Arithmetic of quaternary quadratic forms

by

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Introduction. In this paper we give a general treatment of certain aspects of the arithmetic theory of quaternary quadratic forms. Although we will concentrate on class number questions, our hope is that the framework provided here will also be suitable for the treatment of other arithmetic questions, for example, those pertaining to modular forms of "Nebentypus". We do not employ the classical language in our discussion of quadratic forms. Rather, we adopt the terminology and viewpoint of Eichler's fundamental work [4]. Thus we begin with a regular quadratic vector space \( V \) of dimension four over an algebraic number field \( k \) (a quadernary space over \( k \), in our terminology). The group of primary interest to us will be \( S^+ (V) \), the group of proper similitudes, rather than the special orthogonal group \( O^+ (V) \). Accordingly, the class number we consider here will be the number of similitude classes in an idealcomplex \([14, p. 87]\). We set as our main problem the determination of the number of similitude classes in an arbitrary idealcomplex \( 2 \) of maximal lattices of \( V \). Our viewpoint throughout is to interpret the arithmetic of the quadernary space \( V \) in terms of the arithmetic of its second Clifford algebra \( C_2^+ \). This viewpoint enables us to reduce the main problem, in most cases, to that of determining the class number \( \mathcal{H} \) of a unique idealcomplex \( 3 \) from each \( V \); namely, the idealcomplex containing the maximally integral lattices of \( V \). The precise statement and details of proof are given in § 3.

If the discriminant \( D(V) \) of \( V \) is a square in \( k \), then \( C_2^+ = \mathfrak{M} \mathfrak{W} \), where \( \mathfrak{W} \) is a quaternion algebra over \( k \). In this case \( 3 \) is the only idealcomplex of maximal lattices and \( \mathcal{H} \) is the number of classes of normal ideals of \( \mathfrak{W} \). In my dissertation [16] I derived a formula for \( \mathcal{H} \) in the case where \( k = \mathbb{Q} \), the field of rationals. For this reason the square discriminant case is of no major interest to us here. However, in order to provide a unified approach, we will mention the square discriminant case several times, pointing out how it parallels the nonsquare discriminant case.

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If \( D(V) \) is not a square in \( k \), then \( G_\mathfrak{H} = \mathbb{H}_K \), where \( \mathbb{H} \) is a quaternion algebra over \( k \), and \( K = k(V, D(V)) \). In this case there are infinitely many idealcomplexes of maximal lattices of \( V \) (§ 3, Proposition 7). However, replacing \( V \) by a suitable similar space, if necessary, we may assume that \( L = \mathbb{H} \), except if \( K \) is a subfield of the Hilbert class field of \( k \), in which case we must also consider those \( L \) which contain \( p \)-maximal lattices of \( V \), where \( p \) is a prime of \( k \) which remains prime in \( K \). In § 4 we show, under the assumption that \( k \) has class number 1, that \( H \) is equal to a certain “generalized type number” associated to \( \mathbb{H}_K \). As a simple application, we give in § 5 an elementary proof of the Minkowski-Siegel formula for definite quaternion spaces over \( \mathbb{Q} \). If, in addition to \( k \) having class number 1, we assume that \( V \) is isotropic for every finite prime \( p \) of \( k \), then \( H = t_p \), the type number of Eichler orders of level \( \mathbb{H}_K \), where \( \mathbb{H} \) is the product of all finite non-split primes of \( \mathbb{H} \) which remain prime in \( K \). Using the Selberg trace formula, we derive elementary formulas for \( t_p \) (§ 6, Theorems 1, 2) assuming \( k = \mathbb{Q} \) and the fundamental unit of \( \mathbb{K} \) has norm \(-1\). These formulas were announced earlier in [18].

Interpreting our lattice-theoretic results in the language of quadratic forms, we then obtain elementary formulas for the number of classes of positive definite integral quaternion forms with discriminant \( \mathbb{D}_K \), where \( \mathbb{D}_K \) is the discriminant of a real quadratic extension \( K \) of \( \mathbb{Q} \) having a fundamental unit of norm \(-1\) (§ 6, Corollary 1 to Theorem 5). In the special case where \( \mathbb{D}_K \) is a prime (§ 6, Corollary to Theorem 1) this result also follows from results of Peters [14] and Tamagawa (unpublished) and can also be found in the paper of Kitaoka [8]. The nature of our formulas suggest that class number questions for quaternion forms are “reducible”, in some sense, to class number questions for binary forms. It would be desirable to establish such a relation between quaternion forms and binary forms directly. In particular, one might then have a method of constructing representatives for classes of positive definite quaternion forms which is more effective than the usual reduction technique (cf. [21]).

§ 1. Quaternion spaces. In this section \( k \) denotes either a global or local field of characteristic \( \neq 2 \). A quaternion space over \( k \) is an ordered pair \((V, q)\), where \( V \) is a vector space of dimension four over \( k \) and \( q : V \to k \) is a quadratic mapping. When there is no danger of confusion we will denote \((V, q)\) simply by \( V \). The symmetric bilinear form \( B \) associated to \( q \) is defined by:

\[
B(v, w) = q(v + w) - q(v) - q(w), \quad v, w \in V.
\]

We assume that \( B \) is nondegenerate, so that \( \det[B(v_i, v_j)] \neq 0 \) for any basis \( \{v_i\} \) of \( V \) over \( k \), \( i, j = 1, 2, 3, 4 \). Let \( k^2 \) denote the multiplicative group of non-zero elements in \( k \), \((k^2)^2\) the subgroup of squares of \( k^2 \).

