big)}}{\Phi(q)}$ *number of* $\gamma_n(\chi_1)$'*s in* $t_1 < \gamma_n(\chi_1) < t_2$, *there is no zero of* $L(s,\chi)$ *in* $\gamma_n(\chi_1) \leqslant \operatorname{Im} s \leqslant \gamma_{n+1}(\chi_1)$.

In Theorem 2, "almost all" means the number of exceptional characters is $o(q)$.

THEOREM 2′. *Under the same hypothesis to* t_1 *and* t_2 *and* $\Phi(q)$ *as Theorem* 2, *for each primitive character* χ *to modulus* q, $L(s,\chi)$ *and* $L(s,\chi_1)$ *have at least* $\dfrac{\sqrt{4\log\big((t_2-t_1)\log q\big)}}{\Phi(q)}$ *number of noncoincident zeros in* $t_1 < \operatorname{Im} s < t_2$ *for almost all primitive characters* χ_1 *to modulus* q.

These are our answers to (i) and (ii). About (iii) we can show

THEOREM 3. *For each* $\gamma_n(\chi)$ *in* $0 \leqslant \gamma_n(\chi) \leqslant q^{1/4-a}$, $\gamma_n(\chi_1) \leqslant \gamma_n(\chi) \leqslant \gamma_{n+1}(\chi_1)$ *for almost no* χ_1, *where* χ *and* χ_1'*s are primitive characters to modulus* q *and* $a > 0$.

We may remark here that "almost no" in the above Theorem 3 means the number of such χ_1's is $o(q)$. As in (V) if we define $\Delta_n(\chi_1,\chi)$ by $n-m$ such that $\gamma_m(\chi_1) \leqslant \gamma_n(\chi) \leqslant \gamma_{m+1}(\chi_1)$. Then Theorem 3 is a special case of the following

THEOREM 4. *For each* n *in* $1 \leqslant n \leqslant q^{1/4-b}$, *for each primitive character* χ *to modulus* q *and for any positive increasing function* $\Phi(q)$ *which tends to* ∞ *as* q *tends to* ∞,

$$|\Delta_n(\chi_1,\chi)| > \frac{\sqrt{2\log\log q}}{2\pi\Phi(q)}$$

for almost all primitive characters χ_1 *to modulus* q, *where* $b > 0$.

These come from mean value estimates of

$$S(t_2,\chi_1)-S(t_1,\chi_1)-\big(S(t_2,\chi_2)-S(t_1,\chi_2)\big) \quad \text{or} \quad S(t,\chi_2)-S(t,\chi_1),$$

where

$$S(t,\chi) = \frac{1}{\pi}\arg L(\tfrac{1}{2}+it,\chi)$$

as usual. (Cf. Lemma 1 and Lemma 2 in § 2.) We will prove our Theorems 1 and 1′ in § 3 and Theorems 2, 2′, 3, and 4 in § 4. In this paper we may assume for simplicity that q is a prime as in [6]. Other cases come in the same way. (Cf. [6].)

§ 2. Lemmas

2.1. We need two lemmas.

LEMMA 1. *Let* $|t|, |t+h| \leqslant q^{1/4-a}$, *where* $a > 0$. *Then for* $h > 0$,

$$\sum_{\chi_1}{}' \sum_{\chi_2}{}' \Big(\big(S(t+h,\chi_1)-S(t,\chi_1)\big)-\big(S(t+h,\chi_2)-S(t,\chi_2)\big)\Big)^l$$
$$= \begin{cases} c(k)\,q^2\big(4\log(3+h\log q)\big)^k + \\ \qquad\qquad + O\Big((Ak)^{6k}q^2\big(\log(3+h\log q)\big)^{k-1/2}\Big) & \text{for} \quad l = 2k \\ O\Big((Ak)^{6k}q^2\big(\log(3+h\log q)\big)^{k-1}\Big) & \text{for} \quad l = 2k-1, \end{cases}$$

where χ_1 *and* χ_2 *run over all nonprincipal characters to modulus* q, $c(k) = \dfrac{2k!}{(2\pi)^{2k}k!}$ *and* A'*s are some positive absolute constants.*

