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Arithmetic of quaternary quadratic forms

by

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Introduction. In this paper we give a general treatment of certain aspects of the arithmetic theory of quaternary quadratic forms. Although we will concentrate on class number questions, our hope is that the framework provided here will also be suitable for the treatment of other arithmetic questions, for example, those pertaining to modular forms of "Nebentypus". We do not employ the classical language in our discussion of quadratic forms. Rather, we adopt the terminology and viewpoint of Eichler's fundamental work [4]. Thus we begin with a regular quadratic vector space V of dimension four over an algebraic number field k (a *quaternary space* over k, in our terminology). The group of primary interest to us will be $S^+(V)$, the group of proper similitudes, rather than the special orthogonal group $O^+(V)$. Accordingly, the class number we consider here will be the number of similitude classes in an idealcomplex ([4], p. 87). We set as our main problem the determination of the number of similitude classes in an arbitrary idealcomplex $\mathfrak{L}$ of maximal lattices of V. Our viewpoint throughout is to interpret the arithmetic of the quaternary space V in terms of the arithmetic of its second Clifford algebra C_V^+. This viewpoint enables us to reduce the main problem, in most cases, to that of determining the class number H of a *unique* idealcomplex $\mathfrak{I}$ from each V, namely, the idealcomplex containing the maximally integral lattices of V. The precise statement and details of proof are given in § 3.

If the discriminant $D(V)$ of V is a square in k, then $C_V^+ = \mathfrak{A}\oplus\mathfrak{A}$, where $\mathfrak{A}$ is a quaternion algebra over k. In this case $\mathfrak{I}$ is the only idealcomplex of maximal lattices and H is the number of classes of normal ideals of $\mathfrak{A}$. In my dissertation [16] I derived a formula for H in the case where $k = Q$, the field of rationals. For this reason the square discriminant case is not of major interest to us here. However, in order to provide a unified approach, we will mention the square discriminant case several times, pointing out how it parallels the nonsquare discriminant case.

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If $D(V)$ is not a square in k, then $C_V^+ = \mathfrak{A}_K$, where $\mathfrak{A}$ is a quaternion algebra over k, and $K = k(\sqrt{D(V)})$. In this case there are infinitely many idealcomplexes of maximal lattices of V (§ 3, Proposition 7). However, replacing V by a suitable similar space, if necessary, we may assume that $\mathfrak{L} = \mathfrak{J}$, except if K is a subfield of the Hilbert class field of k, in which case we must also consider those $\mathfrak{L}$ which contain $\mathfrak{p}$-maximal lattices of V, where $\mathfrak{p}$ is a prime of k which remains prime in K. In § 4 we show, under the assumption that k has class number 1, that H is equal to a certain "generalized type number" associated to $\mathfrak{A}_K$. As a simple application, we give in § 5 an elementary proof of the Minkowski–Siegel formula for definite quaternary spaces over $\boldsymbol{Q}$. If, in addition to k having class number 1, we assume that $V_\mathfrak{p}$ is isotropic for every finite prime $\mathfrak{p}$ of k, then $H = t_\delta$, the type number of Eichler orders of level δ in $\mathfrak{A}_K$, where δ is the product of all finite nonsplit primes of $\mathfrak{A}$ which remain prime in K. Using the Selberg trace formula, we derive elementary formulas for t_δ (§ 6, Theorems 1, 2) assuming $k = \boldsymbol{Q}$ and the fundamental unit of K has norm -1. These formulas were announced earlier in [18].

Interpreting our lattice-theoretic results in the language of quadratic forms, we then obtain elementary formulas for the number of classes of positive definite integral quaternary forms with discriminant Δ_K, where Δ_K is the discriminant of a real quadratic extension K of $\boldsymbol{Q}$ having a fundamental unit of norm -1 (§ 6, Corollary 1 to Theorem 3). In the special case where Δ_K is a prime (§ 6, Corollary to Theorem 1) this result also follows from results of Peters [14] and Tamagawa (unpublished) and can also be found in the paper of Kitaoka [8]. The nature of our formulas suggest that class number questions for quaternary forms are "reducible", in some sense, to class number questions for binary forms. It would be desirable to establish such a relation between quaternary forms and binary forms directly. In particular, one might then have a method of constructing representatives for classes of positive definite quaternary forms which is more effective than the usual reduction theory technique (cf. [21]).

§ 1. Quaternary spaces. In this section k denotes either a global or local field of characteristic $\neq 2$. A *quaternary space* over k is an ordered pair (V, q), where V is a vector space of dimension four over k and $q: V \to k$ is a quadratic mapping. When there is no danger of confusion we will denote (V, q) simply by V. The symmetric bilinear form B associated to q is defined by:

$$B(v, w) = q(v+w) - q(v) - q(w), \qquad v, w \in V.$$

We assume that B is nondegenerate, so that $\det[B(v_i, v_j)] \neq 0$ for any basis $\{v_i\}$ of V over k, $i, j = 1, 2, 3, 4$. Let $k^\times$ denote the multiplicative group of non-zero elements in k, $(k^\times)^2$ the subgroup of squares of $k^\times$.

The coset of $\det[B(v_i, v_j)]$ in $k^\times/(k^\times)^2$ is independent of the choice of basis $\{v_i\}$ of V over k. We call this coset the *discriminant* of V and denote it by $D(V)$. Whenever it is convenient, we will feel free to identify $D(V)$ with any of its coset representatives.

A *similitude* σ of V is a linear automorphism of V satisfying $q(\sigma(v)) = a_\sigma q(v)$ for every $v \in V$, where $a_\sigma \in k^\times$ is independent of v. The number a_σ is called the *norm* of σ and is denoted accordingly by $n(\sigma)$. The similitudes of V form a subgroup of $\mathrm{GL}(V)$ which we denote by $S(V)$. An *orthogonal transformation* of V is a similitude of V having norm 1. The group of all orthogonal transformations of V will be denoted by $O(V)$. If σ is a similitude, then $\det(\sigma) = \pm n(\sigma)^2$. We say that σ is *proper* or *improper* according as the plus sign or minus sign holds. In particular, an orthogonal transformation σ is proper if and only if $\det(\sigma) = 1$. We denote the group of proper similitudes by $S^+(V)$ and the group of proper orthogonal transformations by $O^+(V)$.

We proceed now to classify the quaternary spaces over k up to similarity. As the existing literature on this subject is inadequate for our purposes, we will have to draw upon results contained in some unpublished lecture notes of Tamagawa. Our classification depends on an isometry classification of quaternary spaces over k which represent 1.

Let us denote the Clifford algebra of V by C_V. We recall that C_V is a graded associative algebra of dimension 16 over k and V may be regarded as the subspace of homogeneous elements of degree 1. The multiplication in C_V is uniquely determined by the condition $v^2 = q(v)$ for all $v \in V$. The subspace of C_V spanned by all homogeneous elements of even degree is a subalgebra of dimension 8 which we denote by C_V^+.

Let x_1, x_2, x_3, x_4 be an orthogonal basis of V over k and put $z = x_1x_2x_3x_4$. Then $z^2 = D(V)$ and the center K of C_V^+ is given by $K = k + kz$. If $D(V)$ is a square in $k^\times$, then $K \cong k \oplus k$. If $D(V)$ is not a square in $k^\times$, then $K \cong k(\sqrt{D(V)})$, a quadratic extension of k.

Notation. For any associative ring R with 1 let $R^\times$ denote the multiplicative group of all invertible elements in R.

The *Clifford group* Γ_V of V is the subgroup of $C_V^\times$ of all γ such that $\gamma V \gamma^{-1} = V$. We put $\Gamma_V^+ = \Gamma_V \cap C_V^+$. To each $\gamma \in \Gamma_V$ we associate $\sigma_\gamma \in O(V)$ by setting $\sigma_\gamma(v) = \gamma v \gamma^{-1}$, $v \in V$. Then the homomorphism $\gamma \mapsto \sigma_\gamma$ maps Γ_V onto $O(V)$ and Γ_V^+ onto $O^+(V)$. The mapping defined by $v_1 \ldots v_n \mapsto v_n \ldots v_1$, $v_1, \ldots, v_n \in V$, gives an involution of C_V which we denote by $\alpha \mapsto \alpha^*$. One can easily show that Γ_V^+ is the set of all $\gamma \in C_V^+$ such that $\alpha\alpha^* \in k^\times$ (cf. [4], p. 32).

In the remainder of this section we will assume that q represents 1. Then we can choose the orthogonal basis x_1, x_2, x_3, x_4 with $q(x_1) = 1$. Let $\mathfrak{A}$ be the set of all elements in C_V^+ which commute with x_1. Then $\mathfrak{A}$

is a subalgebra of C_V^+ with basis 1, $\lambda = x_2x_3$, $\mu = x_3x_4$, $\nu = x_2x_4$, satisfying the relations:

$$\lambda^2 = -q(x_2)q(x_3), \quad \mu^2 = -q(x_3)q(x_4), \quad \lambda\mu = q(x_3)\nu = -\mu\lambda.$$

In other words, $\mathfrak{A}$ is a quaternion algebra over k. The restriction of the involution $a\mapsto a^*$ to $\mathfrak{A}$ is the canonical involution of $\mathfrak{A}$. We regard $\mathfrak{A}$ as a quaternary space over k with quadratic mapping equal to its reduced norm mapping N, defined by $N(a) = aa^*$, $a \in \mathfrak{A}$. It is clear that $C_V^+ = \mathfrak{A}_K$, where $\mathfrak{A}_K = \mathfrak{A}\otimes_k K$.

We must distinguish between the cases where $D(V)$ is or is not a square in k. First suppose $D(V)$ is a square in k. Then $K = ke_1 + ke_2$ and $C_V^+ = \mathfrak{A}e_1 + \mathfrak{A}e_2$, where e_1, e_2 are the orthogonal idempotents of K. One easily verifies the following relations

$$x_1e_1 = e_2x_1, \quad x_1e_2 = e_1x_1.$$

Define a linear mapping $\varphi\colon \mathfrak{A}\to C_V$ by $\varphi(\xi) = (\xi e_1 + \xi^* e_2)x_1$, $\xi \in \mathfrak{A}$. It is easy to see that $\varphi(\xi)^* = \varphi(\xi)$. Thus the homogeneous components of $\varphi(\xi)$ are symmetric and of odd degree. This implies that $\varphi(\xi) \in V$ for every $\xi \in \mathfrak{A}$. Furthermore, one can easily show that $q(\varphi(\xi)) = \varphi(\xi)^2 = N(\xi)$ for $\xi \in \mathfrak{A}$. It follows that φ is an isometry of $\mathfrak{A}$ onto V. Let γ be an element of C_V^+. Then $\gamma = ae_1 + \beta e_2$, $a, \beta \in \mathfrak{A}$, and $\gamma\gamma^* = N(a)e_1 + N(\beta)e_2$. Hence Γ_V^+ is the set of all $ae_1 + \beta e_2$ with $a, \beta \in \mathfrak{A}^\times$ and $N(a) = N(\beta)$. Furthermore, if $\xi \in \mathfrak{A}$ and $\gamma = ae_1 + \beta e_2 \in \Gamma_V^+$, then $\gamma\varphi(\xi)\gamma^{-1} = \varphi(a\xi\beta^{-1})$.

PROPOSITION 1. *Let V be a quaternary space over k which represents* 1. *If $D(V)$ is a square in k, then*

(a) *V is isometric to a quaternion algebra $\mathfrak{A}$ over k, the quadratic mapping on $\mathfrak{A}$ being its reduced norm N.*

(b) *The proper orthogonal transformations of $\mathfrak{A}$ are all the mappings of the form $\xi\mapsto a\xi\beta^{-1}$, where $a, \beta \in \mathfrak{A}^\times$ and $N(a) = N(\beta)$.*

(c) *The proper similitudes of $\mathfrak{A}$ are all the mappings of the form $\xi\mapsto a\xi\beta$, where $a, \beta \in \mathfrak{A}^\times$.*

Proof. To prove (c) we note first that the mapping $\xi\mapsto a\xi\beta$, $a, \beta \in \mathfrak{A}^\times$ is certainly a proper similitude of norm $N(a)N(\beta)$. Suppose σ is a proper similitude of $\mathfrak{A}$. Choose $\gamma \in \mathfrak{A}^\times$ such that $N(\gamma) = n(\sigma)$. Then $\gamma^{-1}\sigma \in O^+(\mathfrak{A})$, so there exist $a, \beta \in \mathfrak{A}^\times$ such that $\gamma^{-1}\sigma(\xi) = a\xi\beta^{-1}$ for all $\xi \in \mathfrak{A}$. Then $\sigma(\xi) = (\gamma a)\xi\beta^{-1}$ for all $\xi \in \mathfrak{A}$.

Remark. It is evident that the improper orthogonal transformations are of the form $\xi\mapsto a\xi^*\beta^{-1}$, $a, \beta \in \mathfrak{A}^\times$, $N(a) = N(\beta)$, and the improper similitudes are of the form $\xi\mapsto a\xi^*\beta$, $a, \beta \in \mathfrak{A}^\times$.

Now suppose $D(V)$ is not a square in k. Then $K \cong k(\sqrt{D(V)})$, a quadratic extension of k. Let $a\mapsto\bar{a}$ be the conjugation automorphism of K over k. The quadratic mapping q extends uniquely to a quadratic mapping on $V_K = V\otimes_k K$ and it is easy to see that $C_{V_K} = C_V\otimes_k K$. Thus conjugation on K extends to a k-automorphism $a\mapsto\bar{a}$ of C_{V_K} having C_V as its algebra of fixed elements. It is clear that $\mathfrak{A}_K$ is the set of all elements in $C_{V_K}^+$ which commute with x_1. Since $D(V_K)$ is a square in K, we have $C_{V_K}^+ = \mathfrak{A}_K e_1 + \mathfrak{A}_K e_2$, where e_1, e_2 are the orthogonal idempotents of the center of $C_{V_K}^+$. Furthermore, we have an isometry $\varphi\colon \mathfrak{A}_K\to V_K$ defined by $\varphi(\xi) = (\xi e_1 + \xi^* e_2)x_1$. The restriction of φ^{-1} to V then gives an isometry of V onto a k-subspace W of $\mathfrak{A}_K$. We note that $\overline{\varphi(\xi)} = \varphi(\bar{\xi}^*)$. Since V is the subspace of V_K of all elements fixed by the conjugation, we must have $W = \{\xi \in \mathfrak{A}_K | \bar{\xi}^* = \xi\}$, the quadratic mapping on W being the restriction of N.

Let $\gamma \in C_{V_K}^+$. We have already seen that $\gamma \in \Gamma_{V_K}^+$ if and only if $\gamma = ae_1 + \beta e_2$, where $a, \beta \in \mathfrak{A}_K^\times$ and $N(a) = N(\beta)$. From $V = V_K\cap C_V$ it is easy to see that $\gamma \in \Gamma_V^+$ if and only if $\gamma \in \Gamma_{V_K}^+$ and $\gamma = \bar{\gamma} = \bar{\beta}e_1 + \bar{a}e_2$. It follows that $\gamma \in \Gamma_V^+$ if and only if $\gamma = ae_1 + \bar{a}e_2$ and $N(a) = N(\bar{a}) = \overline{N(a)} \neq 0$, which is to say $N(a) \in k^\times$; in this case, $\gamma\varphi(\xi)\gamma^{-1} = \varphi(a\xi\bar{a}^{-1})$.

PROPOSITION 2. *Let V be a quaternary space over k which represents* 1. *Suppose $D(V)$ is not a square in k and put $K = k(\sqrt{D(V)})$. Then*

(a) *There is a unique quaternion algebra $\mathfrak{A}$ over k such that V is isometric to the quaternary space $W = \{\xi \in \mathfrak{A}_K |\ \bar{\xi}^* = \xi\}$, where $\xi\to\xi^*$ is the canonical involution, $\xi\to\bar{\xi}$ is the extension of the conjugation on K, and the quadratic mapping on W is the restriction of the reduced norm N of $\mathfrak{A}_K$.*

(b) *The proper orthogonal transformations of W are all the mappings of the form $\xi\mapsto a\xi\bar{a}^{-1}$, where $a \in \mathfrak{A}_K^\times$ and $N(a) \in k^\times$.*

Proof. The uniqueness of $\mathfrak{A}$ (up to k-isomorphism) follows from the following observation. Let $\mathfrak{A}'$ be any quaternion algebra over k, K' any quadratic extension of k. Then $W' = \{\xi \in \mathfrak{A}'_K |\ \bar{\xi}^* = \xi\}$ is a quaternary space with nonsquare discriminant $D(W')$, $K' = k(\sqrt{D(W')})$, and if $x_1 \in W'$ is any element representing 1, then $\mathfrak{A}'$ is k-isomorphic to the subalgebra of $C_{W'}^+$ commuting with x_1.

Remarks. 1. From (b) of Proposition 2 we obtain a one-to-one correspondence between quaternary spaces V over k with nonsquare discriminant representing 1 and ordered pairs $(\mathfrak{A}, K)$, where $\mathfrak{A}$ is a quaternion algebra over k and K is a quadratic extension of k. If k is a number field, this may be viewed as a global analogue to the well-known local classification of quadratic spaces by their discriminants and Witt invariants.

2. We note that Proposition 2 is valid in the square discriminant case if we take conjugation to be $(a, \beta)\mapsto(\beta, a)$ on $\mathfrak{A}_K \cong \mathfrak{A}\oplus\mathfrak{A}$. Then (a), (b) of Proposition 2 correspond to (a), (b) of Proposition 1.

3. The improper orthogonal transformations of W are the mappings of the form $\xi\mapsto a\xi^*\bar{a}^{-1}$, where $a \in \mathfrak{A}_K$ and $N(a) \in k^\times$.

To determine the proper similitudes of W we proceed as follows. By K-linearity and the fact that $W_K = \mathfrak{A}_K$, any $\sigma \in S^+(W)$ can be uniquely extended to an element of $S^+(\mathfrak{A}_K)$, which we also denote by σ. In this way $S^+(W)$ can be identified with the subgroup of $S^+(\mathfrak{A}_K)$ consisting of all σ such that $\sigma(W) \subset W$. Suppose $\sigma \in S^+(W)$. Then (c) of Proposition 1 implies that $\sigma(\xi) = \alpha\xi\beta$ for all $\xi \in \mathfrak{A}_K$, where $\alpha, \beta \in \mathfrak{A}_K^\times$. Then for every $\xi \in W$ we must have $\alpha\xi\beta = \overline{(\alpha\xi\beta)}^* = \bar{\beta}^*\bar{\xi}^*\bar{\alpha}^* = \bar{\beta}^*\xi\bar{\alpha}^*$. Taking $\xi = 1$, we see that $\alpha^{-1}\bar{\beta}^* = (\bar{\alpha}^*\beta^{-1})^{-1}$. Thus $(\bar{\alpha}^*\beta^{-1})^{-1}\xi\bar{\alpha}^*\beta^{-1} = \xi$ for all $\xi \in W$. By K-linearity, the latter must be true for all $\xi \in \mathfrak{A}_K$. Since $\mathfrak{A}_K$ is central simple over K, we must have $\bar{\alpha}^*\beta^{-1} = c^{-1} \in K^\times$, i.e. $\beta = c\bar{\alpha}^*$, $c \in K^\times$. Then, for every $\xi \in W$, $\sigma(\xi) = c\alpha\xi\bar{\alpha}^* = \overline{(c\alpha\xi\bar{\alpha}^*)}^* = \bar{c}\alpha\xi\bar{\alpha}^*$. It follows that $c \in k^\times$. We conclude that any proper similitude of W must be of the form $\xi \mapsto c\alpha\xi\bar{\alpha}^*$, where $c \in k^\times$, $\alpha \in \mathfrak{A}_K^\times$. Conversely, any such mapping is a proper similitude; in fact, if $n_{K/k}: K \to k$ is the norm mapping, $N(c\alpha\xi\bar{\alpha}^*) = n_{K/k}(cN(\alpha))N(\xi)$. This proves

PROPOSITION 3. *Let V be as in Proposition 2.*

(a) *The proper similitudes of W are all the mappings of the form $\xi \mapsto c\alpha\xi\bar{\alpha}^*$, where $c \in k^\times$ and $\alpha \in \mathfrak{A}_K^\times$.*

(b) *The norm of the proper similitude $\xi \mapsto c\alpha\xi\bar{\alpha}^*$ is $n_{K/k}(cN(\alpha))$.*

Remarks. 1. A similitude $\xi \mapsto c\alpha\xi\bar{\alpha}^*$ is an orthogonal transformation if and only if $n_{K/k}(cN(\alpha)) = 1$. By Hilbert's Theorem 90, the latter is true if and only if $cN(\alpha) = \bar{b}/b$ for some $b \in K^\times$. Putting $\beta = b\alpha$, we see that $N(\beta) = n_{K/k}(b)c^{-1} \in k^\times$ and $c\alpha\xi\bar{\alpha}^* = \beta\xi\bar{\beta}^{-1}$. This agrees with (b) of Proposition 2.

2. Proposition 3 is valid when $D(V)$ is a square in k if we take $(a, \beta) \mapsto (\beta, a)$ as the conjugation on $\mathfrak{A}_K$, so that $n_{K/k}(a, b) = ab$, $a, b \in k^\times$. Then (a) of Proposition 3 corresponds to (c) of Proposition 1.

3. The improper similitudes of W are the mappings of the form $\xi \mapsto c\alpha\xi^*\bar{\alpha}^*$, where $c \in k^\times$, $\alpha \in \mathfrak{A}_K^\times$.

Let us identify V with the subspace W of $\mathfrak{A}_K$. For $(c, \alpha) \in k^\times \times \mathfrak{A}_K^\times$ let $\psi(c, \alpha)$ be the similitude $\xi \mapsto c\alpha\xi\bar{\alpha}^*$. Then $\psi: k^\times \times \mathfrak{A}_K^\times \to S^+(V)$ is a surjective homomorphism with kernel $\{(c, x) \in k^\times \times K^\times \mid n_{K/k}(x) = c^{-1}\}$, the graph of $(n_{K/k})^{-1}$, which we denote by $\Gamma(k^\times \times K^\times)$.

COROLLARY. *Let V be a quaternary space over k representing 1. Let $K = k(\sqrt{D(V)})$ if $D(V)$ is not a square in k, $K = k \oplus k$ if $D(V)$ is a square in k. Let $\mathfrak{A}$ be the quaternion algebra uniquely associated to V as in Proposition 2. Then there is a natural isomorphism*

$$S^+(V) \cong k^\times \times \mathfrak{A}_K^\times / \Gamma(k^\times \times K^\times). \tag{1}$$

§ 2. Local considerations. In this section we assume that k is an algebraic number field. Let $\mathfrak{o}$ be the ring of integers of k, $\mathfrak{p}$ a prime of k (finite or infinite). If $\mathfrak{p}$ is finite, we identify it with the prime ideal of $\mathfrak{o}$ uniquely associated to it. We denote by $k_\mathfrak{p}$ the completion of k with respect to $\mathfrak{p}$. If $\mathfrak{p}$ is finite we let $\mathfrak{o}_\mathfrak{p}$ denote the ring of integers of $k_\mathfrak{p}$.