LEMMA 2. *Let* $|t| \leqslant q^{1/4-a}$, *where* $a > 0$. *Then*

$$\sum_{\chi_1}{}' \sum_{\chi_2}{}' \big(S(t,\chi_1)-S(t,\chi_2)\big)^l$$
$$= \begin{cases} c(k)\,q^2(2\log\log q)^k + O\big((Ak)^{6k}q^2(\log\log q)^{k-1/2}\big) & \text{for} \quad l = 2k, \\ O\big(q^2(\log\log q)^{k-1}(Ak)^{6k}\big) & \text{for} \quad l = 2k-1, \end{cases}$$

where $c(k)$ is the same as in Lemma 1 and A's are positive absolute constants.

2.2. As in [6] we write

$$S(t,\chi)=\frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{\chi(p)}{p^{1/2+it}}+R_x(t,\chi)$$

with some remainder term $R_x(t,\chi)$.

Then we know (cf. [6] or the author's "On the zeros of Dirichlet L-functions (II)", hereafter we quote the latter by (II)) that if $|t|\leqslant q^{1/4-a}$ and $q^{a/k}\leqslant x\leqslant q^{1/k}$,

$$\sum_{\chi}{}'\big(R_x(t,\chi)\big)^{2k}\ll (Ak)^{6k}q, \tag{1}$$

where χ runs over all nonprincipal characters to modulus q. Also we see that if we write

$$F_\alpha(x)=\sum_{p<x}\frac{|a(p)|^{2\alpha}}{p^{\alpha}}$$

for complex numbers $a(p)$ and real positive number α, and if $F_\alpha(x)\ll 1$ for $\alpha\geqslant 2$ and $F_{1/2}(x)\ll q^{c}$, then for $x=q^{1/2kc}$

$$\sum_{\chi}{}'\left|\frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{a(p)\chi(p)}{p^{1/2+it}}\right|^{l} \tag{2}$$
$$=\begin{cases}\dfrac{2b!}{(2\pi)^{2b}b!}\,qF_1(x)^b+O\big(b^b qF_1(x)^{0\vee(b-2)}\big)\\ \qquad\qquad \textit{if } F_1(x)\to\infty \textit{ as } x\to\infty \textit{ and if } l=2b\leqslant 2k,\\ O\big(b^b qF_1(x)^b\big)\quad \textit{if}\quad l=2b+1\leqslant 2k,\end{cases}$$

where $0\vee(b-2)=\operatorname{Max}\{0,b-2\}$. (Cf. (II).)

2.3. Proof of lemmas. Using the above notations

$$\big(S(t+h,\chi_1)-S(t,\chi_1)\big)-\big(S(t+h,\chi_2)-S(t,\chi_2)\big)$$
$$=f(\chi_1)-f(\chi_2)+\big(R_x(t+h,\chi_1)-R_x(t,\chi_1)\big)-\big(R_x(t+h,\chi_2)-R_x(t,\chi_2)\big),$$

where we write

$$f(\chi)=f_x(\chi)=\frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{\chi(p)a(p)}{p^{1/2+it}}$$

with $a(p)=\exp(-ih\log p)-1$. Now using (1) in 2.2, taking $x=q^{1/k}$,

$$\Big(\sum_{\chi_2}{}'\sum_{\chi_1}{}'\Big(\big(S(t+h,\chi_1)-S(t,\chi_1)\big)-\big(S(t+h,\chi_2)-S(t,\chi_2)\big)\Big)^{2k}\Big)^{1/2k} \tag{3}$$
$$=\Big(\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(f(\chi_1)-f(\chi_2)\big)^{2k}\Big)^{1/2k}+O\Big(\Big(\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(R_x(t+h,\chi_1)-R_x(t,\chi_1)\big)^{2k}\Big)^{1/2k}\Big)$$
$$+O\Big(\Big(\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(R_x(t+h,\chi_2)-R_x(t,\chi_2)\big)^{2k}\Big)^{1/2k}\Big)$$
$$=\Big(\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(f(\chi_1)-f(\chi_2)\big)^{2k}\Big)^{1/2k}+O(k^3q^{1/k}).$$

Now

$$\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(f(\chi_1)-f(\chi_2)\big)^{2k}=\sum_{\chi_2}{}'\sum_{\chi_1}{}'\sum_{b=0}^{2k}\binom{2k}{b}(-1)^b f(\chi_1)^b f(\chi_2)^{2k-b}$$
$$=\sum_{b=0}^{2k}\binom{2k}{b}(-1)^b\Big(\sum_{\chi_1}{}'f(\chi_1)^b\Big)\Big(\sum_{\chi_2}{}'f(\chi_2)^{2k-b}\Big)$$
$$=\sum_{m=0}^{k}\binom{2k}{2m}\Big(\sum_{\chi_1}{}'f(\chi_1)^{2m}\Big)\Big(\sum_{\chi_2}{}'f(\chi_2)^{2(k-m)}\Big)$$
$$-\sum_{m=0}^{k-1}\binom{2k}{2m+1}\Big(\sum_{\chi_1}{}'f(\chi_1)^{2m+1}\Big)\Big(\sum_{\chi_2}{}'f(\chi_2)^{2(k-m)-1}\Big).$$

Further since $\sum_{p<x}\frac{|a(p)|^2}{p}=2\log(h\log X)+O(1)$ if $h\log X\to\infty$ as $X\to\infty$, by (2)

$$\sum_{\chi}{}'f(\chi)^{2m}=\frac{2m!}{(2\pi)^{2m}m!}\,q\Big(2\log\Big(\frac{h}{k}\log q\Big)\Big)^m+O\Big(m^m q\Big(\log\Big(\frac{h}{k}\log q\Big)\Big)^{0\vee(m-2)}\Big).$$

Hence

$$\Big(\sum_{\chi_1}{}'f(\chi_1)^{2m}\Big)\Big(\sum_{\chi_2}{}'f(\chi_2)^{2(k-m)}\Big)$$
$$=\frac{2m!\;2(k-m)!}{(2\pi)^{2m}\;m!\,(2\pi)^{2(k-m)}(k-m)!}\,q^2\Big(2\log\Big(\frac{h}{k}\log q\Big)\Big)^k+$$
$$+O\Big(m^m(k-m)^{k-m}q^2\Big(\log\Big(\frac{h}{k}\log q\Big)\Big)^{0\vee(k-2)}\Big)$$

and

$$\sum_{m=0}^{k}\binom{2k}{2m}\Big(\sum_{\chi_1}{}' f(\chi_1)^{2m}\Big)\Big(\sum_{\chi_2}{}' f(\chi_2)^{2(k-m)}\Big)$$

$$= \frac{1}{(2\pi)^{2k}}\left(\sum_{m=0}^{k}\binom{2k}{2m}\frac{2m!}{m!}\frac{2(k-m)!}{(k-m)!}\right)q^2\left(2\log\left(\frac{h}{k}\log q\right)\right)^k + O\left(k^k q^2\left(\log\left(\frac{h}{k}\log q\right)\right)^{0\vee(k-2)}\right)$$

$$= \frac{2k!}{(2\pi)^{2k}k!}q^2\left(4\log\left(\frac{h}{k}\log q\right)\right)^k + O\left(k^k q^2\left(\log\left(\frac{h}{k}\log q\right)\right)^{0\vee(k-2)}\right).$$