Let V be a quaternary space over k representing 1. Put $V_\mathfrak{p} = V \otimes_k k_\mathfrak{p}$. Then the quadratic mapping q on V extends uniquely to a quadratic mapping on $V_\mathfrak{p}$ for each prime $\mathfrak{p}$ of k. We denote the extended form also by q. It is clear that $C_{V_\mathfrak{p}} = C_V \otimes_k k_\mathfrak{p}$, $C^+_{V_\mathfrak{p}} = C^+_V \otimes_k k_\mathfrak{p}$. Furthermore, if K is the center of C^+_V, then $K_\mathfrak{p} = K \otimes_k k_\mathfrak{p}$ is the center of $C^+_{V_\mathfrak{p}}$; if $\mathfrak{A}$ is the quaternion algebra associated to V in the manner of § 1, then $\mathfrak{A}_\mathfrak{p} = \mathfrak{A} \otimes_k k_\mathfrak{p}$ is the quaternion algebra associated to $V_\mathfrak{p}$ and $C^+_{V_\mathfrak{p}} = \mathfrak{A}_{K_\mathfrak{p}} = \mathfrak{A}_K \otimes_k k_\mathfrak{p} = \mathfrak{A}_\mathfrak{p} \otimes_{k_\mathfrak{p}} K_\mathfrak{p}$. The conjugation $\alpha \mapsto \bar{\alpha}$ on $\mathfrak{A}_K$ extends by $k_\mathfrak{p}$-linearity to a $k_\mathfrak{p}$-automorphism of $\mathfrak{A}_{K_\mathfrak{p}}$ which coincides with the one coming from $K_\mathfrak{p}$.

First suppose $D(V)$ is a square in $k_\mathfrak{p}$. Then, according to (a) of Proposition 1, we may identify $V_\mathfrak{p}$ with $\mathfrak{A}_\mathfrak{p}$. Since $k_\mathfrak{p}$ is a local field, there are only two possibilities for $\mathfrak{A}_\mathfrak{p}$. Either $\mathfrak{A}_\mathfrak{p} = M(2, k_\mathfrak{p})$ or $\mathfrak{A}_\mathfrak{p}$ = the unique division algebra of dimension 4 over $k_\mathfrak{p}$. In the former case we say that $\mathfrak{A}_\mathfrak{p}$ is *split* or $\mathfrak{p}$ is a *split prime* of $\mathfrak{A}$; in the latter case we say $\mathfrak{A}_\mathfrak{p}$ is *nonsplit* or $\mathfrak{p}$ is a *nonsplit* (or *ramified*) *prime* of $\mathfrak{A}$. The standard properties of the Hilbert norm residue symbol imply that the set of nonsplit primes of $\mathfrak{A}$ is a finite set with an even number of elements.

Now suppose $D(V)$ is not a square in $k_\mathfrak{p}$. Then, since $K_\mathfrak{p} = k_\mathfrak{p}(\sqrt{D(V)})$ is a quadratic extension of $k_\mathfrak{p}$, it must split $\mathfrak{A}_\mathfrak{p}$, $\mathfrak{A}_{K_\mathfrak{p}} = M(2, K_\mathfrak{p})$. We know from § 1 that, for a fixed $K_\mathfrak{p}$, the quadratic spaces $V_\mathfrak{p}$ with $k_\mathfrak{p}(\sqrt{D(V_\mathfrak{p})}) = K_\mathfrak{p}$ are in one-to-one correspondence with quaternion algebras $\mathfrak{A}_\mathfrak{p}$ over $k_\mathfrak{p}$, $V_\mathfrak{p}$ being identified with $W_\mathfrak{p} = \{\xi \in \mathfrak{A}_{K_\mathfrak{p}} \mid \bar{\xi}^* = \xi\}$. To obtain a realization of a given $V_\mathfrak{p}$ as a $k_\mathfrak{p}$-subspace of $M(2, K_\mathfrak{p})$, it suffices to find a $k_\mathfrak{p}$-embedding of the corresponding $\mathfrak{A}_\mathfrak{p}$ as a subalgebra of $M(2, K_\mathfrak{p})$, for we would then have $M(2, K_\mathfrak{p}) = K_\mathfrak{p}\mathfrak{A}_\mathfrak{p} = \mathfrak{A}_{K_\mathfrak{p}}$.

The fact that $\mathfrak{A}_\mathfrak{p}$ is split by $K_\mathfrak{p}$ implies that $\mathfrak{A}_\mathfrak{p}$ has a basis $\{1, \lambda, \mu, \nu\}$ where $\lambda^2 = c_\mathfrak{p} \in k_\mathfrak{p}^\times$, $\mu^2 = D(V)$, $\lambda\mu = \nu = -\mu\lambda$.

We can then embed $\mathfrak{A}_\mathfrak{p}$ into $M(2, K_\mathfrak{p})$ by

$$\lambda \mapsto \begin{bmatrix} 0 & 1 \\ c_\mathfrak{p} & 0 \end{bmatrix}, \quad \mu \mapsto \begin{bmatrix} \sqrt{D(V)} & 0 \\ 0 & -\sqrt{D(V)} \end{bmatrix}, \quad \nu \mapsto \begin{bmatrix} 0 & -\sqrt{D(V)} \\ c_\mathfrak{p}\sqrt{D(V)} & 0 \end{bmatrix}. \tag{2}$$

This amounts to identifying $\mathfrak{A}_\mathfrak{p}$ with the $k_\mathfrak{p}$-subalgebra of $M(2, K_\mathfrak{p})$ of all elements of the form

$$\begin{bmatrix} x & y \\ c_\mathfrak{p}\bar{y} & \bar{x} \end{bmatrix}, \quad \text{where } x, y \in K_\mathfrak{p}. \tag{3}$$

We note that $\mathfrak{A}_\mathfrak{p}$ will be split or nonsplit according as the norm residue symbol $(c_\mathfrak{p}, D(V))_\mathfrak{p} = 1$ or -1, respectively. One easily checks that the conjugation induced on $M(2, K_\mathfrak{p})$ by this embedding of $\mathfrak{A}_\mathfrak{p}$ is given by

$$(4)\qquad \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto \begin{bmatrix} \bar{w} & c_\mathfrak{p}^{-1}\bar{z} \\ c_\mathfrak{p}\bar{y} & \bar{x} \end{bmatrix}, \qquad x, y, z, w \in K_\mathfrak{p}.$$

It follows that

$$(5)\qquad V_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -c_\mathfrak{p}\bar{y} & d \end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p};\ y \in K_\mathfrak{p} \right\},$$

the quadratic mapping being the determinant

$$(6)\qquad \begin{bmatrix} a & y \\ -c_\mathfrak{p}\bar{y} & d \end{bmatrix} \mapsto ad + c_\mathfrak{p} y\bar{y}.$$

In particular, we see that $V_\mathfrak{p}$ must be isotropic of index 1 if $D(V_\mathfrak{p})$ is not a square in $k_\mathfrak{p}$. We enumerate the various possibilities for $V_\mathfrak{p}$.

(1) If $\mathfrak{A}_\mathfrak{p}$ is split, we may take $c_\mathfrak{p} = 1$ and then

$$(7)\qquad V_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & d \end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p};\ y \in K_\mathfrak{p} \right\}.$$

(2) If $\mathfrak{A}_\mathfrak{p}$ is nonsplit and $\mathfrak{p}$ is infinite, then $k_\mathfrak{p} = \boldsymbol{R}$, $K_\mathfrak{p} = \boldsymbol{C}$ and we may take $c_\mathfrak{p} = -1$. Then

$$(8)\qquad V_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ \bar{y} & d \end{bmatrix} \,\middle|\, a, d \in \boldsymbol{R};\ y \in \boldsymbol{C} \right\}.$$

(3) If $\mathfrak{A}_\mathfrak{p}$ is nonsplit, $\mathfrak{p}$ is finite and $K_\mathfrak{p}$ is unramified over $k_\mathfrak{p}$, we may take $c_\mathfrak{p} = \pi$, a generator of $\mathfrak{p}$. Then

$$(9)\qquad V_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p};\ y \in K_\mathfrak{p} \right\}.$$

(4) If $\mathfrak{A}_\mathfrak{p}$ is nonsplit, $\mathfrak{p}$ is finite and $K_\mathfrak{p}$ is ramified over $k_\mathfrak{p}$, we may take $c_\mathfrak{p} = u_\mathfrak{p}$, any unit of $k_\mathfrak{p}$ such that $(u_\mathfrak{p}, D(V))_\mathfrak{p} = -1$. Then

$$(10)\qquad V_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -u_\mathfrak{p}\bar{y} & d \end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p};\ y \in K_\mathfrak{p} \right\}.$$

We recall that two quadratic spaces (V, q), (V', q') of the same dimension are said to be *similar*, written $V \sim V'$, if there is a linear map $f\colon V \to V'$ and an element $a \in k^\times$ such that $q'(f(v)) = aq(v)$ for all $v \in V$.

One can easily show that $V \sim V'$ implies $C_V^+ \cong C_{V'}^+$ ([1], p. 157, paragraph 12 (a)). For quaternary spaces over a number field k the converse is true.

PROPOSITION 4. *Let V, V' be quaternary spaces over a number field k. Then $V \sim V'$ if and only if $C_V^+ \cong C_{V'}^+$.*

Proof. Multiplying each quadratic mapping by a suitable scalar, we may assume that V, V' both represent 1. If C_V^+, $C_{V'}^+$ are k-isomorphic, then they must have the same center K. Let $\mathfrak{A}$, $\mathfrak{A}'$ be the quaternion algebras uniquely associated to V, V', respectively. Then $\mathfrak{A}_K \cong \mathfrak{A}'_K$ implies that $\mathfrak{A}_\mathfrak{p} \cong \mathfrak{A}'_\mathfrak{p}$ for all primes $\mathfrak{p}$ which split in K. For such $\mathfrak{p}$, $V_\mathfrak{p}$ is isometric to $\mathfrak{A}_\mathfrak{p}$ and $V'_\mathfrak{p}$ is isometric to $\mathfrak{A}'_\mathfrak{p}$. Hence $V_\mathfrak{p}$, $V'_\mathfrak{p}$ are isometric for all $\mathfrak{p}$ which split in K. For all other $\mathfrak{p}$ it is evident from the preceding discussion that $V_\mathfrak{p} \sim V'_\mathfrak{p}$. We conclude that $V \sim V'$, by the Hasse principle for similarity ([13], Theorem 1).

§ 3. Maximal lattices and class numbers. Let k be an algebraic number field or a non-archimedean local field of characteristic $\neq 2$ and let $\mathfrak{o}$ denote its ring of integers. Let V be a quaternary space over k. A *lattice* L in V is a finitely generated $\mathfrak{o}$-submodule of V having rank four. Given a lattice L in V, set

$$L^\# = \{v \in V \mid B(v, L) \subset \mathfrak{o}\}.$$

Then $L^\#$ is a lattice in V, called the *dual* of L, and $L^{\#\#} = L$. The *discriminant* $\Delta(L)$ of L is defined to be the fractional ideal $[L^\#:L]$ of $\mathfrak{o}$ (cf. [2], p. 10). If L happens to be a free $\mathfrak{o}$-module, then

$$\Delta(L) = \big(\det[B(v_i, v_j)]\big),$$

where $\{v_i\}$ is an $\mathfrak{o}$-basis of L, $i, j = 1, 2, 3, 4$.

Suppose L is a lattice in V. The fractional ideal of $\mathfrak{o}$ spanned by the set of all $q(v)$, $v \in L$, is called the *norm* of L and is denoted by $n(L)$. Set

$$L' = \{v \in V \mid B(v, L) \subset n(L)\}.$$

The *reduced discriminant* $\Delta'(L)$ is defined to be $[L':L]$. It is clear that $\Delta'(L)$ is an integral ideal of $\mathfrak{o}$. If L is a free $\mathfrak{o}$-module, as in the local case for example, then

$$\Delta'(L) = \big(\det[n(L)^{-1}B(v_i, v_j)]\big),$$

where $\{v_i\}$ is an $\mathfrak{o}$-basis of L. It is clear from this remark that our definition of reduced discriminant is the same as the one found in [4].

A lattice L in V is *integral* if $n(L)$ is an integral ideal; it is *maximally integral* if, in addition, it is not properly contained in another integral lattice. A lattice L is *maximal* if it is not properly contained in another lattice having the same norm; L is $\mathfrak{i}$-maximal if L is maximal and $n(L) = \mathfrak{i}$, a fractional ideal of $\mathfrak{o}$. Since q represents 1 locally, it is clear that a maximally integral lattice is nothing more than an $\mathfrak{o}$-maximal lattice.

We say that two lattices L, M in V are *similar*, and write $L\sim M$, if $\sigma L = M$ for some $\sigma \in S^+(V)$; if σ can be taken from $O^+(V)$, then we say that L, M are *equivalent*, and write $L \cong M$. The group of all proper similitudes, respectively, proper orthogonal transformations, which map a lattice L onto itself will be denoted by $S^+(L)$, respectively, $O^+(L)$. We call the elements of $S^+(L)$ the *units* of L and the elements of $O^+(L)$ the *orthogonal units* of L.

From now on we suppose that k is a number field. If L is a lattice in V, then $L_\mathfrak{p} = L\otimes_\mathfrak{o}\mathfrak{o}_\mathfrak{p}$ is a lattice in $V_\mathfrak{p}$ for every finite prime $\mathfrak{p}$. It is clear that the following relations hold:

$$(L^\#)_\mathfrak{p} = (L_\mathfrak{p})^\#, \quad \Delta(L_\mathfrak{p}) = \mathfrak{o}_\mathfrak{p}\Delta(L), \quad n(L_\mathfrak{p}) = \mathfrak{o}_\mathfrak{p}n(L),$$
$$(L')_\mathfrak{p} = (L_\mathfrak{p})', \quad \Delta'(L_\mathfrak{p}) = \mathfrak{o}_\mathfrak{p}\Delta'(L).$$

Furthermore, it is clear that L is integral $\Leftrightarrow L_\mathfrak{p}$ is integral for all finite $\mathfrak{p}$, L is maximal $\Leftrightarrow L_\mathfrak{p}$ is maximal for all finite $\mathfrak{p}$, and L is maximally integral $\Leftrightarrow L_\mathfrak{p}$ is maximally integral for all finite $\mathfrak{p}$.

Two lattices L, M in V are in the same *class* if $L \cong M$, in the same *similitude class* if $L\sim M$; they are in the same *genus* if $L_\mathfrak{p} \cong M_\mathfrak{p}$ for all finite $\mathfrak{p}$, in the same *idealcomplex* if $L_\mathfrak{p} \sim M_\mathfrak{p}$ for all finite $\mathfrak{p}$; if σL is in the genus of M for some $\sigma \in S^+(V)$, then L, M are in the same *similitude genus*. The basic finiteness theorems for quadratic spaces state that each genus decomposes into a finite number of classes and each idealcomplex decomposes into a finite number of similitude classes ([4], p. 79).

Let $\mathfrak{G}$ be a genus of lattices in V and $\mathfrak{G}_s$ the similitude genus containing $\mathfrak{G}$. Then the number of similitude classes represented in $\mathfrak{G}$ is the same as the number of similitude classes in $\mathfrak{G}_s$. However, the number of classes in $\mathfrak{G}$ may be greater than the number of similitude classes represented in $\mathfrak{G}$. The precise relation between these two numbers is given in [14], § 2. In particular, if $k = \mathbf{Q}$, the field of rational numbers, then these two numbers coincide if q has signature $\neq 0$.

The collection of maximally integral lattices in V forms a genus which we denote by $\mathfrak{M}$ (cf. [12], p. 240). We denote the common discriminant of all the lattices in $\mathfrak{M}$ by Δ. Then $\mathfrak{M}$ can also be described as the set of all integral lattices in V having discriminant Δ. Let $\mathfrak{I}$ denote the idealcomplex containing $\mathfrak{M}$. We can also describe $\mathfrak{I}$ as the set of all maximal lattices having reduced discriminant Δ (cf. [4], p. 87). The class numbers of interest to us will be H_0, the number of classes in $\mathfrak{M}$, and H, the number of similitude classes in $\mathfrak{I}$. The number of similitude genera in $\mathfrak{I}$ can be expressed in terms of the number of ambiguous ideal classes in K (cf. [14], § 2, Satz 6). In particular, if $k = \mathbf{Q}$ and q has signature $\neq 0$, then $\mathfrak{I}$ decomposes into g^+ similitude genera, where g^+ is the number of strict genera of K. Unfortunately, if q is definite, the various similitude genera

need not have the same number of similitude classes, so that the most we can say about the relation of H_0 to H in the case $k = \mathbf{Q}$, q definite, is that $H_0 \leqslant H$, with equality $\Leftrightarrow g^+ = 1 \Leftrightarrow K$ has prime discriminant.

We proceed now to classify the maximal lattices in V locally. We may assume that V represents 1. Our first step is to classify the maximally integral lattices in V. Since the maximally integral lattices are all locally equivalent, it is enough to exhibit one maximally integral lattice $M_\mathfrak{p}$ for each finite prime $\mathfrak{p}$ of k. As usual, we must consider various cases.

Notation. Let $\mathfrak{O}$ denote the ring of integers in K; for a finite prime $\mathfrak{p}$ of k, let $\mathfrak{O}_\mathfrak{p} = \mathfrak{O}\otimes_\mathfrak{o}\mathfrak{o}_\mathfrak{p}$ denote the ring of integers in $K_\mathfrak{p}$.

1. $D(V)$ is a square in $k_\mathfrak{p}$.

(a) If $V_\mathfrak{p}$ is isotropic, then $V_\mathfrak{p} = \mathfrak{A}_\mathfrak{p} = M(2, k_\mathfrak{p})$ and we may take $M_\mathfrak{p} = M(2, \mathfrak{o}_\mathfrak{p})$.

(b) If $V_\mathfrak{p}$ is anisotropic, then $V_\mathfrak{p} = \mathfrak{A}_\mathfrak{p} =$ the unique quaternion division algebra over $k_\mathfrak{p}$ and we take $M_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$, the maximal order of $\mathfrak{A}_\mathfrak{p}$.

2. $D(V)$ is not a square in $k_\mathfrak{p}$, $K_\mathfrak{p} = k_\mathfrak{p}(\sqrt{D(V)})$.

(a) If $\mathfrak{A}_\mathfrak{p}$ is split, then

$$V_\mathfrak{p} = \left\{\begin{bmatrix} a & y \\ -\bar{y} & d\end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p},\ y \in K_\mathfrak{p}\right\}.$$

We claim

$$M_\mathfrak{p} = \left\{\begin{bmatrix} a & y \\ -\bar{y} & d\end{bmatrix} \,\middle|\, a, d \in \mathfrak{o}_\mathfrak{p},\ y \in \mathfrak{O}_\mathfrak{p}\right\} \tag{11}$$

is maximally integral in $V_\mathfrak{p}$. Suppose not. Then there is an integral lattice $L_\mathfrak{p} \supset M_\mathfrak{p}$, $L_\mathfrak{p} \neq M_\mathfrak{p}$. Take $v \in L_\mathfrak{p}$, $v \notin M_\mathfrak{p}$. Then

$$v = \begin{bmatrix} a & y \\ -\bar{y} & d\end{bmatrix}, \quad a, d \in k_\mathfrak{p},\ y \in K_\mathfrak{p}.$$

Suppose either a or d is not in $\mathfrak{o}_\mathfrak{p}$. By symmetry, we may assume $a \notin \mathfrak{o}_\mathfrak{p}$. Put

$$w = \begin{bmatrix} a & y \\ -\bar{y} & d+1\end{bmatrix}.$$

Then $w \in L_\mathfrak{p}$ but $N(w) = N(v) + a \notin \mathfrak{o}_\mathfrak{p}$, a contradiction. Hence $a, d \in \mathfrak{o}_\mathfrak{p}$, which implies $y \in \mathfrak{O}_\mathfrak{p}$, since $\mathfrak{O}_\mathfrak{p} = \{y \in K_\mathfrak{p} \mid y\bar{y} \in \mathfrak{o}_\mathfrak{p}\}$.

(b) If $\mathfrak{A}_\mathfrak{p}$ is nonsplit, then

$$V_\mathfrak{p} = \left\{\begin{bmatrix} a & y \\ -\pi\bar{y} & d\end{bmatrix} \,\middle|\, a, d \in k_\mathfrak{p},\ y \in K_\mathfrak{p}\right\} \tag{i}$$

if $K_{\mathfrak{p}}$ is unramified over $k_{\mathfrak{p}}$, where π is a generator of $\mathfrak{p}$, and

$$\text{(ii)} \qquad V_{\mathfrak{p}} = \left\{ \begin{bmatrix} a & y \\ -u_{\mathfrak{p}}\bar{y} & d \end{bmatrix} \middle| \; a, d \in k_{\mathfrak{p}}, y \in K_{\mathfrak{p}} \right\}$$

if $K_{\mathfrak{p}}$ is ramified over $k_{\mathfrak{p}}$, where $u_{\mathfrak{p}}$ is a unit of $k_{\mathfrak{p}}$ such that $(u_{\mathfrak{p}}, D(V))_{\mathfrak{p}} = -1$.

Accordingly, reasoning exactly as in 2(a), we may take

$$(12) \qquad \text{(i)} \quad M_{\mathfrak{p}} = \left\{ \begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_{\mathfrak{p}}, y \in \mathfrak{O}_{\mathfrak{p}} \right\},$$

$$(13) \qquad \text{(ii)} \quad M_{\mathfrak{p}} = \left\{ \begin{bmatrix} a & y \\ -u_{\mathfrak{p}}\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_{\mathfrak{p}}, y \in \mathfrak{O}_{\mathfrak{p}} \right\}.$$

Suppose that $\mathcal{O}$ is an order of a quaternion algebra over a local or global number field. The *level* (*Stufe*) of $\mathcal{O}$ is defined to be $n(\mathcal{O}^{\#})^{-1}$. By an *Eichler order* we mean an order of a quaternion algebra over a local or global number field having square-free level (cf. [6], p. 130). Suppose $D(V)$ is not a square in k, so that K is a quadratic extension of k. Let $\mathfrak{p}$ be a finite prime of k which splits in K. We say that an $\mathfrak{O}_{\mathfrak{p}}$-order $\Omega(\mathfrak{p})$ of $\mathfrak{A}_{K_{\mathfrak{p}}} = \mathfrak{A}_{\mathfrak{p}} \oplus \mathfrak{A}_{\mathfrak{p}}$ is an *Eichler order* if it is $\mathfrak{O}_{\mathfrak{p}}$-isomorphic to an order of the form $\mathcal{O}_1 \oplus \mathcal{O}_2$, where $\mathcal{O}_1$, $\mathcal{O}_2$ are Eichler orders of $\mathfrak{A}_{\mathfrak{p}}$.

Let Λ be an $\mathfrak{O}$-lattice of $\mathfrak{A}_K$. For any finite prime $\mathfrak{p}$ of k we put $\Lambda_{\mathfrak{p}} = \Lambda \otimes_{\mathfrak{O}} \mathfrak{O}_{\mathfrak{p}} = \Lambda \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$, an $\mathfrak{O}_{\mathfrak{p}}$-lattice of $\mathfrak{A}_{K_{\mathfrak{p}}}$. We say that a lattice Λ of $\mathfrak{A}_K$ is *symmetric* if $\bar{\Lambda}^* = \Lambda$; similarly, a lattice $\Lambda(\mathfrak{p})$ of $\mathfrak{A}_{K_{\mathfrak{p}}}$ is symmetric if $\overline{\Lambda(\mathfrak{p})}^* = \Lambda(\mathfrak{p})$. It is clear that a lattice Λ of $\mathfrak{A}_K$ is symmetric $\Leftrightarrow \Lambda_{\mathfrak{p}}$ is symmetric for every finite prime $\mathfrak{p}$ of k.

Lemma. *For any finite prime $\mathfrak{p}$ of k there exists an Eichler order $\Omega(\mathfrak{p})$ of $\mathfrak{A}_{K_{\mathfrak{p}}}$ such that $\Omega(\mathfrak{p})$ is symmetric and $M_{\mathfrak{p}} = \Omega(\mathfrak{p}) \cap V_{\mathfrak{p}}$ is a maximally integral lattice in $V_{\mathfrak{p}}$. The order $\Omega(\mathfrak{p})$ is maximal except if $\mathfrak{p}$ ramifies in $\mathfrak{A}$ and remains prime in K.*

Proof. 1. $D(V)$ is a square in $K_{\mathfrak{p}}$. Then $\mathfrak{A}_{K_{\mathfrak{p}}} = \mathfrak{A}_{\mathfrak{p}} \oplus \mathfrak{A}_{\mathfrak{p}}$, $V_{\mathfrak{p}} = \{(\xi, \xi^*) \mid \xi \in \mathfrak{A}_{\mathfrak{p}}\}$. We take $\Omega(\mathfrak{p}) = M(2, \mathfrak{o}_{\mathfrak{p}}) \oplus M(2, \mathfrak{o}_{\mathfrak{p}})$ for case 1(a) and $\Omega(\mathfrak{p}) = \mathcal{O}_{\mathfrak{p}} \oplus \mathcal{O}_{\mathfrak{p}}$ for case 1 (b). For cases 2 (a) and 2 (b) (ii) we take $\Omega(\mathfrak{p}) = M(2, \mathfrak{O}_{\mathfrak{p}})$. For case 2(b) (i) we take

$$\Omega(\mathfrak{p}) = \begin{bmatrix} \mathfrak{O}_{\mathfrak{p}} & \mathfrak{O}_{\mathfrak{p}} \\ \mathfrak{p}\mathfrak{O}_{\mathfrak{p}} & \mathfrak{O}_{\mathfrak{p}} \end{bmatrix}.$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$ be the finite primes of k which ramify in $\mathfrak{A}$ but remain prime in K; let $\mathfrak{q}_1, \ldots, \mathfrak{q}_f$ be the finite primes of k which ramify in $\mathfrak{A}$ but split in K (i.e. the anisotropic finite primes of V). Put $\delta = \mathfrak{p}_1 \ldots \mathfrak{p}_e \mathfrak{q}_1 \ldots \mathfrak{q}_f$. By an Eichler order of $\mathfrak{A}_K$ of level δ we will mean one of levels $\delta\mathfrak{O}$.

Proposition 5. *Suppose V represents 1 and $D(V)$ is not a square in k. Then there exists a symmetric Eichler order Ω of $\mathfrak{A}_K$ of level δ such that $M = \Omega \cap V$ is a maximally integral lattice in V.*

Proof. Let Φ be an arbitrary Eichler order of $\mathfrak{A}_K$ of level δ. Then, for almost all $\mathfrak{p}$, $\bar{\Phi}_{\mathfrak{p}}^* = \Phi_{\mathfrak{p}}$ and $(\Phi \cap V)_{\mathfrak{p}} = \Phi_{\mathfrak{p}} \cap V_{\mathfrak{p}}$ is a maximally integral lattice in $V_{\mathfrak{p}}$. At the finitely many exceptional primes we replace $\Phi_{\mathfrak{p}}$ by the order $\Omega(\mathfrak{p})$ of the preceding lemma. We then let Ω be the unique order of $\mathfrak{A}_K$ such that $\Omega_{\mathfrak{p}} = \Omega(\mathfrak{p})$ for the exceptional $\mathfrak{p}$ and $\Omega_{\mathfrak{p}} = \Phi_{\mathfrak{p}}$ for all other $\mathfrak{p}$.

At this point let us determine the discriminant Δ of a maximally integral lattice M in V. It is enough to compute the local discriminant $\Delta(M_{\mathfrak{p}})$ for each finite prime $\mathfrak{p}$.

1. $D(V)$ is a square in $k_{\mathfrak{p}}$. Then $M_{\mathfrak{p}}$ can be taken as a maximal order in $\mathfrak{A}_{\mathfrak{p}}$, so $\Delta(M_{\mathfrak{p}})$ is just the usual discriminant of the quaternion algebra $\mathfrak{A}_{\mathfrak{p}}$, namely

$$(14) \qquad \Delta(M_{\mathfrak{p}}) = \begin{cases} \mathfrak{o}_{\mathfrak{p}} & \text{if } \mathfrak{A}_{\mathfrak{p}} \text{ is split,} \\ \mathfrak{p}^2 & \text{if } \mathfrak{A}_{\mathfrak{p}} \text{ is nonsplit.} \end{cases}$$

2. $D(V)$ is not a square in $k_{\mathfrak{p}}$. Denote the discriminants of K, $K_{\mathfrak{p}}$ by Δ_K, $\Delta_{K_{\mathfrak{p}}}$, respectively. Let $\{x_1, x_2\}$ be an $\mathfrak{o}_{\mathfrak{p}}$-basis of $\mathfrak{O}_{\mathfrak{p}}$ and $\{x_1', x_2'\}$ its dual basis, i.e. $\mathrm{tr}_{K_{\mathfrak{p}}/k_{\mathfrak{p}}}(x_i x_j') = \delta_{ij}$, $i, j = 1, 2$, where $\mathrm{tr}_{K_{\mathfrak{p}}/k_{\mathfrak{p}}}\colon K_{\mathfrak{p}} \to k_{\mathfrak{p}}$ is the trace mapping. For cases 2(a) and 2(b) (ii) we have

$$M_{\mathfrak{p}} = \left\{ \begin{bmatrix} a & y \\ -u_{\mathfrak{p}}\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_{\mathfrak{p}}, y \in \mathfrak{O}_{\mathfrak{p}} \right\},$$

where $u_{\mathfrak{p}}$ is a unit of $\mathfrak{o}_{\mathfrak{p}}$. Thus an $\mathfrak{o}_{\mathfrak{p}}$-basis for $M_{\mathfrak{p}}$ is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & x_1 \\ -u_{\mathfrak{p}}\bar{x}_1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x_2 \\ -u_{\mathfrak{p}}\bar{x}_2 & 0 \end{bmatrix}.$$

The dual of this basis with respect to the bilinear form B is given by

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & u_{\mathfrak{p}}^{-1}\bar{x}_1' \\ -x_1' & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & u_{\mathfrak{p}}^{-1}\bar{x}_2' \\ -x_2' & 0 \end{bmatrix}.$$

It follows directly that $\Delta(M_{\mathfrak{p}}) = \Delta_{K_{\mathfrak{p}}} = \Delta_K \mathfrak{o}_{\mathfrak{p}}$. For case 2(b) (i) we have

$$M_{\mathfrak{p}} = \left\{ \begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_{\mathfrak{p}}, y \in \mathfrak{O}_{\mathfrak{p}} \right\},$$

where $(\pi) = \mathfrak{p}$.

Then

$$\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \begin{bmatrix}0&x_1\\-\pi\bar{x}_1&0\end{bmatrix},\quad \begin{bmatrix}0&x_2\\-\pi\bar{x}_2&0\end{bmatrix}$$

is an $\mathfrak{o}_\mathfrak{p}$-basis for $M_\mathfrak{p}$ and

$$\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&\pi^{-1}\bar{x}_1'\\-x_1'&0\end{bmatrix},\quad \begin{bmatrix}0&\pi^{-1}\bar{x}_2'\\-x_2'&0\end{bmatrix}$$

is its dual basis. Hence $\Delta(M_\mathfrak{p}) = \mathfrak{p}^2\Delta_K = \mathfrak{p}^2$. We have proved the following:

PROPOSITION 6. *Let V be a quaternary space over k representing 1. Let Δ be the discriminant of the genus $\mathfrak{M}$ of maximally integral lattices in V.*

(a) *If $D(V)$ is a square in k, then $\Delta = (\mathfrak{q}_1\ldots\mathfrak{q}_f)^2$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_f$ are the anisotropic finite primes of V.*

(b) *If $D(V)$ is not a square in k, $K = k(\sqrt{D(V)})$, and $\mathfrak{A}$ is the quaternion algebra over k associated to V, then $\Delta = \Delta_K\delta^2$, where Δ_K is the discriminant of K and δ is the product of all finite primes of k which ramify in $\mathfrak{A}$ but not in K.*

We now complete the local classification of maximal lattices in V by exhibiting a representative $M_\mathfrak{p}$ for each local similitude class of maximal lattices in $V_\mathfrak{p}$, where $\mathfrak{p}$ is a finite prime.

If $D(V)$ is a square in $k_\mathfrak{p}$, then every element of $k_\mathfrak{p}$ is the norm of a proper similitude. Hence every maximal lattice in $V_\mathfrak{p}$ is similar to a maximally integral lattice. Thus we may take $M_\mathfrak{p}$ = a maximally integral lattice in $V_\mathfrak{p}$. If $D(V)$ is not a square in $k_\mathfrak{p}$, then we must distinguish between the cases $K_\mathfrak{p}$ ramified over $k_\mathfrak{p}$ and $K_\mathfrak{p}$ unramified over $k_\mathfrak{p}$. If $K_\mathfrak{p}$ is ramified over $k_\mathfrak{p}$, then (b) of Proposition 3 shows that some prime element π of $k_\mathfrak{p}$ is the norm of a similitude. If follows that every maximal lattice in $V_\mathfrak{p}$ is similar to a maximally integral lattice $M_\mathfrak{p}$. If $K_\mathfrak{p}$ is unramified over $k_\mathfrak{p}$, then the elements of $k_\mathfrak{p}^\times$ which are the norms of similitudes are those having even $\mathfrak{p}$-order. Hence every maximal lattice is similar either to a maximally integral lattice or to a maximal lattice of norm $\mathfrak{p}$.

If $\mathfrak{A}_\mathfrak{p}$ is split, then it is easy to verify that

$$M_\mathfrak{p} = \left\{\begin{bmatrix}a&\pi y\\-\pi\bar{y}&\pi d\end{bmatrix}\;\middle|\; a, d\in\mathfrak{o}_\mathfrak{p}, y\in\mathfrak{O}_\mathfrak{p}\right\} \tag{15}$$

is $\mathfrak{p}$-maximal, where π is a prime element of $k_\mathfrak{p}$. It is clear that $\Delta'(M_\mathfrak{p}) = \mathfrak{p}^2$.

If $\mathfrak{A}_\mathfrak{p}$ is nonsplit, then we take

$$M_\mathfrak{p} = \left\{\begin{bmatrix}\pi a&y\\-\pi\bar{y}&d\end{bmatrix}\;\middle|\; a, d\in\mathfrak{o}_\mathfrak{p}, y\in\mathfrak{O}_\mathfrak{p}\right\}. \tag{16}$$

From $\Delta'(M_\mathfrak{p}) = \mathfrak{o}_\mathfrak{p}$ it follows that $M_\mathfrak{p}$ must be $\mathfrak{p}$-maximal ([4], p. 50).

Summarizing, we have shown:

PROPOSITION 7. *Let V be a quaternary space over k representing 1. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_f$ be the anisotropic finite primes of V.*

(a) *If $D(V)$ is a square in k, then V has a unique idealcomplex of maximal lattices $\mathfrak{I}$, the set of all maximal lattices of reduced discriminant $(\mathfrak{q}_1\ldots\mathfrak{q}_f)^2$.*

(b) *If $D(V)$ is not a square in k, then for every finite set $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$ of primes of k which remain prime in $K = k(\sqrt{D(V)})$ there is a (unique) idealcomplex of maximal lattices in V having reduced discriminant*

$$\Delta_K(\mathfrak{q}_1\ldots\mathfrak{q}_f)^2(\mathfrak{p}_1\ldots\mathfrak{p}_e)^2.$$

Every maximal lattice of V lies in one of these idealcomplexes.

We see that $\Delta_K(\mathfrak{q}_1\ldots\mathfrak{q}_f)^2$ is the "smallest" reduced discriminant of maximal lattices in V, hence the "smallest" reduced discriminant of all lattices in V. Accordingly, it is natural to call $\Delta_K(\mathfrak{q}_1\ldots\mathfrak{q}_f)^2$ the *fundamental discriminant* of V, denoted by Δ_V. Then Proposition 4 can be rephrased as: $V\sim V'$ if and only if $D(V) = D(V')$, $\Delta_V = \Delta_{V'}$, and $V_\mathfrak{p}$, $V_\mathfrak{p}'$ have the same Witt index for every infinite prime $\mathfrak{p}$ of k.

If V, V' are quaternary spaces which are similar by a mapping $f\colon V\to V'$, then f gives a one-to-one correspondence between similitude classes and idealcomplexes of V and those of V'. In particular, since f preserves maximal lattices and reduced discriminants, f must take an idealcomplex of maximal lattices in V to the idealcomplex of maximal lattices in V' of the same reduced discriminant. Suppose $\mathfrak{L}$ is an idealcomplex of maximal lattices of V. Let Δ' be the reduced discriminant of $\mathfrak{L}$. We can choose $L\in\mathfrak{L}$ such that $L_\mathfrak{p}$ is either maximally integral or $\mathfrak{p}$-maximal for every finite prime $\mathfrak{p}$. Let S denote the set of finite primes $\mathfrak{p}$ for which $L_\mathfrak{p}$ is not maximally integral. If S has an even number of elements, let $\mathfrak{A}'$ be the quaternion algebra over k obtained by taking $\mathfrak{A}_\mathfrak{p}'$ different from $\mathfrak{A}_\mathfrak{p}$ for all $\mathfrak{p}\in S$; if S has an odd number of elements and at least one prime $\mathfrak{r}$ of k (finite or infinite) ramifies in K, then let $\mathfrak{A}'$ be the one obtained by changing $\mathfrak{A}_\mathfrak{p}$ for all $\mathfrak{p}\in S\cup\{\mathfrak{r}\}$. Then the quaternary space V' associated to $\mathfrak{A}'$ is similar to V and its maximally integral lattices have discriminant Δ'. Thus if at least one prime of k ramifies in K, then for purposes of studying similitude classes of maximal lattices, there is no loss of generality in restricting ourselves to the idealcomplex $\mathfrak{I}$ of V, provided we let V vary. However, if every prime of k is unramified in K (i.e. K is a subfield of the Hilbert class field of k), then there exist idealcomplexes of maximal lattices which are never mapped to $\mathfrak{I}$ by any similarity mapping. Such an idealcomplex will be called *intransigent*. We note that intransigent idealcomplexes exist only if k has even class number. To study similitude classes of intransigent idealcomplexes we need only consider those having reduced discriminant $\Delta/\mathfrak{p}_0^2$, where $\mathfrak{p}_0$ is a finite prime of k which remains prime in K and ramifies in $\mathfrak{A}$. Such an idealcomplex contains a lattice M such that $M_\mathfrak{p}$ is maximally integral for $\mathfrak{p} \neq \mathfrak{p}_0$, but

(17) $$M_{\mathfrak{p}_0} = \left\{\begin{bmatrix} \pi a & y \\ -\pi\bar{y} & d\end{bmatrix} \;\middle|\; a, d \in \mathfrak{o}_{\mathfrak{p}_0}, y \in \mathfrak{O}_{\mathfrak{p}_0}\right\},$$

where $(\pi) = \mathfrak{p}_0$. Then $M_{\mathfrak{p}_0} = \Lambda_{\mathfrak{p}_0} \cap V_{\mathfrak{p}_0}$, where

(18) $$\Lambda_{\mathfrak{p}_0} = \begin{bmatrix} \pi\mathfrak{O}_{\mathfrak{p}_0} & \mathfrak{O}_{\mathfrak{p}_0} \\ \pi\mathfrak{O}_{\mathfrak{p}_0} & \mathfrak{O}_{\mathfrak{p}_0}\end{bmatrix} = \begin{bmatrix} \mathfrak{O}_{\mathfrak{p}_0} & \mathfrak{O}_{\mathfrak{p}_0} \\ \mathfrak{O}_{\mathfrak{p}_0} & \mathfrak{O}_{\mathfrak{p}_0}\end{bmatrix}\begin{bmatrix} 0 & 1 \\ \pi & 0\end{bmatrix},$$

a left ideal of $M(2, \mathfrak{O}_{\mathfrak{p}_0})$. We note that $\Lambda_{\mathfrak{p}_0}$ is symmetric. Then Proposition 5 can be reformulated for intransigent idealcomplexes to read: There exists a symmetric lattice Λ with left order of level $\delta/\mathfrak{p}_0$, such that $M = \Lambda \cap V$ is a $\mathfrak{p}_0$-maximal lattice of V. Because of this formal resemblance, and in order to simplify our notation, we extend the use of $\mathfrak{J}, H$ by permitting them to denote, respectively, an intransigent idealcomplex, and the number of similitude classes therein. Consistent with Proposition 5, we let Ω denote the left order of Λ. Note that, although Λ is symmetric, Ω is not.

§ 4. The relation of H to a generalized type number. In this section we determine the local groups of units $S^+(M_\mathfrak{p})$ for a maximal lattice M in V and use this information to relate H to a certain generalized type number associated to the quaternion algebra $\mathfrak{A}_K$.

According to our discussion in § 3, we may assume that either (1) M is maximally integral, $M = \Omega \cap V$, where Ω is a symmetric Eichler order of $\mathfrak{A}_K$ of level δ, or (2) M is $\mathfrak{p}_0$-maximal, $M = \Lambda \cap V$, where Λ is a symmetric lattice whose left order Ω is an Eichler order of level $\delta/\mathfrak{p}_0$. Furthermore, by throwing the primes which ramify in K or $\mathfrak{A}$ into the exceptional set mentioned in the proof of Proposition 5, we may assume that $M_\mathfrak{p}$, $\Omega_\mathfrak{p}$, $\Lambda_\mathfrak{p}$ for these $\mathfrak{p}$ are in the standard forms given in (17), (18), and in the proof of the lemma preceding Proposition 5.

Let $\mathfrak{p}$ be a prime of k. By the corollary to Proposition 3, we have

$$S^+(V_\mathfrak{p}) = k_\mathfrak{p}^\times \times \mathfrak{A}_{K_\mathfrak{p}}^\times / \Gamma(k_\mathfrak{p}^\times \times K_\mathfrak{p}^\times).$$

If $\mathfrak{p}$ is a finite prime, let $U_\mathfrak{p}$, $U(\mathfrak{O}_\mathfrak{p})$, $U(\Omega_\mathfrak{p})$ denote the unit groups of $\mathfrak{o}_\mathfrak{p}, \mathfrak{O}_\mathfrak{p}, \Omega_\mathfrak{p}$, respectively. If $\mathfrak{p}$ is infinite, put $\mathfrak{o}_\mathfrak{p} = k_\mathfrak{p}$, $\mathfrak{O}_\mathfrak{p} = K_\mathfrak{p}$, $\Omega_\mathfrak{p} = \mathfrak{A}_{K_\mathfrak{p}}$, and let $U_\mathfrak{p}$, $U(\mathfrak{O}_\mathfrak{p})$, $U(\Omega_\mathfrak{p})$ denote $k_\mathfrak{p}^\times$, $K_\mathfrak{p}^\times$, $\mathfrak{A}_{K_\mathfrak{p}}^\times$, respectively. Put

$$\Gamma\big(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\big) = \{(c, x) \in U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p}) \mid n_{K_\mathfrak{p}/k_\mathfrak{p}}(x) = c^{-1}\}.$$

Then

$$U_\mathfrak{p} \times U(\Omega_\mathfrak{p}) \cap \Gamma(k_\mathfrak{p}^\times \times K_\mathfrak{p}^\times) = \Gamma\big(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\big).$$

Hence we have a natural embedding

$$U_\mathfrak{p} \times U(\Omega_\mathfrak{p}) / \Gamma\big(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\big) \subset S^+(V_\mathfrak{p}).$$

LEMMA. *Let Ω, M be chosen as in Proposition* 5 *or its analogue for intransigent idealcomplexes. Then*

$$U_\mathfrak{p} \times U(\Omega_\mathfrak{p}) / \Gamma\big(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\big) \subset S^+(M_\mathfrak{p}).$$

Proof. Let $u \in U_\mathfrak{p}$, $\varepsilon \in U(\Omega_\mathfrak{p})$. Suppose $M_\mathfrak{p}$ is maximally integral. Then $M_\mathfrak{p} = \Omega_\mathfrak{p} \cap V_\mathfrak{p} = u\varepsilon\Omega_\mathfrak{p}\bar{\varepsilon}^* \cap V_\mathfrak{p} = u\varepsilon(\Omega_\mathfrak{p} \cap V_\mathfrak{p})\bar{\varepsilon}^* = u\varepsilon M_\mathfrak{p}\bar{\varepsilon}^*$. Now suppose $M_\mathfrak{p}$ is $\mathfrak{p}$-maximal. Then $M_\mathfrak{p} = \Omega_\mathfrak{p}\Pi \cap V_\mathfrak{p}$, where

$$\Pi = \begin{bmatrix} 0 & 1 \\ \pi & 0\end{bmatrix}.$$

We note that $\bar{\Omega}_\mathfrak{p}^* = \Pi^{-1}\Omega_\mathfrak{p}\Pi$. Then $\bar{\varepsilon}^* \in U(\bar{\Omega}_\mathfrak{p}^*)$ and

$$M_\mathfrak{p} = \Omega_\mathfrak{p}\Pi \cap V_\mathfrak{p} = u\varepsilon\Omega_\mathfrak{p}\Pi \cap V_\mathfrak{p} = u\varepsilon\Pi\bar{\Omega}_\mathfrak{p}^* \cap V_\mathfrak{p} = u\varepsilon\Pi\bar{\Omega}_\mathfrak{p}^*\bar{\varepsilon}^* \cap V_\mathfrak{p}$$
$$= u\varepsilon\Omega_\mathfrak{p}\Pi\bar{\varepsilon}^* \cap V_\mathfrak{p} = u\varepsilon M_\mathfrak{p}\bar{\varepsilon}^*.$$

Let $\mathcal{O}$ be an Eichler order in a quaternion algebra $\mathfrak{A}$ over a local or global number field k. The *normalizer* $\mathfrak{N}(\mathcal{O})$ of $\mathcal{O}$ is defined to be the group of all $a \in \mathfrak{A}^\times$ such that $a\mathcal{O}a^{-1} = \mathcal{O}$. If k is a global number field and $\mathfrak{p}$ is a finite prime of k, then

(19) $$[\mathfrak{N}(\mathcal{O}_\mathfrak{p}) : k_\mathfrak{p}^\times U(\mathcal{O}_\mathfrak{p})] = \begin{cases} 1 & \text{if } \mathcal{O}_\mathfrak{p} \text{ is of level } \mathfrak{o}_\mathfrak{p}, \\ 2 & \text{if } \mathcal{O}_\mathfrak{p} \text{ is of level } \mathfrak{p}. \end{cases}$$

In the case where $\mathcal{O}_\mathfrak{p}$ is of level $\mathfrak{p}$, the non-trivial coset $\bmod k_\mathfrak{p}^\times U(\mathcal{O}_\mathfrak{p})$ can be represented by a generator Π of the unique two-sided prime ideal of $\mathcal{O}_\mathfrak{p}$ (cf. [6], § 2).