Similarly using (2)

$$\sum_{m=0}^{k-1}\binom{2k}{2m+1}\Big(\sum_{\chi_1}{}' f(\chi_1)^{2m+1}\Big)\Big(\sum_{\chi_2}{}' f(\chi_2)^{2(k-m)-1}\Big) = O\left(k^k q^2\left(\log\left(\frac{h}{k}\log q\right)\right)^{k-1}\right).$$

Hence we get

$$\sum_{\chi_2}{}'\sum_{\chi_1}{}'\big(f(\chi_1)-f(\chi_2)\big)^{2k} = \frac{2k!}{(2\pi)^{2k}k!}q^2\left(4\log\left(\frac{h}{k}\log q\right)\right)^k + O\left(k^k q^2\left(\log\left(\frac{h}{k}\log q\right)\right)^{k-1}\right)$$

if $h\log q\to\infty$ as $q\to\infty$.

Hence from (3) we see that

$$\sum_{\chi_2}{}'\sum_{\chi_1}{}'\Big(\big(S(t+h,\chi_1)-S(t,\chi_1)\big)-\big(S(t+h,\chi_2)-S(t,\chi_2)\big)\Big)^{2k}$$

$$= \frac{2k!}{(2\pi)^{2k}k!}q^2\big(4\log(h\log q)\big)^k + O\Big((Ak)^{6k}q^2\big(\log(h\log q)\big)^{k-1/2}\Big)$$

if $h\log q\to\infty$ as $q\to\infty$.

Hence we get our Lemma 1 for even power case. Odd power case comes similarly and we omit it. Proof of Lemma 2 is completely similar and we omit it.

§ 3. Proof of Theorems 1 and 1'.

3.1. To prove Theorem 1 we use Lemma 1. $A_1, A_2, \ldots$ are positive absolute constants in the following. We put $h = 2\pi C/\log q$ and

$$f(\chi_1,\chi_2) = \big(S(t+h,\chi_1)-S(t,\chi_1)\big)-\big(S(t+h,\chi_2)-S(t,\chi_2)\big).$$

We write $E_M = \{(\chi_1,\chi_2)\neq(\chi_0,\chi_0): f(\chi_1,\chi_2) > M\}$ for $M \geqslant 0$, where χ_0 is the principal character to modulus q. And let $\varphi_M(\chi_1,\chi_2)$ be the characteristic function of E_M. Now

$$\sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}\varphi_0(\chi_1,\chi_2) = \sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}\varphi_0(\chi_1,\chi_2)\varphi_M(\chi_1,\chi_2) + \sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}\varphi_0(\chi_1,\chi_2)\big(1-\varphi_M(\chi_1,\chi_2)\big)$$

$$\leqslant \sqrt{|E_M|}\sqrt{\sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2(2l+1)}} + M^{2l+1}q^2.$$

On the other hand

$$\sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}\varphi_0(\chi_1,\chi_2) = \tfrac12\sum_{\chi_1}{}'\sum_{\chi_2}{}' |f(\chi_1,\chi_2)|^{2l+1} + \tfrac12\sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}$$

$$\geqslant \frac12\,\frac{\Big(\sum_{\chi_1}{}'\sum_{\chi_2}{}' |f(\chi_1,\chi_2)|^{2l}\Big)^{(2l-1)/2(l-1)}}{\Big(\sum_{\chi_1}{}'\sum_{\chi_2}{}' |f(\chi_1,\chi_2)|^{2}\Big)^{1/2(l-1)}} + \frac12\sum_{\chi_1}{}'\sum_{\chi_2}{}' f(\chi_1,\chi_2)^{2l+1}.$$

Taking $l = (\log\log C)^{1-\varepsilon}$ with some arbitrary small positive ε and $C > C_0$, we get by Lemma 1

$$A_1\big((2l+1)^{2l+1}q^2(\log C)^{2l+1}\big)^{1/2}\sqrt{|E_M|} \geqslant A_2\frac{\big(l^l q^2(\log C)^l\big)^{(2l-1)/2(l-1)}}{(q^2\log C)^{1/2(l-1)}} - M^{2l+1}q^2(e\pi^2)^{l+1/2}$$

Hence we get

$$|E_M| \geqslant q^2\left(A_3\frac{l^{l\frac{(2l-1)}{2(l-1)}}}{(2l+1)^{l+1/2}} - A_4\frac{M^{2l+1}(e\pi^2)^{l+1/2}}{(2l+1)^{l+1/2}(\log C)^{l+1/2}}\right)^2,$$

provided

$$\big(A_2 l^{l(2l-1)/2(l-1)}(\log C)^{l+1/2}\big)^{1/(2l+1)} > M.$$

Hence for $M = A_5(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon'}$ with positive ε', we get

$$|E_M| \geqslant A_6 q^2 e^{-(\log\log C)^{1-\varepsilon}}.$$

3.2. Now as is well-known the number $N(t,\chi)$ of zeros of $L(s,\chi)$ in $0 < \operatorname{Re} s < 1$, $0 \leqslant \operatorname{Im} s \leqslant t$, possible zeros on $\operatorname{Im} s = 0$ or t counted one half only, has an asymptotic formula (Riemann–von Mangoldt formula):

$$N(t,\chi) = \frac{t}{2\pi}\log t - \frac{1+\log\frac{2\pi}{q}}{2\pi}t - \frac{\chi(-1)}{8} + S(t,\chi) - S(0,\chi) + O\left(\frac{1}{1+t}\right) \quad \text{for} \quad t > 0.$$

Hence from 3.1, we have

COROLLARY. *Let* $0 < t_1 < t_2 < q^{1/4-a}$, *where* $a > 0$. *Let* $(t_2 - t_1)\log q = C > C_0$. *Then for at least* $A_6 q^2 \exp\left(-(\log\log C)^{1-\varepsilon}\right)$ *pairs* (χ_1, χ_2) *of characters to modulus* q,

$$\left(N(t_2, \chi_1) - N(t_1, \chi_1)\right) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right) > A_7 (\log C)^{1/2} (\log\log C)^{1/2+\varepsilon'}.$$

Same statement is true for

$$\left(N(t_2, \chi_1) - N(t_1, \chi_1)\right) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right) < -A_7 (\log C)^{1/2} (\log\log C)^{1/2+\varepsilon'}$$

where A_6 *and* A_7 *are positive absolute constants,* ε *and* ε' *are suitable small positive numbers and* C_0 *is a suitable large constant.*

From this corollary we get our Theorem 1. We may remark here that if we use $\sum_{\chi_1}' \sum_{\chi_2}' f(\chi_1, \chi_2)\varphi_0(\chi_1, \chi_2)$ instead of $\sum_{\chi_1}' \sum_{\chi_2}' f(\chi_1, \chi_2)^{2l+1}\varphi_0(\chi_1, \chi_2)$ in 3.1, we get

COROLLARY. *Under the same assumption on* t_1 *and* t_2 *as above, for at least* $A_8 q^2$ *pairs* (χ_1, χ_2) *of characters to modulus* q,

$$N(t_2, \chi_1) - N(t_1, \chi_1) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right) > A_9 (\log C)^{1/2}.$$

Same statement is true for

$$N(t_2, \chi_1) - N(t_1, \chi_1) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right) < -A_9 (\log C)^{1/2}.$$