We proceed to determine coset representatives for

$$\big(U_\mathfrak{p} \times U(\Omega_\mathfrak{p}) / \Gamma\big(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\big)\big) \backslash S^+(M_\mathfrak{p}).$$

Notation. Suppose $a, b \in k_\mathfrak{p}^\times$, $\alpha, \beta \in \mathfrak{A}_{K_\mathfrak{p}}^\times$. Then $(a, \alpha) \equiv (b, \beta)$ will mean

$$(ab^{-1}, \alpha\beta^{-1}) \in \Gamma(k_\mathfrak{p}^\times \times K_\mathfrak{p}^\times).$$

1. $D(V)$ is a square in $k_\mathfrak{p}$. Then

$$V_\mathfrak{p} = \{(\xi, \xi^*) \mid \xi \in \mathfrak{A}_\mathfrak{p}\}, \quad M_\mathfrak{p} = \{(\xi, \xi^*) \mid \xi \in \mathcal{O}_\mathfrak{p}\},$$

where $\mathcal{O}_\mathfrak{p}$ is a maximal order of $\mathfrak{A}_\mathfrak{p}$. From $\bar{\Omega}_\mathfrak{p}^* = \Omega_\mathfrak{p}$ and $\Omega_\mathfrak{p} \cap V_\mathfrak{p} = M_\mathfrak{p}$ we deduce $\Omega_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \oplus \mathcal{O}_\mathfrak{p}$. Suppose $(c, \alpha) \in k_\mathfrak{p}^\times \times \mathfrak{A}_{K_\mathfrak{p}}^\times$ and $\psi(c, \alpha) \in S^+(M_\mathfrak{p})$ (cf. Corollary, Proposition 3). Then $\alpha = (\beta, \gamma)$, $\beta, \gamma \in \mathfrak{A}_\mathfrak{p}^\times$, and

$$M_\mathfrak{p} = c\alpha M_\mathfrak{p}\bar{\alpha}^* = (c\beta\mathcal{O}_\mathfrak{p}\gamma^* \oplus c\gamma\mathcal{O}_\mathfrak{p}\beta^*) \cap V_\mathfrak{p}.$$

Thus $c\beta\mathcal{O}_\mathfrak{p}\gamma^* = \mathcal{O}_\mathfrak{p}$, which implies $\beta, \gamma^* \in \mathfrak{N}(\mathcal{O}_\mathfrak{p})$. According to (19), if $\mathfrak{A}_\mathfrak{p}$ is split we may assume $\beta, \gamma^* \in k_\mathfrak{p}^\times$, i.e. $\beta, \gamma \in k_\mathfrak{p}^\times$. Then $c\beta\mathcal{O}_\mathfrak{p}\gamma^* = c\beta\gamma\mathcal{O}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$, which implies $c = (\beta\gamma)^{-1}u$, $u \in U_\mathfrak{p}$. Hence $(c, \alpha) \equiv (u, 1) \in U_\mathfrak{p} \times U(\Omega_\mathfrak{p})$. We conclude that

$$S^+(M_\mathfrak{p}) = U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p}))$$

if $\mathfrak{A}_\mathfrak{p}$ is split. If $\mathfrak{A}_\mathfrak{p}$ is nonsplit we have the additional possibility $\alpha = (\Pi, \Pi)$, $c = N(\Pi)^{-1}$, which gives

$$[S^+(M_\mathfrak{p}) : U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p}))] = 2.$$

2. $D(V)$ is not a square in $k_\mathfrak{p}$. Suppose that $c\alpha M_\mathfrak{p}\bar{\alpha}^* = M_\mathfrak{p}$, $(c, \alpha) \in k_\mathfrak{p}^\times \times \mathfrak{A}_{K_\mathfrak{p}}^\times$. Multiplying (c, α) by a suitable element of $\Gamma(k_\mathfrak{p}^\times \times K_\mathfrak{p}^\times)$ we may take $\alpha \in \Omega_\mathfrak{p}$. Let π be a prime element of $\mathfrak{O}_\mathfrak{p}$. If $\Omega_\mathfrak{p} = M(2, \mathfrak{O}_\mathfrak{p})$, then, multiplying α on the left by a suitable unit of $\Omega_\mathfrak{p}$ ([6], p. 132), we may assume that

$$\alpha = \begin{bmatrix} \pi^i & x \\ 0 & \pi^j \end{bmatrix}, \quad i, j \geqslant 0 \tag{20}$$

where $x \in \mathfrak{O}_\mathfrak{p}$ is reduced mod π^j. If, on the other hand,

$$\Omega_\mathfrak{p} = \begin{bmatrix} \mathfrak{O}_\mathfrak{p} & \mathfrak{O}_\mathfrak{p} \\ \pi\mathfrak{O}_\mathfrak{p} & \mathfrak{O}_\mathfrak{p} \end{bmatrix},$$

then we have the additional possibility

$$\alpha = \begin{bmatrix} 0 & \pi^i \\ \pi^{j+1} & x \end{bmatrix}, \quad i, j \geqslant 0 \tag{21}$$

where x is reduced mod π^{i+1}. For the sake of simplicity we write $n = n_{K_\mathfrak{p}/k_\mathfrak{p}}$ in the ensuing discussion.

(a) Suppose $\mathfrak{A}_\mathfrak{p}$ is split and $K_\mathfrak{p}$ is unramified over $k_\mathfrak{p}$. Then

$$M_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_\mathfrak{p}, y \in \mathfrak{O}_\mathfrak{p} \right\}$$

and $\Omega_\mathfrak{p}$ is a maximal order of $\mathfrak{A}_{K_\mathfrak{p}} = M(2, K_\mathfrak{p})$. We claim that $\Omega_\mathfrak{p} = M(2, \mathfrak{O}_\mathfrak{p})$. Put $\Lambda_\mathfrak{p} = \mathfrak{O}_\mathfrak{p} M_\mathfrak{p}$, an $\mathfrak{O}_\mathfrak{p}$-lattice in $\mathfrak{A}_{K_\mathfrak{p}}$. Then $\Omega_\mathfrak{p} \supset \Lambda_\mathfrak{p}$, $M(2, \mathfrak{O}_\mathfrak{p}) \supset \Lambda_\mathfrak{p}$. Furthermore,

$$\Delta(\Lambda_\mathfrak{p}) = \mathfrak{O}_\mathfrak{p}\Delta(M_\mathfrak{p}) = \mathfrak{O}_\mathfrak{p} = \Delta(\Omega_\mathfrak{p}) = \Delta(M(2, \mathfrak{O}_\mathfrak{p})).$$

Hence $\Omega_\mathfrak{p} = \Lambda_\mathfrak{p} = M(2, \mathfrak{O}_\mathfrak{p})$. Taking $\pi \in k_\mathfrak{p}$, we have

(22)
$$c\begin{bmatrix} \pi^i & x \\ 0 & \pi^j \end{bmatrix}\begin{bmatrix} a & y \\ -\bar{y} & d \end{bmatrix}\begin{bmatrix} \pi^i & 0 \\ -\bar{x} & \pi^j \end{bmatrix} = c\begin{bmatrix} a\pi^{2i} - dn(x) - \pi^i(x\bar{y} + \bar{x}y) & d\pi^j x + \pi^{i+j}y \\ -(d\pi^j\bar{x} + \pi^{i+j}\bar{y}) & d\pi^{2j} \end{bmatrix}.$$

It follows immediately that $c\pi^{2j} = u \in U_\mathfrak{p}$. Taking $y = 0$, $a = 0$, $d = 1$, we see that $\pi^{2j} | n(x)$. Since x is reduced mod π^j, this implies $x = 0$. Then $c\pi^{2i} \in U_\mathfrak{p}$, which shows that $i = j$ and $(c, \alpha) = (c\pi^{2j}, 1)(\pi^{-2j}, \pi^j) \equiv (u, 1)$. We conclude that

$$S^+(M_\mathfrak{p}) = U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})).$$

(b) Suppose $K_\mathfrak{p}$ is ramified over $k_\mathfrak{p}$. We may assume that $M_\mathfrak{p}$, $\Omega_\mathfrak{p}$ are in the standard forms

$$M_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -u_\mathfrak{p}\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_\mathfrak{p}, y \in \mathfrak{O}_\mathfrak{p} \right\}, \quad u_\mathfrak{p} \in U_\mathfrak{p},$$

and $\Omega_\mathfrak{p} = M(2, \mathfrak{O}_\mathfrak{p})$. Following the same argument as in 2 (a) (but using $n(\pi)^i$, $n(\pi)^j$ instead of π^{2i}, π^{2j}), we deduce once again that

$$S^+(M_\mathfrak{p}) = U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})).$$

(c) Suppose $\mathfrak{A}_\mathfrak{p}$ is nonsplit and $K_\mathfrak{p}$ is unramified over $k_\mathfrak{p}$. Let π be a prime element of $\mathfrak{o}_\mathfrak{p}$.

(i) $M_\mathfrak{p}$ is maximally integral. We take $M_\mathfrak{p}$, $\Omega_\mathfrak{p}$ in the standard forms

$$M_\mathfrak{p} = \left\{ \begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix} \middle| \; a, d \in \mathfrak{o}_\mathfrak{p}, y \in \mathfrak{O}_\mathfrak{p} \right\}, \quad \Omega_\mathfrak{p} = \begin{bmatrix} \mathfrak{O}_\mathfrak{p} & \mathfrak{O}_\mathfrak{p} \\ \pi\mathfrak{O}_\mathfrak{p} & \mathfrak{O}_\mathfrak{p} \end{bmatrix}.$$

If $\alpha = \begin{bmatrix} \pi^i & x \\ 0 & \pi^j \end{bmatrix}$, x reduced mod π^j, then

$$c\begin{bmatrix} \pi^i & x \\ 0 & \pi^j \end{bmatrix}\begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix}\begin{bmatrix} \pi^i & 0 \\ -\pi\bar{x} & \pi^j \end{bmatrix} \tag{23}$$
$$= c\begin{bmatrix} a\pi^{2i} - d\pi n(x) - \pi^{i+1}(x\bar{y} + \bar{x}y) & d\pi^j x + \pi^{i+j}y \\ -\pi(d\pi^j\bar{x} + \pi^{i+j}\bar{y}) & d\pi^{2j} \end{bmatrix}.$$

As before, we have $c\pi^{2j} = u \in U_\mathfrak{p}$. This implies $\pi^{2j} | \pi n(x)$, i.e. $\pi^{2j-1} | n(x)$. However, $n(x)$ has even p-order. Hence we must have $\pi^{2j} | n(x)$, which shows that $x = 0$, $i = j$, and $(c, \alpha) \equiv (u, 1)$.

If $\alpha = \begin{bmatrix} 0 & \pi^i \\ \pi^{j+1} & x \end{bmatrix}$, x reduced mod π^{i+1}, then

$$c\begin{bmatrix} 0 & \pi^i \\ \pi^{j+1} & x \end{bmatrix}\begin{bmatrix} a & y \\ -\pi\bar{y} & d \end{bmatrix}\begin{bmatrix} 0 & -\pi^j \\ -\pi^{i+1} & \bar{x} \end{bmatrix} \tag{24}$$
$$= c\begin{bmatrix} -d\pi^{2i+1} & d\pi^i\bar{x} + \pi^{i+j+1}\bar{y} \\ -\pi(d\pi^i x + \pi^{i+j+1}y) & -a\pi^{2j+1} + dn(x) + \pi^{j+1}(x\bar{y} + \bar{x}y) \end{bmatrix}.$$

Then $c\pi^{2i+1} = u \in U_\mathfrak{p}$ and $\pi^{2i+1}|n(x)$, which implies $\pi^{2(i+1)})|n(x)$. Hence $x = 0, i = j$, and $(c, a) \equiv (u\pi^{-1}, \Pi)$, where

$$\Pi = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}.$$

We conclude that

$$\left[S^+(M_\mathfrak{p}) : U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma\left(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\right)\right] = 2.$$

(ii) $M_\mathfrak{p}$ is $\mathfrak{p}$-maximal. Here we take $M_\mathfrak{p}$, $\Omega_\mathfrak{p}$ in the standard forms

$$M_\mathfrak{p} = \left\{ \begin{bmatrix} \pi a & y \\ -\pi\bar{y} & d \end{bmatrix} \middle|\; a, d \in \mathfrak{o}_\mathfrak{p}, y \in \mathfrak{O}_\mathfrak{p} \right\},$$

$\Omega_\mathfrak{p} = M(2, \mathfrak{O}_\mathfrak{p})$. We need only consider

$$a = \begin{bmatrix} \pi^i & x \\ 0 & \pi^j \end{bmatrix}, \quad x \text{ reduced} \bmod \pi^j.$$

This case can be treated by assuming that $\pi|a$ in (23). We note that this assumption does not affect the argument following (23). Hence the same conclusion is valid, and we must have

$$S^+(M_\mathfrak{p}) = U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma\left(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\right).$$

We summarize the preceding discussion in

PROPOSITION 8. *Let Ω, M be chosen as in Proposition 5 or its analogue for intransigent idealcomplexes.*

(a) *If $\Omega_\mathfrak{p}$ is of level $\mathfrak{O}_\mathfrak{p}$, then*

$$S^+(M_\mathfrak{p}) = U_\mathfrak{p} \times U(\Omega_\mathfrak{p})/\Gamma\left(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\right).$$

(b) *If $\Omega_\mathfrak{p}$ is of level $\mathfrak{p}$, then*

$$\left[S^+(M_\mathfrak{p}) : U_\mathfrak{p} \times (\Omega_\mathfrak{p})/\Gamma\left(U_\mathfrak{p} \times U(\mathfrak{O}_\mathfrak{p})\right)\right] = 2,$$

the non-trivial coset being represented by $\psi\left(N(\Pi)^{-1}, (\Pi, \Pi)\right)$ if $\mathfrak{p}$ splits in K, and by $\psi\left(N(\Pi)^{-1}, \Pi\right)$ if $\mathfrak{p}$ remains prime in K.

Remark. It is clear that Proposition 8 remains valid if we replace M by an arbitrary lattice $L \in \mathfrak{J}$ and Ω by the left order of L.

Let $J_k, J_K, J_{\mathfrak{A}_K}$ denote the idele groups of $k, K, \mathfrak{A}_K$, respectively. We have natural inclusions $J_k \subset J_K \subset J_{\mathfrak{A}_K}$ which are compatible with the natural inclusions $k^\times \subset K^\times \subset \mathfrak{A}_K^\times$. The norm mapping $n_{K/k}: K^\times \to k^\times$ extends in the usual way to a mapping $n_{K/k}: J_K \to J_k$. We put

$$\Gamma(J_k \times J_K) = \{(c, x) \in J_k \times J_K |\; n_{K/k}(x) = c^{-1}\},$$

(25) $$S^+(V)_\mathrm{A} = J_k \times J_{\mathfrak{A}_K}/\Gamma(J_k \times J_K).$$

Then

$$k^\times \times \mathfrak{A}_K^\times \cap \Gamma(J_k \times J_K) = \Gamma(k^\times \times K^\times)$$

implies that we have a natural embedding

(26) $$S^+(V) \subset S^+(V)_\mathrm{A}.$$

Let V_A denote the adelization of V. For each lattice L in V put $\tilde{L} = \prod_\mathfrak{p} L_\mathfrak{p}$, where $L_\mathfrak{p} = V_\mathfrak{p}$ for infinite $\mathfrak{p}$. Then $\tilde{L}$ is an open subgroup of V_A. Conversely, if $L(\mathfrak{p})$ is a lattice in $V_\mathfrak{p}$ for each finite prime $\mathfrak{p}$ and $L(\mathfrak{p}) = V_\mathfrak{p}$ for each infinite $\mathfrak{p}$, and if $\prod_\mathfrak{p} L(\mathfrak{p})$ is an open subgroup of V_A, then there exists a lattice L in V such that $L_\mathfrak{p} = L(\mathfrak{p})$ for every $\mathfrak{p}$. Furthermore, L is uniquely determined by $\tilde{L}$, since $L = \tilde{L} \cap V$. For $a \in J_{\mathfrak{A}_K}$, $a = (a_\mathfrak{p})$, put $\bar{a}^* = (\bar{a}_\mathfrak{p}^*)$. Then $S^+(V)_\mathrm{A}$ acts transitively on $\mathfrak{J}$ by

$$L \to ca\tilde{L}\bar{a}^* \cap V, \quad c \in J_k,\ a \in J_{\mathfrak{A}_K}.$$

This action is compatible with the action of $S^+(V)$ on $\mathfrak{J}$ already defined and the embedding (26). Thus $L_1, L_2 \in \mathfrak{J}$ are similar if and only if they are in the same orbit under the action of the subgroup $S^+(V)$. For any lattice $L \in \mathfrak{J}$ let $S^+(\tilde{L})$ denote the isotropy group of L under the action of $S^+(V)_\mathrm{A}$. Then

(27) $$H = \operatorname{card}\left(S^+(V) \backslash S^+(V)_\mathrm{A}/S^+(\tilde{L})\right).$$

Choose Ω, M as in Proposition 8. Set

$$U_k = \prod_\mathfrak{p} U_\mathfrak{p}, \quad U_K = \prod_\mathfrak{p} U(\mathfrak{O}_\mathfrak{p}), \quad U(\tilde{\Omega}) = \prod_\mathfrak{p} U(\Omega_\mathfrak{p}),$$

$$\mathfrak{N}(\tilde{\Omega}) = \prod_\mathfrak{p} \mathfrak{N}(\Omega_\mathfrak{p}),$$

where $\mathfrak{N}(\Omega_\mathfrak{p}) = \mathfrak{A}_{K_\mathfrak{p}}^\times$ for infinite $\mathfrak{p}$. Define $\mathfrak{N}_s(\tilde{\Omega})$ to be the group of all $\nu \in J_{\mathfrak{A}_K}$ such that $\nu\tilde{\Omega}\bar{\nu}^* = n\tilde{\Omega}$ for some $n \in J_k$. We call n a *multiplier* of ν and denote the set of all multipliers of ν by $m(\nu)$. Then $m(\nu) = nU_k$, and we may regard m as a homomorphism $m: \mathfrak{N}_s(\tilde{\Omega}) \to J_k/U_k$. It is clear that $\mathfrak{N}_s(\tilde{\Omega}) = \prod_\mathfrak{p} \mathfrak{N}_s(\Omega_\mathfrak{p})$, where $\mathfrak{N}_s(\Omega_\mathfrak{p}) = \{\nu_\mathfrak{p} \in \mathfrak{A}_{K_\mathfrak{p}}^\times |\; \nu_\mathfrak{p}\Omega_\mathfrak{p}\bar{\nu}_\mathfrak{p}^* = n_\mathfrak{p}\Omega_\mathfrak{p},\ n_\mathfrak{p} \in k_\mathfrak{p}^\times\}$.

If $\nu_\mathfrak{p} \in \mathfrak{N}_s(\Omega_\mathfrak{p})$, then $\nu_\mathfrak{p}\Omega_\mathfrak{p}\nu_\mathfrak{p}^{-1} = \Omega_\mathfrak{p}$, since $\nu_\mathfrak{p}\Omega_\mathfrak{p}\nu_\mathfrak{p}^{-1}$ is the left order of $\nu_\mathfrak{p}\Omega\bar{\nu}_\mathfrak{p}^*$ and $\Omega_\mathfrak{p}$ is the left order of $n_\mathfrak{p}\Omega_\mathfrak{p}$. Thus we have

(28) $$K_\mathfrak{p}^\times U(\Omega_\mathfrak{p}) \subset \mathfrak{N}_s(\Omega_\mathfrak{p}) \subset \mathfrak{N}(\Omega_\mathfrak{p})$$

and

(29) $$J_K U(\tilde{\Omega}) \subset \mathfrak{N}_s(\tilde{\Omega}) \subset \mathfrak{N}(\tilde{\Omega}).$$

It follows from (28) that $\mathfrak{N}_s(\Omega_\mathfrak{p}) = K_\mathfrak{p}^\times U(\Omega_\mathfrak{p})$ if $\Omega_\mathfrak{p}$ has level $\mathfrak{O}_\mathfrak{p}$. If $\Omega_\mathfrak{p}$ has level $\mathfrak{p}$ and $\mathfrak{p}$ remains prime in K, then $\Pi\Omega_\mathfrak{p}\overline{\Pi}^* = N(\Pi)\Omega_\mathfrak{p}$, where

$$\Pi = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}.$$

Hence $\mathfrak{N}_s(\Omega_\mathfrak{p}) = \mathfrak{N}(\Omega_\mathfrak{p})$ in that case. If, however, $\mathfrak{p}$ splits in K, then

$$[\mathfrak{N}(\Omega_\mathfrak{p}):K_\mathfrak{p}^\times U(\Omega_\mathfrak{p})] = 4 \quad \text{but} \quad [\mathfrak{N}_s(\Omega_\mathfrak{p}):K_\mathfrak{p}^\times U(\Omega_\mathfrak{p})] = 2,$$

because

$$(\Pi, \Pi)\Omega_\mathfrak{p}(\overline{\Pi, \Pi})^* = N(\Pi)\Omega_\mathfrak{p},$$
$$(\Pi, 1)\Omega_\mathfrak{p}(\overline{\Pi, 1})^* = (\Pi, \Pi)\Omega_\mathfrak{p} = (1, \Pi)\Omega_\mathfrak{p}(\overline{1, \Pi})^*.$$

In particular, $\mathfrak{N}(\tilde{\Omega})/\mathfrak{N}_s(\tilde{\Omega})$ is an elementary abelian 2-group of order 2^f.

The usual *type number* $t(\Omega)$ is defined by

$$(30) \qquad t(\Omega) = \operatorname{card}\big(\mathfrak{A}_K^\times/K^\times \diagdown J_{\mathfrak{A}_K}/J_K/\mathfrak{N}(\tilde{\Omega})/J_K\big).$$

We generalize this notion slightly by introducing

$$(31) \qquad t_s(\Omega) = \operatorname{card}\big(\mathfrak{A}_K^\times/K^\times \diagdown J_{\mathfrak{A}_K}/J_K/\mathfrak{N}_s(\tilde{\Omega})/J_K\big).$$

Then $t_s(\Omega) = t(\Omega)$ if $f = 0$, that is, if $V_\mathfrak{p}$ is isotropic for every finite prime $\mathfrak{p}$ of k.

The mapping $(c, a) \mapsto a \bmod J_K$, $c \in J_k$, $a \in J_{\mathfrak{A}_K}$, induces a homomorphism

$$\varrho: S^+(V)_\mathrm{A} \to J_{\mathfrak{A}_K}/J_K.$$

Then $\varrho(S^+(V)) = \mathfrak{A}_K^\times J_K/J_K = \mathfrak{A}_K^\times/K^\times$ and Proposition 8 shows that

$$\varrho(S^+(\tilde{M})) = \mathfrak{N}_s(\tilde{\Omega})/J_K.$$

This implies that $t_s(\Omega) \leqslant H$. Suppose $(a, \alpha), (b, \beta) \in J_k \times J_{\mathfrak{A}_K}$ and $\beta = \gamma\alpha\nu$ where $\gamma \in \mathfrak{A}_K^\times$, $\nu \in \mathfrak{N}_s(\tilde{\Omega})$ with multiplier n. Let h denote the class number of k and let $c_1, \ldots, c_h$ be a complete set of representatives for $J_k/k^\times U_k$. Then $ba^{-1}n = dc_i u$ for some $i = 1, 2, \ldots, h$, where $d \in k^\times$, $u \in U_k$, and we have $(b, \beta) = (c_i, 1)(d, \gamma)(a, \alpha)(n^{-1}u, \nu)$. We conclude that $t_s(\Omega) \leqslant H \leqslant ht_s(\Omega)$. In particular, $H = t_s(\Omega)$ if k has class number 1. If k does not have class number 1 it seems difficult to give the precise relation between $t_s(\Omega)$ and H, the major obstacle being the fact that elements of $\mathfrak{N}_s(\tilde{\Omega})$ need not have principal multipliers. However, we can say a little bit in case $h = 2$.