3.3. For a given χ to modulus q, if we assume

$$\left|N(t_2, \chi) - N(t_1, \chi) - \left(N(t_2, \chi_1) - N(t_1, \chi_1)\right)\right| < A_7 (\log C)^{1/2} (\log\log C)^{1/2+\varepsilon'}$$

for almost all characters χ_1 to modulus q, then

$$\begin{aligned}\left|N(t_2, \chi_1) - N(t_1, \chi_1) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right)\right| &\leqslant \left|N(t_2, \chi) - N(t_1, \chi) - \left(N(t_2, \chi_1) - N(t_1, \chi_1)\right)\right| + \\ &\quad + \left|N(t_2, \chi) - N(t_1, \chi) - \left(N(t_2, \chi_2) - N(t_1, \chi_2)\right)\right| \\ &\leqslant 2A_7 (\log C)^{1/2} (\log\log C)^{1/2+\varepsilon'}\end{aligned}$$

for almost all pairs (χ_1, χ_2). But this contradicts the first corollary in 3.1. Hence for positive proportion of characters χ_1 to modulus q

$$\left|N(t_2, \chi) - N(t_1, \chi) - \left(N(t_2, \chi_1) - N(t_1, \chi_1)\right)\right| > A_7 (\log C)^{1/2} (\log\log C)^{1/2+\varepsilon'}$$

for a given χ. This proves Theorem 1.

§ 4. Proof of Theorems 2, 2′, 3 and 4. Since the pattern of proofs are similar we will prove only Theorem 4.

4.1. Let χ, χ_1, and χ_2 be nonprincipal characters to modulus q. By the definition of $\Delta_n(\chi_1, \chi)$,

$$\begin{aligned}\Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi) &= \left(N(\gamma_n(\chi), \chi) - N(\gamma_n(\chi), \chi_1)\right) - \\ &\quad - \left(N(\gamma_n(\chi), \chi) - N(\gamma_n(\chi), \chi_2)\right) + O(1) \\ &= S(\gamma_n(\chi), \chi_2) - S(\gamma_n(\chi), \chi_1) + O(1)\end{aligned}$$

by Riemann–von Mangoldt formula for $N(\gamma_n(\chi), \chi_i)$. Hence by Lemma 2 if $0 \leqslant \gamma_n(\chi) \leqslant q^{1/4-a}$, then

$$\begin{aligned}&\sum_{\chi_1}' \sum_{\chi_2}' \left(\Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi)\right)^l \\ &= \sum_{\chi_1}' \sum_{\chi_2}' \left(S(\gamma_n(\chi), \chi_2) - S(\gamma_n(\chi), \chi_1) + O(1)\right)^l \\ &= \begin{cases} \dfrac{2k!}{(2\pi)^{2k} k!} q^2 (2\log\log q)^k + O\left((Ak)^{6k} q^2 (\log\log q)^{k-1/2}\right) & \text{if } l = 2k, \\ O\left((Ak)^{6k} q^2 (\log\log q)^{k-1}\right) & \text{if } l = 2k-1. \end{cases}\end{aligned}$$

Now if we put

$$F_q(u) = \frac{1}{q^2}\left|\left\{(\chi_1, \chi_2),\ \chi_i \text{ is a nonprincipal character to modulus } q \text{ and } -\infty < \Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi) \leqslant \frac{u\sqrt{2\log\log q}}{2\pi}\right\}\right|,$$

the above asymptotic formula yields

$$\int_{-\infty}^{\infty} u^l dF_q(u) = \mu_l(q) = \begin{cases} \dfrac{2k!}{k!} + O\left(\dfrac{(Ak)^{6k}}{\sqrt{\log\log q}}\right) & \text{if } l = 2k, \\ O\left(\dfrac{(Ak)^{6k}}{\sqrt{\log\log q}}\right) & \text{if } l = 2k-1. \end{cases}$$

Since

$$\mu_l(q) \to \mu_l = \begin{cases} \dfrac{2k!}{k!} & \text{if } l = 2k, \\ 0 & \text{if } l = 2k-1 \end{cases}$$

as $q \to \infty$ and the distribution function determined by $\{\mu_l;\ l = 0, 1, 2, \ldots\}$ is $\int_{-\infty}^{u} \frac{1}{\sqrt{4\pi}} e^{-x^2/4} dx$,

$$\lim_{q\to\infty} F_q(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{4\pi}} e^{-x^2/4} dx$$

(cf. 4.24 and 3.4 of [5]).