Let $\alpha_j \in J_{\mathfrak{A}_K}$, $j = 1, \ldots, t_s(\Omega)$, represent all the (generalized) type classes for Ω. Suppose $a, b \in J_k$ and (a, α_i), (b, α_j) represent the same similitude class. Then $i = j$ and $ba^{-1} \in k^\times m(\mathfrak{N}_s(\tilde{\Omega}))$. Putting $h' = [J_k : k^\times m(\mathfrak{N}_s(\tilde{\Omega}))]$, we conclude that $h' t_s(\Omega) \leqslant H$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$ be all the finite primes $\mathfrak{p}$ of k which remain prime in K and for which $\Omega_\mathfrak{p}$ is of level $\mathfrak{p}$. For each $i = 1, \ldots, e$ let π_i be an idele of k whose $\mathfrak{p}_i$-th component is a prime element and whose other components are 1. Let $\langle\pi_1, \ldots, \pi_e\rangle$ be the subgroup of J_k generated by $\pi_1, \ldots, \pi_e$. It is easy to see that

$$m(\mathfrak{N}_s(\tilde{\Omega})) = U_k n_{K/k}(J_K)\langle\pi_1, \ldots, \pi_e\rangle.$$

We know by class field theory that $[J_k : k^\times n_{K/k}(J_K)] = 2$, $\pi_i \notin k^\times n_{K/k}(J_K)$, $i = 1, \ldots, e$. Hence $h' = 1$ except if $e = 0$ and $U_k \subset k^\times n_{K/k}(J_K)$, in which case $h' = 2$. If we assume, in addition, that $h = 2$, then K is the Hilbert class field of k and $H = 2t_s(\Omega)$. We have proved:

Proposition 9. *Let Ω be chosen as in Proposition 5 or its analogue for intransigent idealcomplexes.*

(a) *If k has class number 1, then $H = t_s(\Omega)$.*

(b) *If K is the Hilbert class field of k and Ω is a maximal order, then $H = 2t_s(\Omega)$.*

Remark. Part (a) is valid even if $D(V)$ is a square in k, $K = k \oplus k$. In that case $t_s(\Omega)$ can be interpreted as the number of classes of normal ideals of $\mathfrak{A}$ (cf. [17], Proposition 1).

§ 5. Some weight computations. We assume from now on that $k = \boldsymbol{Q}$, the field of rational numbers. In general, we will make the convention that a fractional ideal of $\boldsymbol{Q}$ will be identified with the unique positive rational number generating it. This will be the case, in particular, for the norm $n(L)$ of a lattice L and the level δ of an order Ω. We will make exceptions for Δ_K and the discriminant and reduced discriminant of a lattice L, defining the latter two by

$$\Delta(L) = \det[B(v_i, v_j)], \qquad \Delta'(L) = \det[n(L)^{-1}B(v_i, v_j)],$$

where $\{v_i\}$ is a $\boldsymbol{Z}$-basis of L. It is clear from these definitions that $\Delta(L)$, $\Delta'(L)$ will be negative if q is indefinite and of signature $\neq 0$. By taking into account the signs of the discriminants, it is easy to verify that (b) of Proposition 6 is still valid.

Throughout this section we assume q is positive definite and $D(V)$ is not a square in $\boldsymbol{Q}$. Thus K is a real quadratic extension of $\boldsymbol{Q}$ and $\mathfrak{A}$ is a definite quaternion algebra over $\boldsymbol{Q}$. We choose Ω, M as in Proposition 5.

For each prime p of $\boldsymbol{Q}$, $\mathfrak{p}$ of K, let the corresponding normalized valuation be denoted by $|\ |_p$, $|\ |_\mathfrak{p}$, respectively. Put

$$J_{\boldsymbol{Q}}^1 = \Big\{(a_p) \in J_{\boldsymbol{Q}} \,\Big|\, \prod_p |a_p|_p = 1\Big\},$$
$$J_K^1 = \Big\{(x_\mathfrak{p}) \in J_K \,\Big|\, \prod_\mathfrak{p} |x_\mathfrak{p}|_\mathfrak{p} = 1\Big\},$$
$$J_{\mathfrak{A}_K}^1 = \Big\{(\alpha_\mathfrak{p}) \in J_{\mathfrak{A}_K} \,\Big|\, \prod_\mathfrak{p} |N(\alpha_\mathfrak{p})|_\mathfrak{p} = 1\Big\}.$$

By the product formula, $Q^\times \subset J_Q^1$, $K^\times \subset J_K^1$, $\mathfrak{A}_K^\times \subset J_{\mathfrak{A}_K}^1$. Set

$$U_Q^1 = U_Q \cap J_Q^1, \quad U_K^1 = U_K \cap J_K^1, \quad U^1(\tilde{\Omega}) = U(\tilde{\Omega}) \cap J_{\mathfrak{A}_K}^1,$$
$$\mathfrak{N}^1(\tilde{\Omega}) = \mathfrak{N}(\tilde{\Omega}) \cap J_{\mathfrak{A}_K}^1, \quad \mathfrak{N}_s^1(\tilde{\Omega}) = \mathfrak{N}_s(\tilde{\Omega}) \cap J_{\mathfrak{A}_K}^1.$$

Put

$$G = \mathfrak{A}_K^\times / K^\times, \quad G_{\mathrm{A}}^1 = J_{\mathfrak{A}_K}^1 / J_K^1, \quad G^1(\tilde{\Omega}) = \mathfrak{N}^1(\tilde{\Omega}) / J_K^1, \quad G_s^1(\tilde{\Omega}) = \mathfrak{N}_s^1(\tilde{\Omega}) / J_K^1.$$

Then G_{A}^1 is a locally compact group, G is a discrete subgroup of G_{A}^1 and, since $\mathfrak{A}_K$ is totally definite, $G^1(\tilde{\Omega})$, $G_s^1(\tilde{\Omega})$ and $G \backslash G_{\mathrm{A}}^1$ are compact spaces. It follows, in particular, that G_{A}^1 is unimodular. Similarly, if we put

$$\Gamma(J_Q^1 \times J_K^1) = \Gamma(J_Q \times J_K) \cap J_Q^1 \times J_K^1, \quad S^+(V)_{\mathrm{A}}^1 = J_Q^1 \times J_{\mathfrak{A}_K}^1 / \Gamma(J_Q^1 \times J_K^1),$$
$$S^+(\tilde{M})^1 = S^+(\tilde{M}) \cap S^+(V)_{\mathrm{A}}^1,$$

then $S^+(V)_{\mathrm{A}}^1$ is a locally compact unimodular group, $S^+(V)$ is a discrete subgroup of $S^+(V)_{\mathrm{A}}^1$, and $S^+(\tilde{M})^1$, $S^+(V) \backslash S^+(V)_{\mathrm{A}}^1$ are compact spaces. The homomorphism $\varrho: S^+(V)_{\mathrm{A}} \to J_{\mathfrak{A}_K}/J_K$, when restricted to $S^+(V)_{\mathrm{A}}^1$, gives a one-to-one correspondence between $S^+(V) \backslash S^+(V)_{\mathrm{A}}^1 / S^+(\tilde{M})^1$ and $G \backslash G_{\mathrm{A}}^1 / G_s^1(\tilde{\Omega})$.

Furthermore, we have

$$H = \operatorname{card}\big(S^+(V) \backslash S^+(V)_{\mathrm{A}}^1 / S^+(\tilde{M})^1\big), \quad t_s(\Omega) = \operatorname{card}\big(G \backslash G_{\mathrm{A}}^1 / G_s^1(\tilde{\Omega})\big).$$

Let λ be the Haar measure on G_{A}^1 such that $\lambda\big(G_s^1(\tilde{\Omega})\big) = 1$. For simplicity, let λ also denote the right invariant measure on $G \backslash G_{\mathrm{A}}^1$ which "lifts" to λ by means of the local homeomorphism: $G_{\mathrm{A}}^1 \to G \backslash G_{\mathrm{A}}^1$.

Let $M_1, \ldots, M_H$ represent the similitude classes in $\mathfrak{I}$. We may assume that $\tilde{M}_j = a_j \tilde{M} \bar{a}_j^*$, $a_j \in J_{\mathfrak{A}_K}^1$, $j = 1, \ldots, H$. Put $g_j = a_j \bmod J_K^1$, $j = 1, \ldots, H$. Then we have a disjoint double coset decomposition

$$G_{\mathrm{A}}^1 = \bigcup_{j=1}^{H} G g_j G_s^1(\tilde{\Omega})$$

which shows that

$$\lambda(G \backslash G_{\mathrm{A}}^1) = \sum_{j=1}^{H} \lambda\big(G \backslash G g_j G_s^1(\tilde{\Omega})\big) = \sum_{j=1}^{H} \lambda\Big(G \backslash G\big(g_j G_s^1(\tilde{\Omega}) g_j^{-1}\big)\Big)$$
$$= \sum_{j=1}^{H} \frac{\lambda\big(g_j G_s^1(\tilde{\Omega}) g_j^{-1}\big)}{\operatorname{card}\big(G \cap g_j G_s^1(\tilde{\Omega}) g_j^{-1}\big)} = \sum_{j=1}^{H} \frac{1}{\operatorname{card}\big(G \cap g_j G_s^1(\tilde{\Omega}) g_j^{-1}\big)}.$$

We note that $\varrho\big(S^+(\tilde{M}_j)^1\big) = g_j G_s^1(\tilde{\Omega}) g_j^{-1}$. Hence

$$\varrho\big(S^+(M_j)\big) = \varrho\big(S^+(V) \cap S^+(\tilde{M}_j)^1\big) = G \cap g_j G_s^1(\tilde{\Omega}) g_j^{-1}.$$

Since q is definite, $S^+(M_j) = O^+(M_j)$, $j = 1, \ldots, H$. Let ϱ_j be the restriction of ϱ to $O^+(M_j)$. Then

$$O^+(M_j)/\operatorname{Ker} \varrho_j \cong G \cap g_j G_s^1(\tilde{\Omega}) g_j^{-1}, \quad j = 1, \ldots, H.$$

To determine $\operatorname{Ker} \varrho_j$, suppose $c \in Q^\times$, $x \in K^\times$, and $cxM_j\bar{x}^* = M_j$. Then $cn_{K/Q}(x)M_j = M_j$ and we must have $n_{K/Q}(x) = \pm c^{-1}$. It follows that $\operatorname{Ker} \varrho_j = \{\pm 1\}$ and

(32) $$\lambda(G \backslash G_{\mathrm{A}}^1) = 2 \sum_{j=1}^{H} \frac{1}{\operatorname{card}\big(O^+(M_j)\big)}.$$

We write $\sum_{j=1}^{H} \frac{1}{\operatorname{card}\big(O^+(M_j)\big)} = M(\mathfrak{I})$, the *Minkowski–Siegel weight* (*Mass*) of $\mathfrak{I}$.

If, on the other hand, we choose the Haar measure λ' on G_{A}^1 such that $\lambda'\big(G^1(\tilde{\Omega})\big) = 1$, then the same line of reasoning shows that

(33) $$\lambda'(G \backslash G_{\mathrm{A}}^1) = \sum_{n=1}^{t(\Omega)} \frac{1}{[\mathfrak{N}(\Omega_n) : K^\times]},$$

where the Ω_n, $n = 1, \ldots, t(\Omega)$ are representatives for the types of Eichler orders of level δ in $\mathfrak{A}_K$. We know by a result of Eichler ([6], p. 137, (16), (18)) that

(34) $$\sum_{n=1}^{t(\Omega)} \frac{1}{[\mathfrak{N}(\Omega_n) : K^\times]} = \frac{2\zeta_K(2)\, \Delta_K^{3/2}}{2^{e+2f}(2\pi)^4} \prod_{i=1}^{e} (p_i^2 + 1) \prod_{k=1}^{f} (q_k - 1)^2$$

where ζ_K is the Dedekind zeta function of K, $p_1, \ldots, p_e$ are the isotropic primes of V dividing δ, and $q_1, \ldots, q_f$ are the anisotropic primes of V dividing δ. The value $\zeta_K(2)$ can be expressed in terms of generalized Bernoulli numbers ([11], p. 135) as

(35) $$\zeta_K(2) = \frac{\pi^4}{\Delta_K^{3/2}} B^2 B_\chi^2,$$

where $B^2 = 1/6$ and B_χ^2 is the generalized Bernoulli number associated to the Kronecker symbol $\chi(m) = \left(\frac{\Delta_K}{m}\right)$,

(36) $$B_\chi^2 = \frac{\sum_{m=1}^{\Delta_K} \left(\frac{\Delta_K}{m}\right) (\Delta_K B + n - \Delta_K)^2}{\Delta_K} = \frac{\sum_{m=1}^{\Delta_K} \left(\frac{\Delta_K}{m}\right) m^2}{\Delta_K}.$$

Using the fact that $\lambda(G \backslash G_{\mathrm{A}}^1) = 2^f \lambda'(G \backslash G_{\mathrm{A}}^1)$, we conclude

Proposition 10. *Let λ be the Haar measure on $G^1_{\mathbf{A}}$ such that $\lambda(G^1_s(\tilde{\Omega}))=1$. Then*

$$(37)\quad \lambda(G\backslash G^1_{\mathbf{A}}) = 2M(\mathfrak{I}) = \frac{\prod_{i=1}^{e}(p_i^2+1)\prod_{k=1}^{f}(q_k-1)^2}{3\cdot 2^{e+f+4}}\;\frac{\left[\sum_{m=1}^{\Delta_K}\left(\frac{\Delta_K}{m}\right)m^2\right]}{\Delta_K}.$$

Remarks. 1. The character sum appearing in (37) can be further simplified as follows (cf. [10], § 6)

$$\frac{\sum_{m=1}^{\Delta_K}\left(\frac{\Delta_K}{m}\right)m^2}{\Delta_K} =$$

$$\begin{array}{ll}
(38) & \\
(39) & = \\
(40) & \\
(41) &
\end{array}
\begin{cases}
\dfrac{4\sum_{m=1}^{(\Delta_K-1)/2}\left(\frac{\Delta_K}{m}\right)m}{\left(\left(\frac{\Delta_K}{2}\right)-4\right)} & \text{if}\quad \Delta_K\equiv 1\,(\mathrm{mod}\,4),\\
4\left(\dfrac{2}{\Delta_K/4}\right)\sum_{m=1}^{(\Delta_K/4-1)/2}(-1)^m\left(\dfrac{m}{\Delta_K/4}\right)m & \text{if}\quad \Delta_K\equiv 4\,(\mathrm{mod}\,8),\\
2\sum_{m=1}^{\Delta_K/8}\left(\dfrac{\Delta_K}{m}\right)\left[\dfrac{\Delta_K}{4}+\left((-1)^{(m-1)/2}-1\right)m\right] & \text{if}\quad \Delta_K\equiv 0\,(\mathrm{mod}\,8),\ \Delta_K>8\\
2 & \text{if}\quad \Delta_K=8.
\end{cases}$$

2. Exactly the same sort of reasoning can be used to evaluate $M(\mathfrak{I})$ in the square discriminant case (cf. [17], § 7).

Let $\mathfrak{G}_r$, $r=1,\ldots,g^+$, be the similitude genera contained in $\mathfrak{I}$, where g^+ is the number of strict genera of K. Let $M(\mathfrak{G}_r)$ denote the Minkowski–Siegel weight of $\mathfrak{G}_r$, $r=1,\ldots,g^+$, and $M(\mathfrak{M})$ the Minkowski–Siegel weight of $\mathfrak{M}$, the genus of maximally integral lattices. Then

$$M(\mathfrak{I}) = \sum_{r=1}^{g^+} M(\mathfrak{G}_r).$$

We claim that all the $\mathfrak{G}_r$ have the same weight, so that $M(\mathfrak{I})=g^+M(\mathfrak{M})$. Let $\boldsymbol{Q}^\times_+$ denote the subgroup of $\boldsymbol{Q}^\times$ of positive rationals. We extend the notion of the norm of a similitude by setting $n(\sigma)=n(\sigma(M))$, $\sigma\in S^+(V)^1_{\mathbf{A}}$. It is clear that $n\colon S^+(V)^1_{\mathbf{A}}\to\boldsymbol{Q}^\times_+$ is a homomorphism. We recall that two maximal lattices L_1, L_2 are in the same similitude genus if and only if $n(L_2)/n(L_1)=n_{K/k}(x)$, $x\in K^\times$ ([14], p. 338). This implies that $\mathcal{N}=n^{-1}(\boldsymbol{Q}^\times_+\cap n_{K/k}(K^\times))$ is a normal subgroup of $S^+(V)^1_{\mathbf{A}}$ of index g^+ which contains both $S^+(V)$ and $S^+(\tilde{M})^1$. In particular, $\mathcal{N}$ must be an open normal subgroup of $S^+(V)^1_{\mathbf{A}}$. The similitude genera $\mathfrak{G}_r$, $r=1,\ldots,g^+$, are in one-to-one correspondence with the cosets of $S^+(V)^1_{\mathbf{A}}\bmod\mathcal{N}$. Let $\sigma_r\in S^+(V)^1_{\mathbf{A}}$ be a representative for the coset corresponding to $\mathfrak{G}_r$, $r=1,\ldots,g^+$. Then the similitude classes in $\mathfrak{G}_r$ are in one-to-one correspondence with the double cosets of $S^+(V)\backslash\mathcal{N}\sigma_r/S^+(\tilde{M})^1$. If we choose the Haar measure μ on $S^+(V)^1_{\mathbf{A}}$ such that $\mu(S^+(\tilde{M})^1)=1$, then

$$M(\mathfrak{G}_r)=\mu(S^+(V)\backslash\mathcal{N}\sigma_r)=\mu(S^+(V)\backslash\mathcal{N})=M(\mathfrak{M}),\qquad r=1,\ldots,g^+.$$

Let t denote the number of distinct primes dividing Δ_K. Then $g^+=2^{t-1}$, which shows

Corollary. *Let $\mathfrak{M}$ be the genus of maximally integral lattices in V. Let t be the number of distinct primes dividing Δ_K. Then*

$$(42)\qquad M(\mathfrak{M}) = \frac{\prod_{i=1}^{e}(p_i^2+1)\prod_{k=1}^{f}(q_k-1)^2}{3\cdot 2^{e+f+t+4}}\;\frac{\left[\sum_{m=1}^{\Delta_K}\left(\frac{\Delta_K}{m}\right)m^2\right]}{\Delta_K}.$$

Let $O^+(V)_{\mathbf{A}}$ be the adelization of $O^+(V)$. Let $\tau=\prod_p\tau_p$ be the Tamagawa measure on $O^+(V)_{\mathbf{A}}$. The Minkowski–Siegel theorem states that

$$(43)\qquad M(\mathfrak{M})\left(\prod_p\tau_p(O^+(M_p))\right)=2=\tau(O^+(V)\backslash O^+(V)_{\mathbf{A}}).$$

It turns out that the product of local measures appearing in (43) can be computed in an elementary fashion. This computation, together with the preceding corollary, gives an elementary proof of the Minkowski–Siegel theorem in this special case. Having already done this for the square discriminant case in [16], § 5, we obtain in this manner an elementary proof that the Tamagawa number of $O^+(V)$ is 2 for any definite quaternary space V over $\boldsymbol{Q}$. As this particular aspect is not of primary importance to us here, we will only sketch the means by which the local measures $\tau_p(O^+(M_p))$ can be computed.

For this purpose it is convenient to take the description of $O^+(V)$ provided by (b) of Proposition 2. Put

$$\hat{\mathfrak{A}}^\times_K=\{\alpha\in\mathfrak{A}^\times_K \mid N(\alpha)\in\boldsymbol{Q}^\times\},\qquad \hat{\mathfrak{A}}^\times_{K_p}=\{\alpha_p\in\mathfrak{A}^\times_{K_p}\mid N(\alpha_p)\in\boldsymbol{Q}^\times_p\}.$$

Then (b) of Proposition 2 shows that we have canonical isomorphisms

$$\hat{\mathfrak{A}}^\times_K/\boldsymbol{Q}^\times\cong O^+(V),\qquad \hat{\mathfrak{A}}^\times_{K_p}/\boldsymbol{Q}^\times_p\cong O^+(V_p).$$

Put $\hat{U}(\Omega_p)=U(\Omega_p)\cap\hat{\mathfrak{A}}^\times_{K_p}$. Since $\hat{U}(\Omega_p)\cap\boldsymbol{Q}^\times_p=U_p$, we may regard $\hat{U}(\Omega_p)/U_p$ as a subgroup of $O^+(V_p)$. In fact, we have

$$\hat{U}(\Omega_p)/U_p\subset O^+(M_p).$$

One can show that $O^+(M_p) = \hat{U}(\Omega_p)/U_p$ except if $p \mid \delta\Delta_K$, in which case

$$[O^+(M_p) : \hat{U}(\Omega_p)/U_p] = 2.$$

Let K_A, $(\mathfrak{A}_K)_A$ be the adele rings of K, $\mathfrak{A}_K$, respectively. The Tamagawa measure $\sigma = \prod_{\mathfrak{p}} \sigma_{\mathfrak{p}}$ on K_A is given by: $\sigma_{\mathfrak{p}}(\mathfrak{O}_{\mathfrak{p}}) = 1$ for $\mathfrak{p} < \infty$, $\sigma_{\mathfrak{p}} = \Delta_K^{-1/4}$ times ordinary Lebesgue measure for $\mathfrak{p}$ infinite. For each infinite prime $\mathfrak{p}$ of K we have $\mathfrak{A}_{K_{\mathfrak{p}}} = \boldsymbol{H}$, the Hamilton quaternions. Any $a \in \boldsymbol{H}$ can be written $a = x_0 + x_1 i + x_2 j + x_3 k$, where $1, i, j, k$ is the standard basis of $\boldsymbol{H}$. We define a measure $\omega = \prod_{\mathfrak{p}} \omega_{\mathfrak{p}}$ on $(\mathfrak{A}_K)_A$ by setting $\omega_{\mathfrak{p}}(\Omega_{\mathfrak{p}}) = 1$ for $\mathfrak{p} < \infty$, $\omega_{\mathfrak{p}} = dx_0 dx_1 dx_2 dx_3$ for $\mathfrak{p}$ infinite. The Tamagawa measure on $(\mathfrak{A}_K)_A$ is $c\omega$, where $c = 16/(\delta\Delta_K)^2$.

Put $\mathfrak{A}_K^{(1)} = \{a \in \mathfrak{A}_K \mid N(a) = 1\}$, $\mathfrak{A}_{K_p}^{(1)} = \{a_p \in \mathfrak{A}_{K_p} \mid N(a_p) = 1\}$, $\Omega_p^{(1)} = \Omega_p \cap \mathfrak{A}_{K_p}^{(1)}$. Let $\nu = \prod_{\mathfrak{p}} \nu_{\mathfrak{p}}$ be the Haar measure on $(\mathfrak{A}_K^{(1)})_A$, the adelization of $\mathfrak{A}_K^{(1)}$, obtained from σ and ω by the usual limiting procedure. Then $c\nu$ is the Tamagawa measure on $(\mathfrak{A}_K^{(1)})_A$. For each prime p of $\boldsymbol{Q}$ let $\nu_p = \prod_{\mathfrak{p} \mid p} \nu_{\mathfrak{p}}$.

If $p \nmid \delta$, then $\Omega_p^{(1)} = \mathrm{SL}(2, \mathfrak{O}_p)$ and it is well known that

$$\nu_p(\Omega_p^{(1)}) = (1-p^{-2})\left(1 - \left(\frac{\Delta_K}{p}\right)p^{-2}\right) \tag{44}$$

where $\left(\frac{\Delta_K}{p}\right)$ is the Kronecker symbol.

If $p = p_i$, $i = 1, \ldots, e$, then it is easy to show that

$$\nu_p(\Omega_p^{(1)}) = \frac{p^2}{p^2+1}(1-p^{-2})(1+p^{-2}). \tag{45}$$

If $p = q_k$, $k = 1, \ldots, f$, then

$$\nu_p(\Omega_p^{(1)}) = \frac{p^2}{(p-1)^2}(1-p^{-2})^2. \tag{46}$$

Finally,

$$\nu_p(\Omega_\infty^{(1)}) = \Delta_K^{1/2}\pi^4. \tag{47}$$

The inclusion $\mathfrak{A}_K^{(1)} \subset \hat{\mathfrak{A}}_K^\times$ induces a natural mapping $\mathfrak{A}_K^{(1)} \to \hat{\mathfrak{A}}_K^\times/\boldsymbol{Q}^\times$ with kernel $\{\pm 1\}$ which is an isogeny at the local level: $\mathfrak{A}_{K_p}^{(1)} \to \hat{\mathfrak{A}}_{K_p}^\times/\boldsymbol{Q}_p^\times$. The restriction: $\Omega_p^{(1)} \to \hat{U}(\Omega_p)/U_p$ is an isogeny with cokernel of order $[U_p : (U_p)^2]$ for $p < \infty$ and order 1 for $p = \infty$. Since $[U_p : (U_p)^2] = 2$ for $p \neq 2$, but $[U_2 : (U_2)^2] = 4$, we have

$$\prod_p \tau_p(\hat{U}(\Omega_p)/U_p) = c\prod_p \nu_p(\Omega_p^{(1)}). \tag{48}$$

Hence

$$\prod_p \tau_p(O^+(M_p)) = c2^{e+f+t}\prod_p \nu_p(\Omega_p^{(1)}) \tag{49}$$

$$= \left(\frac{\Delta_K^{3/2}\prod_{i=1}^{e}(p_i^2+1)\prod_{k=1}^{f}(q_k-1)^2\zeta_K(2)}{2^{e+f+t+4}\pi^4}\right)^{-1} = 2M(\mathfrak{M})^{-1}.$$

§ 6. Class number formulas. Let us first dispose of the indefinite case.

Proposition 11. *Suppose V is indefinite. Let t be the number of distinct primes dividing Δ_K.*

(a) *If q has signature $\neq 0$, or if K has an element of norm -1, then $H = 2^{t-1}$.*

(b) *If q has signature 0 and K does not have an element of norm -1, then $H = 2^{t-2}$.*

Proof. By the theorem of Kneser ([9], p. 330), every similitude genus of maximal lattices contains just one similitude class. It follows that H is the number of similitude classes in $\mathfrak{J}$. We noted in § 3 that $\mathfrak{J}$ contains $g^+ = 2^{t-1}$ similitude classes if q has signature $\neq 0$. Suppose q has signature 0. Then K is a real quadratic extension of $\boldsymbol{Q}$, and, since $\mathfrak{A}_K$ is totally indefinite,

$$n(S^+(V)) = n_{K/k}(N(\mathfrak{A}_K^\times)) = n_{K/k}(K^\times).$$

It follows that $L_1, L_2 \in \mathfrak{J}$ are in the same similitude genus $\Leftrightarrow n(L_2)/n(L_1) = n_{K/k}(x)$, $x \in K^\times$. We conclude that $H = g$, the number of genera of K. It is well known that $g = 2^{t-1}$ if $n_{K/k}$ represents -1, $g = 2^{t-2}$ if it does not.

Now suppose that V is definite. We impose the following additional conditions:

(i) *V_p is isotropic for every finite prime p.*

(ii) *The fundamental unit of K has norm -1.*

Condition (i) implies that $\mathfrak{A}_K$ is split at every finite prime of K. Hence $\delta = p_1 \ldots p_e$ and $H = t_\delta$, the type number of Eichler orders of level δ in $\mathfrak{A}_K$. We could then apply the general formula for the type number of Eichler orders in a totally definite quaternion algebra ([6], [14], [15]) to obtain a formula for H. Unfortunately, this formula is rather complicated and not very explicit. However, the imposition of condition (ii) greatly simplifies the computation of the terms appearing in the Selberg trace formula expression for H, and results in explicit formulas for H of a very elementary sort (cf. Theorems 1 and 2). In this section we will only state these formulas. Their actual derivation will be left to the remaining sections. We note that condition (ii) implies that Δ_K only has prime divisors p of the form $p\equiv 1 \pmod 4$ or $p=2$.

Notation. For any positive integer m, let $\lambda(m)$ denote the number of primes dividing m; let $h(-m)$ denote the class number of $Q(\sqrt{-m})$.

THEOREM 1. *Suppose V is a definite quaternary space over Q satisfying conditions* (i) *and* (ii). *Let $\mathfrak{I}$ be the idealcomplex containing the maximally integral lattices of V. Let H be the number of similitude classes in $\mathfrak{I}$ and Δ the reduced discriminant of $\mathfrak{I}$. Let δ be defined by $\Delta = \Delta_K\delta^2$, where $K = Q(\sqrt{\Delta})$ (cf. § 3, Proposition 6). Denote the square-free kernel of Δ by D. If D is odd and $\Delta > 5$, then*

$$H = 2M(\mathfrak{I}) + c_1 h(-D) + c_3 h(-3D) + \sum_{n|\delta,\, d|D} 2^{-\lambda(n)-\sigma(nd)} h(-nd)h(-nD/d), \tag{50}$$

where $nd \neq 1, 3$; $d < \sqrt{D}$ and

$$c_1 = \begin{cases} \frac{1}{8} & \text{if } 2\nmid\delta, \\ \frac{3}{16} & \text{if } 2|\delta, \end{cases}$$

$$c_3 = \begin{cases} \frac{1}{6} & \text{if } 3\nmid\delta, \\ \frac{5}{6} & \text{if } 3|\delta,\ D\equiv 1 \pmod 8, \\ \frac{1}{3} & \text{if } 3|\delta,\ D\equiv 5 \pmod 8 \end{cases}$$

and if $D\equiv 1 \pmod 8$,

$$\sigma(m) = \begin{cases} -2 & \text{if } m\equiv 3 \pmod 8, \\ 0 & \text{if } m\equiv 7 \pmod 8, \\ 2 & \text{if } m\equiv 1 \pmod 4 \end{cases}$$

while if $D\equiv 5 \pmod 8$,

$$\sigma(m) = \begin{cases} 0 & \text{if } m\equiv 3 \pmod 4, \\ 2 & \text{if } m\equiv 2 \pmod 4,\ 2|\delta, \\ 2 & \text{if } m\equiv 1 \pmod 4,\ 2\nmid\delta, \\ 3 & \text{if } m\equiv 1 \pmod 4,\ 2|\delta. \end{cases}$$

Furthermore,

$$M(\mathfrak{I}) = \frac{\prod_{p|\delta}(p^2+1)}{3\cdot 2^{e+3}\left(\left(\frac{D}{2}\right)-4\right)}\left[\sum_{m=1}^{(D-1)/2}\left(\frac{D}{m}\right)m\right], \tag{51}$$

where $\left(\frac{D}{m}\right)$ is the Kronecker symbol.

COROLLARY. *Suppose that $\Delta = p$, a prime greater than 5. Then*

$$H = \frac{\sum_{m=1}^{(p-1)/2}\left(\frac{p}{m}\right)m}{12\left(\left(\frac{p}{2}\right)-4\right)} + \frac{h(-p)}{8} + \frac{h(-3p)}{6}. \tag{52}$$

Remarks. 1. If $\Delta = p$, a prime, then $\Delta_K = p$, $p\equiv 1 \pmod 4$. It is well known in this case that the fundamental unit has norm -1 ([3], p. 185).

2. Tamagawa has shown, under the assumption of the Corollary, that $H = h(\mathfrak{A}_K)/h(K)$, where $h(\mathfrak{A}_K)$ is the ideal class number of $\mathfrak{A}_K$ and $h(K)$ is the class number of K. Combining this with Peters' formula for $h(\mathfrak{A}_K)$ ([14], p. 363), we obtain another proof of (52). Still another proof of (52) can be found in [8].

THEOREM 2. *Suppose V is a definite quaternary space over Q satisfying conditions* (i) *and* (ii). *Let $\mathfrak{I}, H, \Delta, \delta, D$ be defined as in Theorem* 1. *If D is even, then*

$$H = 2M(\mathfrak{I}) + \tfrac{5}{8}h(-D) + c_3 h(-3D) + \sum_{n|\delta,\, d|D} c_{nd} 2^{-\lambda(n)-\sigma(nd)} h(-nd)h(-nD/d), \tag{53}$$

where $nd > 3$, d is odd and

$$c_3 = \begin{cases} \frac{1}{6} & \text{if } 3\nmid\delta, \\ \frac{7}{12} & \text{if } 3|\delta \end{cases}$$

and for $m > 3$,

$$c_m = \begin{cases} 5 & \text{if } m\equiv 3 \pmod 8, \\ 1 & \text{if } m\equiv 7 \pmod 8, \\ 3 & \text{if } m\equiv 1 \pmod 4, \end{cases}$$

$$\sigma(m) = \begin{cases} 1 & \text{if } m\equiv 3 \pmod 8, \\ 0 & \text{if } m\equiv 7 \pmod 8, \\ 2 & \text{if } m\equiv 1 \pmod 4. \end{cases}$$

Furthermore,

$$M(\mathfrak{I}) = \begin{cases} \dfrac{\prod_{p|\delta}(p^2+1)}{3\cdot 2^{e+4}}\left[\sum_{m=1}^{D/2}\left(\frac{D}{m}\right)\left(D+\left((-1)^{(m-1)/2}-1\right)m\right)\right] & \text{if } D\neq 2, \quad (54) \\ \dfrac{\prod_{p|\delta}(p^2+1)}{3\cdot 2^{e+4}} & \text{if } D = 2. \quad (55) \end{cases}$$

At this point it seems appropriate to interpret our lattice-theoretic results in the traditional language of quadratic forms. A *quaternary (quadratic) form* over $\boldsymbol{Q}$ is a homogeneous polynomial $f = f(X_1, X_2, X_3, X_4)$ of degree 2 with coefficients in $\boldsymbol{Q}$. The *discriminant* $\Delta(f)$ of f is defined by

$$(56) \qquad \Delta(f) = \det\left[\frac{\partial^2 f}{\partial X_i \partial X_j}\right], \quad i, j = 1, 2, 3, 4.$$

We always assume $\Delta(f) \neq 0$. Given a quaternary form f over $\boldsymbol{Q}$ we can define a quaternary space V_f over $\boldsymbol{Q}$ by evaluating f at the elements of $\boldsymbol{Q}^4$. We assume that V_f is not negative definite.

A quaternary form f over $\boldsymbol{Q}$ is said to be *integral* if all of its coefficients are integers. An integral form is *primitive* if the g.c.d. of its coefficients is 1. Two quaternary forms f, f' over $\boldsymbol{Q}$ are *equivalent*, written $f \cong f'$, if there is an element $\sigma \in \mathrm{SL}(4, \boldsymbol{Z})$ such that $f'(X_1, X_2, X_3, X_4) = f(\sigma X_1, \sigma X_2, \sigma X_3, \sigma X_4)$, where $\sigma X_i = \sum_{j=1}^{4} a_{ij} X_j$ if $\sigma = (a_{ij})$. If $f \cong f'$, then $\Delta(f) = \Delta(f')$, f is integral $\Leftrightarrow f'$ is integral, and f is primitive $\Leftrightarrow f'$ is primitive. Classical reduction theory shows that the number of equivalence classes of integral quaternary forms having a fixed discriminant is finite. It is this notion of class number which is of primary interest from the classical point of view. We proceed now to relate it to the lattice-theoretic notion.

Let V be a quaternary space over $\boldsymbol{Q}$ with quadratic mapping $q: V \to \boldsymbol{Q}$. We fix an ordered basis v_1, v_2, v_3, v_4 for V and call any other ordered basis w_1, w_2, w_3, w_4 of V *positively oriented* if the linear automorphism defined by $v_i \mapsto w_i$, $i = 1, 2, 3, 4$, has positive determinant. Let L be a lattice in V with positively oriented $\boldsymbol{Z}$-basis x_1, x_2, x_3, x_4. Define f_L to be the unique quaternary form such that

$$(57) \qquad f_L(a_1, a_2, a_3, a_4) = n(L)^{-1} q\left(\sum_{i=1}^{4} a_i x_i\right)$$

for all $a_i \in \boldsymbol{Q}$, $i = 1, 2, 3, 4$. Then f_L is a primitive integral form and $\Delta(f_L) = \Delta'(L)$. It is clear that a different choice of positively oriented $\boldsymbol{Z}$-basis of L will yield an equivalent form. Hence the equivalence class $\{f_L\}$ is uniquely determined by L.

Two lattices L, M are said to be *strictly similar*, written $L \approx M$, if $\sigma L = M$ for some $\sigma \in S^+(V)$ with $n(\sigma) > 0$. Strict similarity can differ from ordinary similarity only if q has signature 0, as all proper similitudes have positive norms when q has signature $\neq 0$. Suppose L, M are lattices in V and $L \approx M$. Then $\sigma L = M$, where $\sigma \in S^+(V)$, $n(\sigma) > 0$. It follows that $n(M) = n(\sigma) n(L)$. Let x_1, x_2, x_3, x_4 be a positively oriented $\boldsymbol{Z}$-basis of L. Then $\sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4)$ is a positively oriented $\boldsymbol{Z}$-basis of M

and we have

$$f_M(a_1, a_2, a_3, a_4) = n(M)^{-1} q\left(\sum_{i=1}^{4} a_i \sigma(x_i)\right)$$
$$= n(\sigma) n(M)^{-1} q\left(\sum_{i=1}^{4} a_i x_i\right) = f_L(a_1, a_2, a_3, a_4)$$

for all $a_i \in \boldsymbol{Q}$, $i = 1, 2, 3, 4$. Hence $f_L = f_M$, and we have a well-defined mapping $\{L\} \mapsto \{f_L\}$ from strict similitude classes of lattices in V to equivalence classes of primitive quaternary forms. Suppose L, M are lattices in V with $f_L \cong f_M$. We may assume, without loss of generality, that $f_L = f_M$. Then there exist positively oriented $\boldsymbol{Z}$-bases x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 of L, M respectively, such that

$$(58) \qquad n(L)^{-1} q\left(\sum_{i=1}^{4} a_i x_i\right) = n(M)^{-1} q\left(\sum_{i=1}^{4} a_i y_i\right)$$

for all $a_i \in \boldsymbol{Q}$, $i = 1, 2, 3, 4$. Define $\sigma \in \mathrm{GL}(V)$ by $\sigma(x_i) = y_i$, $i = 1, 2, 3, 4$. Then (58) implies that $\sigma \in S^+(V)$ and $n(\sigma) = n(M) n(L)^{-1} > 0$, which shows $L \approx M$. Thus $\{L\} \mapsto \{f_L\}$ defines a one-to-one correspondence between strict similitude classes of lattices in V and equivalence classes of primitive quaternary forms. This correspondence is "discriminant preserving" in the sense that $\Delta'(L) = \Delta(f_L)$. Furthermore, it is clear that any primitive quaternary form f arises in this manner (simply take $V = V_f$, $L = \boldsymbol{Z}^4$).

If two quaternary spaces over $\boldsymbol{Q}$ are similar by a positive factor of similarity, they yield the same classes of primitive forms. Therefore, in order to obtain all classes of primitive (non-negative definite) quaternary forms, it suffices to fix one quaternary space (V, q) for each possible fundamental discriminant when q is positive definite or the signature of q is 0; when q is indefinite with signature $\neq 0$, we must also take $(V, -q)$.

Notation. For any integer m let $H^+(m)$, $H^-(m)$ denote, respectively, the number of classes of positive definite integral quaternary forms with discriminant m, the number of classes of indefinite integral quaternary forms with discriminant m.

THEOREM 3. *Suppose K is a quadratic extension of $\boldsymbol{Q}$ and V is a quaternary space with $\Delta = \Delta_K$.*

(a) *If V is positive definite, then $H^+(\Delta_K) = H$.*

(b) *If V is indefinite with signature $\neq 0$, then $H^-(\Delta_K) = 2H$.*

(c) *If V has signature 0 and the fundamental unit of K has norm -1, then $H^-(\Delta_K) = H$.*

(d) *If V has signature 0 and the fundamental unit of K has norm 1, then $H^-(\Delta_K) = 2H$.*

Proof. *If* $L \in \mathfrak{I}$, then f_L is an integral quaternary form with $\Delta(f_L) = \Delta'(L) = \Delta_K$. Conversely, suppose f is an integral quaternary form with $\Delta(f) = \Delta_K$. Then f must be primitive since Δ_K is not divisible by the fourth power of a prime. If V is positive definite, then $f = f_L$ for some lattice L in V, and $\Delta'(L) = \Delta(f_L) = \Delta_K$ implies that $L \in \mathfrak{I}$. If V is indefinite of signature $\neq 0$, then $f = f_L$ for some lattice L in (V, q) or $(V, -q)$. The classes of forms coming from (V, q) are disjoint from those coming from $(V, -q)$, as they have different signatures. Hence their total number is $2H$. If V has signature 0, then $f = f_L$ for some $L \in \mathfrak{I}$ and $H^-(\Delta_K)$ = the number of strict similitude classes in $\mathfrak{I}$. We must compare the latter number with H. Suppose $L, M \in \mathfrak{I}$ and $\sigma L = M$, where $\sigma \in S^+(V)$ and $n(\sigma) < 0$. Then $L \approx M$ if and only if there is a $\tau \in S^+(L)$ with $n(\tau) = -1$. From the remark following Proposition 8 it is evident that $\tau \in S^+(L)$ implies $n(\tau) = n_{K/k}(u)$ for some unit u of $\mathfrak{O}$. On the other hand, since $\mathfrak{A}_K$ is totally indefinite, the strong approximation theorem of Eichler ([5], p. 239) shows that any unit u of $\mathfrak{O}$ can be expressed as $N(\varepsilon)$, where ε is a unit of the left order of L; thus $n_{K/k}(u) = n(\tau)$, where $\tau = \psi(1, \varepsilon) \in S^+(L)$. We conclude that L has a unit of norm -1 if and only if $\mathfrak{O}$ has a unit of norm -1. Assertions (c), (d) follow immediately from this observation.

Corollary 1. *Let K be a real quadratic extension of Q whose fundamental unit has norm -1. Let D denote the square-free kernel of Δ_K.*

(a) *If D is odd, $D > 5$, then*

$$H^+(\Delta_K) = \frac{\sum_{m=1}^{(D-1)/2} \left(\frac{D}{m}\right) m}{12\left(\left(\frac{D}{2}\right) - 4\right)} + \frac{h(-D)}{8} + \frac{h(-3D)}{6} + \frac{1}{4} \sum_{\substack{d \mid D \\ 1 < d < \sqrt{D}}} h(-d) h(-D/d) \tag{59}$$

(b) *If D is even, $D \neq 2$, then*

$$H^+(\Delta_K) = \frac{1}{24} \sum_{m=1}^{D/2} \left(\frac{D}{m}\right) \left(D + \left((-1)^{(m-1)/2} - 1\right) m\right) + \frac{5}{8} h(-D) + \frac{h(-3D)}{6} + \frac{3}{4} \sum_{\substack{d \mid D \\ 1 < d,\, d \text{ odd}}} h(-d) h(-D/d). \tag{60}$$

Corollary 2. *Let K be a quadratic extension of Q. Let t be the number of distinct primes dividing Δ_K.*

(a) *If $\Delta_K > 0$ and $t = 1$, then $H^-(\Delta_K) = 0$.*

(b) *If $\Delta_K > 0$, $t > 1$, and the norm of the fundamental unit of K is -1, then $H^-(\Delta_K) = 2^{t-1}$.*

(c) *If $\Delta_K > 0$ and $n_{K/k}$ does not represent -1, then $H^-(\Delta_K) = 2^{t-1}$.*

(d) *If $\Delta_K < 0$, or if $\Delta_K > 0$, $n_{K/k}$ represents -1, but the fundamental unit of K has norm 1, then $H^-(\Delta_K) = 2^t$.*

Proof. If $\Delta_K > 0$ and $t = 1$, then $\mathfrak{A}$ is nonsplit at ∞. Hence V must be definite and $H^-(\Delta_K) = 0$. The remaining assertions (b), (c), (d) follow from Proposition 11. Note that $t > 1$ if $\Delta_K > 0$ and the fundamental unit of K has norm 1.

§ 7. Normalizers of Eichler orders. The remaining two sections are devoted to the derivation of formulas (50) and (53) for H. We have shown that $H = t_\delta$, the type number of Eichler orders of level δ in $\mathfrak{A}_K$. In order to compute t_δ we must first determine which $a \in \mathfrak{A}_K$ lie in the normalizer $\mathfrak{N}(\Omega)$ (cf. § 4) of some Eichler order Ω of level δ. To do this we fix Ω and use the structure of ambiguous ideals of $\mathfrak{O}$ to put the minimal polynomial of a over K into a standard form. From this we obtain necessary conditions on a which we then show to be sufficient (Proposition 14).

Suppose $a \in \mathfrak{N}(\Omega)$, $a \notin K^\times$. We may assume that a is integral over $\mathfrak{O}$. From our local discussion in § 4 it is evident that the principal ideal $(N(a)) = n\mathfrak{i}^2$, where n is a rational integer dividing δ, and $\mathfrak{i}$ is an integral ideal of $\mathfrak{O}$. Then $\mathfrak{i}^2 = (n^{-1}N(a))$, which implies that $\mathfrak{i}$ lies in an *ambiguous* ideal class of $\mathfrak{O}$. Since the fundamental unit of K has negative norm, $\mathfrak{i}$ is equivalent to an ambiguous ideal $\mathfrak{j}$ of $\mathfrak{O}$, i.e. there is an element $x \in K^\times$ such that $x\mathfrak{i} = \mathfrak{j}$, where $\mathfrak{j}$ is a primitive integral ideal of $\mathfrak{O}$ satisfying the condition $\bar{\mathfrak{j}} = \mathfrak{j}$ ([3], p. 189, Ex. 6). It follows that $(N(ax)) = n\mathfrak{j}^2$, where $\mathfrak{j}$ is ambiguous. Then $\mathfrak{j}^2 = (d)$, where d is a rational integer dividing D ([3], p. 190, Ex. 11). Hence we may assume, without loss of generality, that $(N(a)) = (m)$, where m is a rational integer dividing δD. From the description of the local normalizers $\mathfrak{N}(\Omega_p)$ it is clear that an element of $\mathfrak{N}(\Omega)$ having integral norm must be integral over $\mathfrak{O}$. Thus the minimal polynomial of a over K must be of the form $X^2 + bX + um$, where $b \in \mathfrak{O}$ and u is a unit of $\mathfrak{O}$. As $N(a) = um$ is totally positive and the fundamental unit of K has negative norm, u must be the square of another unit of $\mathfrak{O}$. Hence we may assume that $u = 1$. The minimal polynomial of a then has the form $X^2 + bX + m$, where $b \in \mathfrak{O}$ and $m \mid \delta D$.

Lemma. *Suppose ω is a unit of Ω and $N(\omega) = 1$. Then ω is a root of unity.*

Proof. This is an immediate consequence of the fact that the group of all units of Ω having norm 1 is a finite group ([6], p. 129, Satz 2).

Proposition 12. *If $a \in \mathfrak{N}(\Omega)$, then $a^* = \omega a$, where ω is a root of unity which commutes with a. If $\omega \neq -1$, then $K(a) = K(\omega)$.*

Proof. Put $\omega = \alpha^*\alpha^{-1}$. It is clear that ω commutes with α. Furthermore, since $\Omega^* = \Omega$, α^* must be an element of $\mathfrak{N}(\Omega)$. From the description of local normalizers we see that $\alpha^*\alpha^{-1}$ is a unit of Ω if $\alpha \in \mathfrak{N}(\Omega)$. Then ω is a unit of Ω and $N(\omega) = 1$. The preceding lemma shows that ω must be a root of unity. If $\omega \neq -1$, then $K(\omega), K(\alpha)$ are both proper extensions of K contained in $\mathfrak{A}_K$, and ω commutes with α. Hence $K(\omega) = K(\alpha)$.