Hence for any positive increasing function $\Phi(q)$ which tends to ∞ as $q \to \infty$,

$$|\Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi)| > \frac{\sqrt{2\log\log q}}{2\pi\Phi(q)}$$

for almost all pairs (χ_1, χ_2).

4.2. Now to derive Theorem 4, we assume that

$$|\Delta_n(\chi_1, \chi)| \leqslant \frac{\sqrt{2\log\log q}}{2\pi\Phi(q)}$$

for some positive increasing function $\Phi(q)$ which tends to ∞ as $q\to\infty$, and for positive proposition of nonprincipal characters χ_1 to modulus q. Then if χ_1 and χ_2 are such characters, then

$$|\Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi)| \leqslant |\Delta_n(\chi_1, \chi)| + |\Delta_n(\chi_2, \chi)| \leqslant \frac{\sqrt{2\log\log q}}{\pi\Phi(q)}.$$

Since the number of such pairs (χ_1, χ_2) is at least Aq^2 with some positive absolute constant, this contradicts with the fact in 4.1. Hence for a given χ to modulus q $|\Delta_n(\chi_1, \chi)| > \frac{\sqrt{2\log - \log q}}{2\pi\Phi(q)}$ for almost all χ_1 to modulus q. This proves Theorem 4. Theorem 3 is a special case of Theorem 4.

4.3. Similarly from Lemma 1 if $(t_2-t_1)\log q\to\infty$ as $q\to\infty$, for any positive increasing function $\Phi(q)$ which tends to ∞ as $q\to\infty$,

$$\big|\big(N(t_2, \chi_1) - N(t_1, \chi_1)\big) - \big(N(t_2, \chi_2) - N(t_1, \chi_2)\big)\big| > \frac{\sqrt{4\log\big((t_2-t_1)\log q\big)}}{2\pi\Phi(q)}$$

for almost all pairs (χ_1, χ_2) of characters to modulus q. From this we can see Theorems 2 and 2′ as before.

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On the zeros of Dirichlet *L*-functions (VII)

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§ 1. Introduction. As an application of the methods which we have used in the author's previous works, [1] and [4] we add some results to Knapowski–Turán's problem which will be explained later. We will quote the above articles by (I) or (V).

Let q be a given fixed positive integer. Assume that $(b, q) = (d, q) = 1$ and $b \not\equiv d (\bmod q)$. Let χ be a character to modulus q. We write

$$g(\chi) = \frac{1}{\varphi(q)}\big(\bar{\chi}(b) - \bar{\chi}(d)\big) \quad \text{and} \quad \mu(\varrho) = \mu_{b,d}(\varrho) = \sum_{\chi} g(\chi) m_{\chi}(\varrho),$$

where χ runs over all characters to modulus q and $m_\chi(\varrho)$ is the multiplicity of ϱ as a zero of Dirichlet L-function $L(s, \chi)$. Knapowski and Turán proposed the following problem in their study of prime numbers: to study

$$f(T) = \sum_{\substack{0<\operatorname{Im}\varrho<T \\ \mu(\varrho)\neq 0}} 1$$

(cf. [6]).

To this problem Kátai (unpublished) and Grosswald ([5]) proved independently the existence of infinitely many ϱ's with $\mu(\varrho) \neq 0$. Later Turán obtained the following results (cf. [10]).

1) For $T > \psi(q)$ we have the inequality

$$f(T) > C_1 \exp\big((\log T)^{1/5}\big).$$

2) Under the assumption of the generalized Riemann hypothesis we have

$$f(T) > C_2 T^{1/2} \quad \text{for} \quad T > \psi(q),$$

where C_ν are numerical constants and $\psi(q)$ an explicit function of q.

Recently Motohashi ([7]) obtained the following results.

1) For $T > \psi(q)$ we have

$$f(T) > T^{1/10}(\log T)^{-3}.$$

Here the estimation is independent of b and d.

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