Corollary. *If $\alpha \in \mathfrak{N}(\Omega)$, $\alpha \notin K^\times$, then $\alpha^* = \omega\alpha$, where $\omega \in K(\alpha)$ and ω is a primitive n-th root of unity for one of the following values of n: $2, 3, 4, 5, 6, 8, 10$.*

Proof. If $\omega \neq -1$, then $[K(\omega):\boldsymbol{Q}] = [K(\alpha):\boldsymbol{Q}] = 4$, which implies $[\boldsymbol{Q}(\omega):\boldsymbol{Q}] | 4$. The only possible values of n for which this is true are $3, 4, 5, 6, 8, 10, 12$. However, $n = 12$ is impossible since K would then have to be $\boldsymbol{Q}(\sqrt{3})$, whose fundamental unit has norm 1.

Notation. If $\alpha \in \mathfrak{A}_K$ and ξ is an algebraic number, then $\alpha \simeq \xi$ will mean that there exist $x, y \in K^\times$ such that $x\alpha$ and $y\xi$ have the same minimal polynomial over K. For $\alpha, \beta \in \mathfrak{A}_K$ the condition $\alpha \simeq \beta$ is equivalent to $\alpha, \beta \bmod K^\times$ being conjugate in $\mathfrak{A}_K^\times/K^\times$.

Let α be an element of $\mathfrak{N}(\Omega)$ with minimal polynomial X^2+bX+m, where $b \in \mathfrak{O}$ and $m | \delta D$. Taking ω as in the corollary, we see that $b = -(\alpha+\alpha^*) = -(1+\omega)\alpha$, $m = \alpha\alpha^* = \alpha^2\omega$. Thus $\alpha^2 = \omega^{-1}m$. If $\omega = -1$ then $\alpha \simeq \sqrt{-m}$. If $\omega \neq -1$ our approach will be to find solutions, if any, of the equation $\alpha^2 = \omega^{-1}m$ in the ring of integers of $K(\omega)$.

We first dispose of the exceptional case where ω is a primitive fifth or tenth root of 1. Let ω be a primitive fifth root of 1, so that $-\omega$ is a primitive tenth root of 1. Since $[\boldsymbol{Q}(\omega):\boldsymbol{Q}] = 4$, we must have $K(\alpha) = \boldsymbol{Q}(\omega)$, $K = \boldsymbol{Q}(\sqrt{5})$. The equation $\alpha^2 = \pm\omega^{-1} = \pm\omega^4$ has a solution only if the $+$sign holds, and the solution is $\alpha = \pm\omega^2 \simeq \omega^2$. Now consider the equation $\alpha^2 = \pm\omega^{-1}m$, $m > 1$. Then $(\omega^3\alpha)^2 = \omega\alpha^2 = \pm m$, and since $\boldsymbol{Q}(\sqrt{5})$ is the only quadratic subfield of $\boldsymbol{Q}(\omega)$, we must have $(\omega^3\alpha)^2 = 5$. We have $\alpha \simeq \omega^{-3} \simeq \omega^2$ once again. As ω is a unit, $\omega \in \mathfrak{N}(\Omega)$ if and only if $\omega \in \Omega$. Since ω generates the ring of integers in $\boldsymbol{Q}(\omega)$, the criterion of Eichler ([6], p. 133) shows that $\omega \in \Omega$ for some Ω of level δ if and only if no prime p dividing δ remains prime in $\boldsymbol{Q}(\omega)$. Suppose $\Delta > 5$, that is, $\delta > 1$. Let $p | \delta$. Since p remains prime in $\boldsymbol{Q}(\sqrt{5})$, we must have $p \equiv 2, 3 \pmod 5$; all such primes remain prime in $\boldsymbol{Q}(\omega)$, as their residue classes are of order 4 in $\boldsymbol{Z}/(5)$ ([2], p. 87). This shows

Lemma 1. *Suppose $\alpha \in \mathfrak{N}(\Omega)$ and $\alpha^* = \omega\alpha$, where ω is a primitive fifth or tenth root of unity. If $\Delta > 5$ then $\alpha \simeq 1$.*

Remark. If $\Delta = 5$, then $\alpha \simeq \omega \simeq -\omega$ certainly does occur in an Eichler order of level 1 in $\mathfrak{A}_K$ (i.e. a maximal order of $\mathfrak{A}_K$).

We are left with the cases where ω is a primitive third, fourth, sixth, or eighth root of 1.

Let ζ be a primitive cube root of 1. Then $-\zeta$ is a primitive sixth root of 1. If ω is a primitive eighth root of 1, then $K = \boldsymbol{Q}(\sqrt{2})$ and $K(\omega) = K(\sqrt{-1})$. Hence matters are reduced to studying solutions of $\alpha^2 = \omega^{-1}m$ in the ring of integers of $K(\sqrt{-1})$ or $K(\sqrt{-3})$. We know that $\sqrt{-1}$, ζ generate the rings of integers of $\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\sqrt{-3})$, respectively. To determine the ring of integers of $K(\sqrt{-1})$, $K(\sqrt{-3})$, it suffices to determine the conductors of the orders $\mathfrak{O}+\mathfrak{O}\sqrt{-1}$, $\mathfrak{O}+\mathfrak{O}\zeta$, respectively. This, in turn, can be done by determining the relative discriminants of $K(\sqrt{-1})$, $K(\sqrt{-3})$ over K and comparing them to the ideals (-4), (-3), respectively, of $\mathfrak{O}$. We do this by means of the next proposition. For any positive integer m let $\Delta(-m)$ denote the discriminant of the imaginary quadratic extension $\boldsymbol{Q}(\sqrt{-m})$.

Proposition 13. *Let m be a positive integer and $K = \boldsymbol{Q}(\sqrt{D})$ a real quadratic extension of $\boldsymbol{Q}$. Put $L = K(\sqrt{-m})$ and denote by $\Delta_{L/K}$ the relative discriminant of L over K. Then*

$$n_{K/\boldsymbol{Q}}(\Delta_{L/K}) = \frac{\Delta(-m)\Delta(-mD)}{\Delta_K}. \tag{61}$$

Proof. By the conductor-discriminant product formula ([2], p. 160), the discriminant of L over $\boldsymbol{Q}$ is given by $\Delta(-m)\Delta(-mD)\Delta_K$. On the other hand, it is also given by $n_{K/\boldsymbol{Q}}(\Delta_{L/K})\Delta_K^2$ ([2], p. 17), hence the result.

Notation. For any finite extension L of K let $\mathcal{O}_L$ denote the ring of integers of L.

Corollary. *Let $K = \boldsymbol{Q}(\sqrt{D})$ be a real quadratic extension of $\boldsymbol{Q}$, where D is square-free and $D \equiv 1$ or $2 \pmod 4$.*

(a) *If $L = K(\sqrt{-1})$, then $\mathcal{O}_L \subset \frac{1}{2}(\mathfrak{O}+\mathfrak{O}\sqrt{-1})$.*

(b) *If $L = K(\sqrt{-3})$, then $\mathcal{O}_L = \mathfrak{O}+\mathfrak{O}\zeta$, $\zeta^3 = 1$.*

Proof. First suppose $L = K(\sqrt{-1})$. If $D \equiv 1 \pmod 4$, then

$$\Delta(-D) = -4\Delta_K, \quad n_{K/\boldsymbol{Q}}(\Delta_{L/K}) = 16.$$

Hence $\Delta_{L/K} = (4)$, which shows that $\mathcal{O}_L = \mathfrak{O}+\mathfrak{O}\sqrt{-1}$. If D is even, then $\Delta(-D) = -\Delta_K$, $n_{K/\boldsymbol{Q}}(\Delta_{L/K}) = 4$. This shows that $\Delta_{L/K} = (2)$ and the conductor of $\mathfrak{O}+\mathfrak{O}\sqrt{-1}$ is $\mathfrak{p}$, where $\mathfrak{p}^2 = (2)$. It follows that $\mathcal{O}_L \subset \frac{1}{2}(\mathfrak{O}+\mathfrak{O}\sqrt{-1})$.

Now suppose $L = K(\sqrt{-3})$. Then $\Delta(-3D) = -3\Delta_K$, $n_{K/\boldsymbol{Q}}(\Delta_{L/K}) = 9$. Hence $\Delta_{L/K} = (3)$ and $\mathcal{O}_L = \mathfrak{O}+\mathfrak{O}\zeta$.

We now complete the determination of all possible $\alpha \in \mathfrak{N}(\Omega)$, where α is assumed to have a minimal polynomial of the form X^2+bX+m,

$b \in \mathfrak{O}$, $m \mid \delta D$. Suppose $m = 1$. Then α must be a root of unity. If $\Delta > 5$, then $\alpha \simeq \sqrt{-1}$, ζ or η, a primitive eighth root of 1. We note that if η is a primitive eighth root of 1, then $K = \boldsymbol{Q}(\sqrt{2})$ and $\eta \simeq 1+\sqrt{-1}$. Now suppose $m > 1$.

(i) If $\omega = \pm\sqrt{-1}$, then, according to (a) of the corollary,

$$\alpha = \tfrac{1}{2}(x+y\sqrt{-1}), \quad \text{where } x, y \in \mathfrak{O}, \quad \text{and} \quad \alpha^2 = \pm m\sqrt{-1}.$$

Taking α^* instead of α, if necessary, we may assume that $\alpha^2 = m\sqrt{-1}$. Then

$$4\alpha^2 = (x+y\sqrt{-1})^2 = x^2-y^2+2xy\sqrt{-1} = 4m\sqrt{-1}.$$

This implies $x = \pm y$, $\pm x^2\sqrt{-1} = 2m\sqrt{-1}$. Hence $x^2 = 2m$. If m is odd we must have $2m = D$, $x = \pm\sqrt{D}$. If m is even, then $(x/2)^2 = m/2$ implies $m/2 = 1$ or D, from which it follows that $x = \pm 2$ or $x = \pm 2\sqrt{D}$. In all cases we have $\alpha \simeq 1+\sqrt{-1}$.

(ii) If $\omega = \pm\zeta^{-1}$, then, according to (b) of the corollary,

$$\alpha = x+y\zeta, \quad \text{where } x, y \in \mathfrak{O}, \quad \text{and} \quad \alpha^2 = \pm m\zeta.$$

Thus

$$\begin{aligned}\alpha^2 &= x^2+y^2\zeta^2+2xy\zeta = x^2+y^2(-\zeta-1)+2xy\zeta\\ &= x^2-y^2+(2xy-y^2)\zeta = \pm m\zeta,\end{aligned}$$

which gives $x = \pm y$. If $x = y$, we must have $y^2 = m$; if $x = -y$, we must have $3y^2 = m$. The equation $y^2 = m$ implies $y = \pm 1$ or $y = \pm\sqrt{D}$, in which case $\alpha \simeq \zeta+1 \simeq \zeta$. The equation $3y^2 = m$ implies $m = 3D$, $y = \pm\sqrt{D}$, in which case $\alpha \simeq \zeta-1 \simeq \zeta\sqrt{-3}$.

(iii) If $\omega = \eta$, then

$$K = \boldsymbol{Q}(\sqrt{2}), \quad K(\alpha) = K(\sqrt{-1}), \quad \eta = \pm\tfrac{1}{2}(\sqrt{2}\pm\sqrt{-2}).$$

We will show that the equation $\alpha^2 = m\eta^{-1}$ is impossible. Taking α^* instead of α, if necessary, we may assume

$$\alpha^2 = \pm\frac{m}{2}(\sqrt{2}+\sqrt{-2}) = \pm\frac{m\sqrt{2}}{2}(1+\sqrt{-1}).$$

Furthermore, multiplying α by $\sqrt{-1}$, if necessary, we may assume

$$\alpha^2 = \frac{m\sqrt{2}}{2}(1+\sqrt{-1}).$$

As before, we can write $\alpha = \frac{1}{2}(x+y\sqrt{-1})$, $x, y \in \mathfrak{O}$. Then

$$4\alpha^2 = x^2-y^2+2xy\sqrt{-1} = 2m\sqrt{2}(1+\sqrt{-1}).$$

Hence $x^2-y^2 = 2m\sqrt{2}$, $xy = m\sqrt{2}$, which implies $x^2-y^2 = 2xy$. Then $x = y(1\pm\sqrt{2})$, $xy = y^2(1\pm\sqrt{2}) = m\sqrt{2}$. Taking norms, we obtain $n_{K/\boldsymbol{Q}}(y)^2 = 2m^2$, an impossibility.

The following lemma is well known (cf. [14], p. 354; [15], p. 35).

LEMMA 2. *Let $k_{\mathfrak{p}}$ be a local field with prime ideal $\mathfrak{p}$. Let $\Omega_{\mathfrak{p}}$ be an Eichler order of level $\mathfrak{p}$ in $M(2, k_{\mathfrak{p}})$. Suppose $\alpha_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$ and $N(\alpha_{\mathfrak{p}})$ has $\mathfrak{p}$-order equal to 1. If $\mathfrak{p}$ divides $T(\alpha_{\mathfrak{p}})$, the reduced trace of $\alpha_{\mathfrak{p}}$, then $\alpha_{\mathfrak{p}} \in \mathfrak{N}(\Omega_{\mathfrak{p}})$.*

LEMMA 3. *Suppose $\alpha \in \Omega$. In order that $\alpha \in \mathfrak{N}(\Omega)$ it is sufficient that*

(a) $\alpha = \sqrt{-m}$, $m \mid \delta D$; $\alpha = \zeta$, $\alpha = 1+\sqrt{-1}$ *if* $2 \mid \delta D$, $\alpha = \zeta\sqrt{-3}$ *if* $3 \mid \delta$.

(b) $\alpha \in \mathfrak{p}\Omega_{\mathfrak{p}}$ *for all primes* $\mathfrak{p}$ *of* K *dividing* $\big(N(\alpha), D\big)$.

Proof. It is enough to show that $\alpha \in \mathfrak{N}(\Omega_{\mathfrak{p}})$ for every finite prime $\mathfrak{p}$ of K. Since $N(\alpha)$ is square-free and $N(\alpha) \mid T(\alpha)$ in each case, Lemma 2 implies that $\alpha \in \mathfrak{N}(\Omega_{\mathfrak{p}})$ for every $\mathfrak{p} \mid \big(N(\alpha), \delta\big)$. If $\mathfrak{p} \nmid N(\alpha)$, then $\alpha \in U(\Omega_{\mathfrak{p}}) \subset \mathfrak{N}(\Omega_{\mathfrak{p}})$. If $\mathfrak{p} \mid \big(N(\alpha), D\big)$, let π be a generator of $\mathfrak{p}$. Then (b) implies that $\alpha = \varepsilon\pi$, where $\varepsilon \in U(\Omega_{\mathfrak{p}})$. Hence $\alpha \in \mathfrak{N}(\Omega_{\mathfrak{p}})$.

PROPOSITION 14. *Suppose $\alpha \in \mathfrak{A}_K^\times$, $\alpha \notin K^\times$. If $\Delta > 5$, then in order that $\alpha \in \mathfrak{N}(\Omega)$ for some Eichler order Ω of level δ in $\mathfrak{A}_K$ it is necessary and sufficient that one of the following holds: $\alpha \simeq \sqrt{-m}$, $m \mid \delta D$; $\alpha \simeq \zeta$, $\alpha \simeq 1+\sqrt{-1}$ if $2 \mid \delta D$, $\alpha \simeq \zeta\sqrt{-3}$ if $3 \mid \delta$.*

Proof. The necessity follows from Lemma 1 and the discussion (i), (ii), (iii). To prove sufficiency there is no loss of generality in replacing each $\simeq$ by an $=$. Then, according to Lemma 3, it is enough to choose Ω so that $\alpha \in \Omega$ and (b) holds. Let $\mathfrak{p} \mid \big(N(\alpha), D\big)$ and let π be a generator of $\mathfrak{p}$. Since α/π is integral over $\mathfrak{O}$ in all cases, we can insure (b) of Lemma 3 by insisting that $\Omega_{\mathfrak{p}}$ be a maximal order containing α/π. Hence it is enough to find an Eichler order Ω of level δ such that $\alpha \in \Omega$. According to the criterion of Eichler ([6], p. 133), we need only show that no prime p dividing δ remains prime in $K(\alpha)$. The only way that this could happen is if p remains prime in $\boldsymbol{Q}(\alpha)$, a quadratic extension of $\boldsymbol{Q}$. But then we would have $\boldsymbol{Q}(\alpha)_p = K_p =$ the unique unramified extension of degree 2 over $\boldsymbol{Q}_p$. This would imply $\alpha \in K_p$, $K(\alpha)_p = K(\alpha)\otimes_K K_p \cong K_p \oplus K_p$. Hence p would split in $K(\alpha)$.

Remark. If $\Delta = \Delta_K = 5$, then α normalizes an Eichler order of level 1 if and only if $\alpha \simeq \sqrt{-1}$, ζ or ω, a fifth root of unity (note that $\sqrt{-5} \simeq \sqrt{-1}$).

§ 8. The Selberg trace formula. In this section we complete the derivation of formulas (50) and (53). We regard the type number t_δ as the trace of a certain convolution operator and then use the Selberg trace formula to express this trace as a finite sum of integrals over adelic homogeneous spaces associated to $G = \mathfrak{A}_K^\times/K^\times$. This sum is indexed by the conjugacy classes in G of elements which normalize Eichler orders of level δ. Explicit representatives for these conjugacy classes were found in § 7. To determine the contribution of each representative, we apply the Chevalley–Hasse–Noether Theorem. As a result, the evaluation of each contribution is reduced to the computation of a sum of local unit indices for certain "admissible orders" coming from a fixed imaginary quadratic extension of K (Proposition 15). The admissible orders and their corresponding unit indices are then readily determined (cf. Proposition 16) to yield our formulas.

We use the notation of § 5, allowing Ω to denote an arbitrary Eichler order of level δ in $\mathfrak{A}_K$. In addition, we introduce $G(\Omega) = \mathfrak{N}(\Omega)/K^\times = G^1(\tilde{\Omega}) \cap G$. We fix such an order Ω and denote by $L_2(G\backslash G_{\mathbf{A}}^1/G^1(\tilde{\Omega}))$ the set of all complex-valued functions on $G_{\mathbf{A}}^1$ which are constant on double cosets $GgG^1(\tilde{\Omega})$, $g \in G_{\mathbf{A}}^1$. Then $L_2(G\backslash G_{\mathbf{A}}^1/G^1(\tilde{\Omega}))$ is a complex vector space of dimension t_δ. Let $F_{\tilde{\Omega}}$ denote the characteristic function of $G^1(\tilde{\Omega})$. Then, with respect to the measure λ, the convolution operator $f \mapsto F_{\tilde{\Omega}} * f$, $f \in L_2(G\backslash G_{\mathbf{A}}^1/G^1(\tilde{\Omega}))$ is the identity mapping. Hence its trace $\mathrm{Tr}(F_{\tilde{\Omega}})$ is equal to t_δ. Fix a representative s from each conjugacy class of G and denote the centralizer of s in G by $G(s)$. Applying the Selberg trace formula (cf. [19], [20]), we obtain

$$t_\delta = \mathrm{Tr}(F_{\tilde{\Omega}}) = \sum_s \int_{G(s)\backslash G_{\mathbf{A}}^1} \psi_s(g')\, d\lambda(g') \tag{62}$$

where $\psi_s(g') = F_{\tilde{\Omega}}(g^{-1}sg)$, $g \in G_{\mathbf{A}}^1$.

If $\gamma \in J_{\mathfrak{A}_K}^1$, we put $\gamma\Omega\gamma^{-1} = \gamma\tilde{\Omega}\gamma^{-1} \cap \mathfrak{A}_K$. A representative s makes a non-zero contribution to the trace sum (62) if and only if $g^{-1}sg \in G^1(\tilde{\Omega})$ for some $g \in G_{\mathbf{A}}^1$. Let $\alpha \in \mathfrak{A}_K$ represent s, and $\gamma \in J_{\mathfrak{A}_K}^1$ represent g. Then

$$g^{-1}sg \in G^1(\tilde{\Omega}) \Leftrightarrow s \in G^1(\gamma\tilde{\Omega}\gamma^{-1}) \Leftrightarrow s \in G(\gamma\Omega\gamma^{-1}),$$

since $s \in G$. We conclude that s will make a non-zero contribution to (62) if and only if α normalizes some Eichler order of $\mathfrak{A}_K$ of level δ. Thus, to evaluate (62), we only need to have α range over the finite set of values consisting of 1 and the ones specified in Proposition 14. If $\alpha = 1$, then s contributes $\lambda(G\backslash G_{\mathbf{A}}^1)$ to the trace sum. This contribution can be evaluated by means of (38), (40), (41), and accounts for the leading term in (50) and (53).

Now suppose $\alpha \notin K^\times$. Put $K_\alpha = K(\alpha)$, $\mathcal{O}_\alpha =$ the ring of integers in K_α. Suppose $g^{-1}sg \in G^1(\tilde{\Omega})$ for $g \in G_{\mathbf{A}}^1$. If $\gamma \in J_{\mathfrak{A}_K}^1$ represents g, then we have seen above that $\alpha \in \mathfrak{N}(\gamma\Omega\gamma^{-1})$. Since α is integral, this implies $\alpha \in \gamma\Omega\gamma^{-1}$, hence $\alpha \in \gamma\Omega\gamma^{-1} \cap K_\alpha = \mathcal{O}$, an order of K_α. Any order of K_α arising in this manner

will be called *admissible* for α. Equivalently, an order $\mathcal{O}$ of K_α is admissible for α if $\alpha \in \mathcal{O}$ and, for some Eichler order Ω of level δ, $\mathcal{O} = \Omega \cap K_\alpha$ and $\alpha \in \mathfrak{N}(\Omega)$. Let $\mathcal{O}$ be admissible for α. We denote by $\langle \Omega | \mathcal{O} \rangle$ the set of all $\gamma \in J_{\mathfrak{A}_K}^1$ such that $\gamma\Omega\gamma^{-1} \cap K_\alpha = \mathcal{O}$. Then the contribution of s to the trace sum is

$$\sum_{\mathcal{O}} \lambda\big(G(s)\backslash(\langle \Omega | \mathcal{O} \rangle / J_K^1)\big), \tag{63}$$

the sum being taken over all orders $\mathcal{O}$ which are admissible for α. It is easy to see that

$$[G(s) : K_\alpha^\times/K^\times] = \begin{cases} 2 & \text{if } \alpha \text{ is pure}, \\ 1 & \text{if } \alpha \text{ is not pure}. \end{cases} \tag{64}$$

Let J_{K_α} be the idele group of K_α. We regard J_{K_α} as a subgroup of $J_{\mathfrak{A}_K}$ in the usual way, and put $J_{K_\alpha}^1 = J_{K_\alpha} \cap J_{\mathfrak{A}_K}^1$. Then the Chevalley–Hasse–Noether Theorem shows that

$$\langle \Omega | \mathcal{O} \rangle = J_{K_\alpha}^1 \gamma_0 \mathfrak{N}^1(\tilde{\Omega}) \tag{65}$$

where γ_0 is a fixed element of $\langle \Omega | \mathcal{O} \rangle$ (cf. [17], § 8, Proposition 8). Let $l_{\mathcal{O}}$ denote the number of primes $p | \delta$ which ramify in K_α but do not divide the conductor of $\mathcal{O}$. Putting $\Omega_1 = \gamma_0 \Omega \gamma_0^{-1}$ and arguing as in § 8 of [17], we see that $\langle \Omega | \mathcal{O} \rangle$ is equal to $2^{e-l_{\mathcal{O}}}$ disjoint right translates of $J_{K_\alpha}^1 U^1(\tilde{\Omega}_1)$. Therefore, we have

$$\begin{aligned} \lambda\big((K_\alpha^\times/K^\times)\backslash(\langle \Omega | \mathcal{O} \rangle / J_K^1)\big) &= 2^{e-l_{\mathcal{O}}} \lambda\big((K_\alpha^\times/K^\times)\backslash(J_{K_\alpha}^1 U^1(\tilde{\Omega}_1)/J_K^1)\big) \\ &= 2^{e-l_{\mathcal{O}}} \lambda\big(K_\alpha^\times J_K^1 \backslash J_{K_\alpha}^1 U^1(\tilde{\Omega}_1)\big) \\ &= \frac{2^{e-l_{\mathcal{O}}}}{h(K)} \lambda\big(K_\alpha^\times U_K^1 \backslash J_{K_\alpha}^1 U^1(\tilde{\Omega}_1)\big). \end{aligned} \tag{66}$$

Let $h(\mathcal{O})$ denote the number of classes of locally principal fractional ideals of $\mathcal{O}$. Then

$$h(\mathcal{O}) = [J_{K_\alpha}^1 : K_\alpha^\times U^1(\tilde{\mathcal{O}})],$$

where

$$U^1(\tilde{\mathcal{O}}) = \prod_p U(\mathcal{O}_p) \cap J_{K_\alpha}^1 = U^1(\tilde{\Omega}_1) \cap J_{K_\alpha}^1.$$

Then formula (66) is equal to

$$2^{e-l_{\mathcal{O}}} \frac{h(\mathcal{O})}{h(K)} \lambda\big(K_\alpha^\times U_K^1 \backslash K_\alpha^\times U^1(\tilde{\Omega})\big). \tag{67}$$

Let $E_K, E_a, E(\mathcal{O})$, denote the unit groups of $\mathfrak{O}, \mathcal{O}_a, \mathcal{O}$, respectively, and put $U_a^1 = U^1(\tilde{\mathcal{O}}_a)$. Then

$$E_K = U_K^1 \cap K^\times, \quad E_a = U_a^1 \cap K_a^\times, \quad E(\mathcal{O}) = U^1(\tilde{\mathcal{O}}) \cap K_a^\times,$$

and

$$(68) \qquad (67) = \frac{2^{e-l_{\mathcal{O}}} h(\mathcal{O})}{h(K)} \frac{\lambda(U_K^1 \setminus U^1(\tilde{\Omega}_1))}{[E(\mathcal{O}):E_K]} = \frac{2^{-l_{\mathcal{O}}} h(\mathcal{O})}{h(K)} \frac{\lambda(G^1(\tilde{\Omega}_1))}{[E(\mathcal{O}):E_K]} = \frac{2^{-l_{\mathcal{O}}} h(\mathcal{O})}{h(K)[E(\mathcal{O}):E_K]}.$$

Let $h(K_a)$ denote the class number of K_a. It is easy to see that

$$(69) \qquad \frac{h(\mathcal{O})}{h(K_a)} = \frac{[U_a^1 : U_a^1(\tilde{\mathcal{O}})]}{[E_a : E(\mathcal{O})]}.$$

Hence

$$(70) \qquad (68) = 2^{-l_{\mathcal{O}}} \frac{h(K_a)[U_a^1 : U^1(\tilde{\mathcal{O}})]}{h(K)[E_a : E_K]}.$$

Applying (64), we obtain

LEMMA. *Let s be represented by an integral element $a \in K^\times$ which normalizes an Eichler order of level δ in $\mathfrak{A}_K$. Then the contribution of s to the trace sum* (62) *is*

$$(71) \qquad \frac{h(K_a)}{2h(K)[E_a:E_K]} \sum_{\mathcal{O}} 2^{-l_{\mathcal{O}}} [U_a^1 : U^1(\tilde{\mathcal{O}})] \quad \text{if} \quad a \text{ is pure},$$

$$(72) \qquad \frac{h(K_a)}{h(K)[E_a:E_K]} \sum_{\mathcal{O}} 2^{-l_{\mathcal{O}}} [U^1 : U^1(\tilde{\mathcal{O}})] \quad \text{if} \quad a \text{ is not pure},$$

where $\mathcal{O}$ ranges over all orders of K_a which are admissible for a.

We denote by W_a the group of roots of unity contained in K_a.

PROPOSITION 15. *Suppose s is represented by a as in Proposition* 14. *Then $K_a = K(\sqrt{-m})$, where $m = 3$ or $m \mid \delta D$, and the contribution of s to the trace sum* (62) *is*

$$(73) \qquad \frac{\varepsilon(a) h(-m) h(-mD)}{2 \operatorname{card}(W_a)} \sum_{\mathcal{O}} 2^{-l_{\mathcal{O}}} [U_a^1 : U^1(\tilde{\mathcal{O}})] \quad \text{if} \quad a \text{ is pure},$$

$$(74) \qquad \frac{\varepsilon(a) h(-m) h(-mD)}{\operatorname{card}(W_a)} \sum_{\mathcal{O}} 2^{-l_{\mathcal{O}}} [U_a^1 : U^1(\tilde{\mathcal{O}})] \quad \text{if} \quad a \text{ is not pure},$$

where $\varepsilon(a) = 1$ if $K_a \neq \mathbf{Q}(\sqrt{-1}, \sqrt{-2})$, $\varepsilon(a) = 2$ if $K_a = \mathbf{Q}(\sqrt{-1}, \sqrt{-2})$, and $\mathcal{O}$ ranges over all orders of K_a which are admissible for a.

Proof. Proposition 14 shows that $K_a = K(\sqrt{-m}) = \mathbf{Q}(\sqrt{-m}, \sqrt{-mD})$, where $m = 3$ or $m \mid \delta D$. The classical formula of Bachmann ([7], p. 74) for the class number of an imaginary bicyclic biquadratic number field shows that

$$(75) \qquad \frac{h(K_a)}{h(K)} = \frac{\varepsilon(a)}{2} [E_a : W_a E_K] h(-m) h(-mD).$$

Noting that

$$\frac{[E_a : W_a E_K]}{[E_a : E_K]} = \frac{2}{\operatorname{card}(W_a)}$$

and applying the lemma, we obtain the result.

Remark. In our final class number formulas no special consideration will be necessary for the case where $K_a = \mathbf{Q}(\sqrt{-1}, \sqrt{-2})$. This is because $\mathbf{Q}(\sqrt{-1}, \sqrt{-2})$ has twice as many roots of unity as $\mathbf{Q}(\sqrt{-1}, \sqrt{-m})$ if $m \neq 2$, $m \mid \delta D$, which cancels the doubling of the factor $\varepsilon(a)$.

To explicitly determine the contributions (73), (74) we must carry out two steps. First, we must determine all admissible orders $\mathcal{O}$ for each a as given in Proposition 14. Second, we must evaluate the unit index $[U_a^1 : U^1(\tilde{\mathcal{O}})]$ for each such $\mathcal{O}$. The first step is an application of Proposition 13. Since $[U_a^1 : U^1(\tilde{\mathcal{O}})] = \prod_{\mathfrak{p}<\infty} [U(\mathcal{O}_{a,\mathfrak{p}}) : U(\mathcal{O}_\mathfrak{p})]$, the second step is reduced to the computation of local indices $[U(\mathcal{O}_{a,\mathfrak{p}}) : U(\mathcal{O}_\mathfrak{p})]$ for finite $\mathfrak{p}$. Such a computation is a special case of the following elementary proposition, whose proof we omit.

PROPOSITION 16. *Let k be a local field of characteristic $\neq 2$, with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p}$. Put $q = \operatorname{card}(\mathfrak{o}/\mathfrak{p})$. Let $k_a = k + ka$, where $a \notin k$, $a^2 \in k$. Denote the maximal order of k_a by $\mathfrak{o}_a$ and the unique $\mathfrak{o}$-order of k_a of conductor $\mathfrak{p}^i$, $i \geq 1$, by $\mathfrak{o}_i$. Then* $[U(\mathfrak{o}_a) : U(\mathfrak{o}_i)] =$

(a) $q^{i-1}(q+1)$ *if k_a is an unramified field extension of k,*

(b) q^i *if k_a is a ramified field extension of k,*

(c) $q^{i-1}(q-1)$ *if k_a is a split extension of k.*

In order to compute $l_{\mathcal{O}}$, we note that p_i ramifies in K_a if and only if p_i ramifies in $\mathbf{Q}(a)$, $i = 1, \ldots, e$. It follows that $l_{\mathcal{O}}$ is the number of p_i, $i = 1, \ldots, e$, which ramify in $\mathbf{Q}(a)$ but do not divide the conductor of $\mathcal{O}$.

Remark. It should be noted that the maximal order $\mathcal{O}_a$ is always admissible for a. This is essentially what we showed when we applied Eichler's criterion ([6], p. 133) in the proof of Proposition 14.

Put $(1, a) = \mathfrak{O} + \mathfrak{O}a$. Assume first that D is odd. Then $D = r_1 \ldots r_t$, where r_j is a prime, $r_j \equiv 1 \pmod 4$, $j = 1, \ldots, t$. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_t$ be the prime ideals of K such that $\mathfrak{r}_j^2 = (r_j)$.

(1) Suppose α is not pure, so that $\alpha = \zeta, 1+\sqrt{-1}$, or $\zeta\sqrt{-3}$. Since D is odd, the proof of the corollary to Proposition 13 shows that $(1, \alpha) = \mathcal{O}_\alpha$ in each of these cases. Thus $\mathcal{O}_\alpha$ is the only admissible order in each case.

(i) $\alpha = \zeta$. Then $l_{\mathcal{O}} = 0$ if $3 \nmid \delta$, $l_{\mathcal{O}} = 1$ if $3 \mid \delta$. Applying (74), we see that α contributes

$$\frac{h(-3D)}{6} \quad \text{if} \quad 3 \nmid \delta, \tag{76}$$

$$\frac{h(-3D)}{12} \quad \text{if} \quad 3 \mid \delta. \tag{77}$$

(ii) $\alpha = 1+\sqrt{-1}$. Then $2 \mid \delta$, $l_{\mathcal{O}} = 1$, and the contribution of α is

$$\frac{h(-D)}{8}. \tag{78}$$

(iii) $\alpha = \zeta\sqrt{-3}$. Then $3 \mid \delta$, $l_{\mathcal{O}} = 1$, and the contribution is

$$\frac{h(-3D)}{12}. \tag{79}$$

Now suppose that α is pure, $\alpha = \sqrt{-m}$, where $m \mid \delta D$. Let us write $m = nd$, where $n \mid \delta$, $d \mid D$. We must consider two separate cases:

(2) If $m \equiv 1, 2 \pmod 4$, then, using the notation of Proposition 13, we have $\Delta(-m) = -4nd$, $\Delta(-mD) = -4nD/d$, $n_{K/\mathbf{Q}}(\Delta_{L/K}) = 16n^2$. Hence $\Delta_{L/K} = (-4n)$, which implies that $(1, \alpha)$ has conductor $\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$. Suppose $\mathfrak{r}$ is a prime ideal dividing d. Let π be a local generator for $\mathfrak{r}$. Then, for any Eichler order Ω_1 of level δ, we have $\alpha \in \mathfrak{N}((\Omega_1)_\mathfrak{r})$ if and only if $\alpha/\pi \in \mathfrak{N}((\Omega_1)_\mathfrak{r})$. It follows that $\mathcal{O}_\alpha$ is the only admissible order for α.

(i) $m \equiv 1 \pmod 4$. Then $l_{\mathcal{O}} = \lambda(n)$ if $2 \nmid \delta$, $l_{\mathcal{O}} = \lambda(n)+1$ if $2 \mid \delta$. Hence, according to (73), the contribution of α is

$$\frac{h(-D)}{8} \quad \text{if} \quad m = 1,\ 2 \nmid \delta, \tag{80}$$

$$\frac{h(-D)}{16} \quad \text{if} \quad m = 1,\ 2 \mid \delta, \tag{81}$$

$$2^{-\lambda(n)-2} h(-nd) h(-nD/d) \quad \text{if} \quad m \geqslant 5,\ 2 \nmid \delta, \tag{82}$$

$$2^{-\lambda(n)-3} h(-nd) h(-nD/d) \quad \text{if} \quad m \geqslant 5,\ 2 \mid \delta. \tag{83}$$

(ii) $m \equiv 2 \pmod 4$. Then $2 \mid \delta$, $l_{\mathcal{O}} = \lambda(n)$, and the contribution is

$$2^{-\lambda(n)-2} h(-nd) h(-nD/d). \tag{84}$$

(3) If $m \equiv 3 \pmod 4$, then $\Delta(-m) = -nd$, $\Delta(-mD) = -nD/d$, $n_{K/\mathbf{Q}}(\Delta_{L/K}) = n^2$. Hence $\Delta_{L/K} = (-n)$, from which it follows that $(1, (1+\alpha)/2)$ has conductor $\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$, and $(1, \alpha)$ has conductor $2\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$. We conclude that the conductor of an admissible order for α must divide (2). In particular, it follows that $l_{\mathcal{O}} = \lambda(n)$ for any admissible $\mathcal{O}$. Since α is a unit in $K_{\alpha,\mathfrak{p}}$ for any $\mathfrak{p}$ dividing (2), any order of conductor dividing (2) must in fact be admissible for α.

(i) $D \equiv 5 \pmod 8$. Then 2 remains prime in K, $K_2 = \mathbf{Q}(\alpha)_2 =$ the unique unramified quadratic extension of $\mathbf{Q}_2$. It follows that $K_{\alpha,2}$ is a split extension of K_2. Let $\mathcal{O}_1$ be the order of K_α of conductor (2). Applying (c) of Proposition 16, we see that $[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = 2^2-1 = 3$. Thus α contributes

$$\frac{h(-3D)}{6} \quad \text{if} \quad m = 3, \tag{85}$$

$$2^{-\lambda(n)} h(-nd) h(-nD/d) \quad \text{if} \quad m > 3. \tag{86}$$

(ii) $D \equiv 1 \pmod 8$. Then 2 splits in K, $(2) = \mathfrak{p}_1\mathfrak{p}_2$, $K_2 = K_{\mathfrak{p}_1} \oplus K_{\mathfrak{p}_2} \cong \mathbf{Q}_2 \oplus \mathbf{Q}_2$. Let $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{12}$ denote the orders of K_α of conductor $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1\mathfrak{p}_2$, respectively.

$(\text{ii})_1$ Suppose $m \equiv 3 \pmod 8$. Then $K_{\alpha,\mathfrak{p}_1}, K_{\alpha,\mathfrak{p}_2}$ are unramified quadratic extensions of $K_{\mathfrak{p}_1}, K_{\mathfrak{p}_2}$, respectively. Applying (a) of Proposition 16, we see that

$$[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_2)] = 3, \quad [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_{12})] = 9.$$

Hence the contribution of α is

$$\tfrac{2}{3} h(-3D) \quad \text{if} \quad m = 3, \tag{87}$$

$$2^{-\lambda(n)+2} h(-nd) h(-nD/d) \quad \text{if} \quad m > 3. \tag{88}$$

$(\text{ii})_2$ Suppose $m \equiv 7 \pmod 8$. Then $K_{\alpha,\mathfrak{p}_1}, K_{\alpha,\mathfrak{p}_2}$ are split extensions of $K_{\mathfrak{p}_1}, K_{\mathfrak{p}_2}$, respectively. Applying (c) of Proposition 16, we see that

$$[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_2)] = [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_{12})] = 1.$$

Hence the contribution of α is

$$2^{-\lambda(n)} h(-nd) h(-nD/d). \tag{89}$$

Adding the contributions of $\sqrt{-1}$ and $1+\sqrt{-1}$ (if any), we obtain the term $c_1 h(-D)$ of (50). Adding the contributions of $\zeta, \zeta\sqrt{-3}, \sqrt{-3}$, we obtain the term $c_3 h(-3D)$ of (50). We note that $\sqrt{-nd} \simeq \sqrt{-nD/d}$. Hence we may assume that $\alpha = \sqrt{-nd}$ with $d < \sqrt{D}$. Then the remaining terms in (50) are accounted for by (82), (83), (84), (86), (88) and (89).

Now assume that D is even. Then $D = 2r_1\ldots r_t$, where r_j is a prime, $r_j \equiv 1 \pmod 4$, $j = 1, \ldots, t$. Let $\mathfrak{r}, \mathfrak{r}_1, \ldots, \mathfrak{r}_t$ be the prime ideals of K such that $\mathfrak{r}^2 = (2)$, $\mathfrak{r}_j^2 = (r_j)$, $j = 1, \ldots, t$. Let π denote a local generator of $\mathfrak{r}$.

(1) Suppose α is not pure.

(i) $\alpha = \zeta$. Then $\mathcal{O}_\alpha$ is the only admissible order, with contribution

$$(90) \qquad \frac{h(-3D)}{6} \quad \text{if} \quad 3 \nmid \delta,$$

$$(91) \qquad \frac{h(-3D)}{12} \quad \text{if} \quad 3 \mid \delta.$$

(ii) $\alpha = \zeta\sqrt{-3}$. Once again $\mathcal{O}_\alpha$ is the only admissible order, with contribution

$$(92) \qquad \frac{h(-3D)}{12}.$$

(iii) $\alpha = 1+\sqrt{-1}$. The proof of the corollary to Proposition 13 shows that $(1, \alpha)$ has conductor $\mathfrak{r}$. Then $1, \alpha/\pi$ generate $\mathcal{O}_{\alpha,\mathfrak{r}}$, which shows that $\mathcal{O}_\alpha$ is the only admissible order. Hence the contribution is

$$(93) \qquad \frac{h(-D)}{4}.$$

Suppose α is pure, $\alpha = \sqrt{-m}$, $m = nd$, $n \mid \delta$, $d \mid D$. Since $\sqrt{-nd} \simeq \sqrt{-nD/d}$ and D is even, we may assume that d is odd. It follows, in particular, that m is odd.

(2) If $m \equiv 1 \pmod 4$, then $\Delta(-m) = -4nd$, $\Delta(-mD) = -4nD/d$, which implies $\Delta_{L/K} = (-2n)$. Hence $\mathfrak{r}$ ramifies in K_α and the conductor of $(1, \alpha)$ is $\mathfrak{r}\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$. The fact that α is a unit in $K_{\alpha,\mathfrak{r}}$ implies that the admissible orders for α are those whose conductor divides $\mathfrak{r}$. In particular, $l_{\mathcal{O}} = \lambda(n)$ for any admissible $\mathcal{O}$. Let $\mathcal{O}_1$ be the order of conductor $\mathfrak{r}$ in K. Applying (b) of Proposition 16, we see that $[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = 2$. We conclude that the contribution of α is

$$(94) \qquad \tfrac{3}{8}h(-D) \quad \text{if} \quad m = 1,$$

$$(95) \qquad 3\cdot 2^{-\lambda(n)-2}\, h(-nd)\, h(-nD/d) \quad \text{if} \quad m \geqslant 5.$$

(3) If $m \equiv 3 \pmod 4$, then $\Delta(-m) = -nd$, $\Delta(-mD) = -4nD/d$, $\Delta_{L/K} = (n)$. It follows that $(1, (1+\alpha)/2)$ has conductor $\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$ and $(1, \alpha)$ has conductor $2\prod_{\mathfrak{r}_j \mid d} \mathfrak{r}_j$. Reasoning as before, we see that the admissible orders are those whose conductor divides $(2) = \mathfrak{r}^2$, and $l_{\mathcal{O}} = \lambda(n)$ for any admissible $\mathcal{O}$. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be the orders of K_α of conductor $\mathfrak{r}, \mathfrak{r}^2$, respectively.

(i) $m \equiv 3 \pmod 8$. Then $\mathfrak{r}$ remains prime in K_α, and (a) of Proposition 16 shows that

$$[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = 3, \qquad [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_2)] = 6.$$

Hence the contribution of α is

$$(96) \qquad \tfrac{5}{12}h(-3D) \quad \text{if} \quad m = 3,$$

$$(97) \qquad 5\cdot 2^{-\lambda(n)-1}\, h(-nd)\, h(-nD/d) \quad \text{if} \quad m > 3.$$

(ii) $m \equiv 7 \pmod 8$. Then $\mathfrak{r}$ splits in K_α, and (c) of Proposition 16 shows that

$$[U_\alpha^1 : U^1(\tilde{\mathcal{O}}_1)] = 1, \qquad [U_\alpha^1 : U^1(\tilde{\mathcal{O}}_2)] = 2.$$

We conclude that α contributes

$$(98) \qquad 2^{-\lambda(n)}\, h(-nd)\, h(-nD/d).$$

Compiling the contributions (90)–(98), we account for all the terms appearing in (53).

Remark. The exceptional case $\Delta = \Delta_K = 5$ can also be handled by the Selberg trace formula. In this case a complete set of α with non-zero contribution to the trace sum is given by $1, \sqrt{-1}, \zeta, \omega, \omega'$, where ω, ω' are two primitive fifth roots of unity which are not conjugate over $\mathbf{Q}(\sqrt{5})$. The corresponding contributions are $2M(\mathfrak{I}) = 1/60$, $1/4$, $1/3$, $1/5$, $1/5$, respectively. Hence $H = 1/60 + 1/4 + 1/3 + 2/5 = 1$, as it should be.

We conclude with the following small table, which bears comparison with the table in [21], p. 146–148.

Δ	H	Δ	H
5	1	52	2
8	1	53	3
13	1	61	3
17	1	65	4
20	1	72	2
29	2	73	3
37	2	85	6
40	4	89	4
41	2	97	4
45	1	101	5

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On the zeros of Dirichlet L-functions (VI)

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§ 1. Here we will see a q-analogue of the author's previous work [4]. We will quote this by (V).

Let $L(s,\chi)$ be a Dirichlet L-function with a character χ to modulus q. We write a nontrivial zero of $L(s,\chi)$ by $\varrho(\chi)=\beta(\chi)+i\gamma(\chi)$. As before for given two Dirichlet L-functions $L(s,\chi_1)$ and $L(s,\chi_2)$, we call ϱ a coincident zero of $L(s,\chi_1)$ and $L(s,\chi_2)$ if $L(\varrho,\chi_1)=L(\varrho,\chi_2)=0$ with the same multiplicity. We call ϱ a noncoincident zero of $L(s,\chi_1)$ and $L(s,\chi_2)$ if ϱ is not a coincident zero. We assume the order is given in the set of ordinates of zeros of $L(s,\chi)$ by $0\leqslant\gamma_n(\chi)\leqslant\gamma_{n+1}(\chi)$. Also in the set $\{\gamma_n(\chi_1),\gamma_m(\chi_2): n=1,2,\ldots,m=1,2,\ldots\}$ the order is given by

$$\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2),\quad\text{if}\quad \gamma_n(\chi_1)<\gamma_m(\chi_2)$$

and

$$\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2)\leqslant\gamma_{n+1}(\chi_1)\leqslant\gamma_{m+1}(\chi_2)\leqslant\ldots$$

if

$$\gamma_n(\chi_1)=\gamma_{n+1}(\chi_1)=\ldots=\gamma_m(\chi_2)=\gamma_{m+1}(\chi_2)=\ldots$$

Now we are concerned with the following problems, which are similar to problems (i), (ii) and (iii) in (V).

(i) *Have different primitive L-functions $L(s,\chi_1)$ and $L(s,\chi_2)$ a coincident zero?*

(ii) *For given positive real numbers t_1 and t_2, and for almost all pairs of primitive characters (χ_1,χ_2) does there exist a zero of $L(s,\chi_2)$ in*

$$\gamma_n(\chi_1)\leqslant \operatorname{Im} s\leqslant\gamma_{n+1}(\chi_1)$$

for each $\gamma_n(\chi_1)$ in $t_1\leqslant\gamma_n(\chi_1)\leqslant t_2$?

(iii) *For some $\gamma_n(\chi_1)$, does it happen that $\gamma_n(\chi)\leqslant\gamma_n(\chi_1)\leqslant\gamma_{n+1}(\chi)$ for almost all primitive characters χ?*

Our answers to these are the following theorems.

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