The coset of \( \det[B(v_i, v_j)] \) in \((k^2)^2\) is independent of the choice of basis \( \{v_i\} \) of \( V \) over \( k \). We call this coset the discriminant of \( V \) and denote it by \( D(V) \). Whenever it is convenient, we will use \( v \) to identify \( D(V) \) with any of its coset representatives.

A similitude \( \sigma \) of \( V \) is a linear automorphism of \( V \) satisfying \( g(\sigma(v)) = a_q g(v) \) for every \( v \in V \), where \( a_q \) is \( k^2 \) independent of \( v \). The number \( a_q \) is called the norm of \( \sigma \) and is denoted accordingly by \( n(\sigma) \). The similitudes of \( V \) form a subgroup of \( \text{GL}(V) \) which we denote by \( \text{Sim}(V) \). An orthogonal transformation of \( V \) is a similitude of \( V \) having norm 1. The group of all orthogonal transformations of \( V \) will be denoted by \( O(V) \). If \( \sigma \) is a similitude, then \( \det(\sigma) = \pm n(\sigma)^2 \). We say that \( \sigma \) is proper or improper according as the plus sign or minus sign holds. In particular, an orthogonal transformation \( \sigma \) is proper if and only if \( \det(\sigma) = 1 \). We denote the group of proper similitudes by \( S^+(V) \) and the group of proper orthogonal transformations by \( O^+(V) \).

We proceed now to classify the quaternion spaces over \( k \) up to similarity. As the existing literature on this subject is inadequate for our purposes, we will have to draw upon results contained in some unpublished lecture notes of Tamagawa. Our classification depends on an isometry classification of quaternion spaces over \( k \) which represent 1.

Let us denote the Clifford algebra of \( V \) by \( C_V \). We recall that \( C_V \) is a graded associative algebra of dimension 16 over \( k \) and \( V \) may be regarded as the subspace of homogeneous elements of degree 1. The multiplication in \( C_V \) is uniquely determined by the condition \( \sigma^2 = g(\sigma) \) for all \( \sigma \in V \). The subspace of \( C_V \) spanned by all homogeneous elements of even degree is a subalgebra of dimension 8 which we denote by \( C_V^0 \).

Let \( a_1, a_2, a_3, a_4 \) be an orthogonal basis of \( V \) over \( k \) and put \( s = a_1a_2a_3a_4 \). Then \( s^2 = D(V) \) and the center \( C_0 \) of \( C_V \) is given by \( C_0 = k + ks \). If \( D(V) \) is a square in \( k^2 \), then \( C_0 \) is a subring of \( k^2 \) and \( k^2 \) is a quadratic extension of \( k \).

Notation. For any associative ring \( R \) with 1 let \( R^x \) denote the multiplicative group of all invertible elements in \( R \).

The Clifford group \( C_V^0 \) of \( V \) is the subgroup of \( C_V^0 \) of all \( \gamma \) such that \( \gamma V \gamma^{-1} = V \). We put \( C_V^0 = \Gamma_V \) and \( C_V^0 = \Gamma_V \). To each \( \gamma \in \Gamma_V \) we associate \( \sigma_\gamma \text{ O}(V) \) by setting \( \sigma_\gamma(v) = \gamma v \gamma^{-1}, \quad v \in V \). Then the homomorphism \( \gamma \mapsto \sigma_\gamma \) maps \( \Gamma_V \) onto \( O(V) \) and \( O(V) \) onto \( \text{GL}(V) \). The mapping defined by \( v_1, \ldots, v_4 \mapsto v_1, \ldots, v_4, v_1, \ldots, v_4 \in V \) gives an involution of \( C_V \) which we denote by \( a \mapsto a^* \). One can easily show that \( \Gamma_V \) is the set of all \( \gamma \in \Gamma_V \) such that \( \gamma a \gamma^* \in k^2 \) (cf. [4], p. 33).

In the remainder of this section we will assume that \( q \) represents 1. Then we can choose the orthogonal basis \( a_1, a_2, a_3, a_4 \) with \( q(a_i) = 1 \). Let \( \mathcal{A} \) be the set of all elements in \( C_V^0 \) which commute with \( a_1 \). Then \( \mathcal{A} \)
is a subalgebra of $C\mathcal{P}$ with basis 1, $\lambda = a_1 a_2, \mu = a_1 a_4, \nu = a_2 a_4$, satisfying the relations:

$$\lambda^2 = -g(a_1)g(a_2), \quad \mu^2 = -g(a_2)g(a_1), \quad \lambda \mu = g(a_2)\nu = -\mu \lambda.$$ 

In other words, $\mathcal{H}$ is a quaternion algebra over $k$. The restriction of the involution $a \mapsto \overline{a}$ to $\mathcal{H}$ is the canonical involution of $\mathcal{H}$. We regard $\mathcal{H}$ as a quaternion space over $k$ with quadratic mapping equal to its reduced norm mapping $N$, defined by $N(a) = a a^*, a \in \mathcal{H}$. It is clear that $C\mathcal{P} = \mathcal{H}_K$, where $\mathcal{H}_K = \mathcal{H} \otimes_k K$.

We must distinguish between the cases where $D(V)$ is or is not a square in $k$. First suppose $D(V)$ is a square in $k$. Then $K = \mathcal{H}_K e_1 + \mathcal{H}_K e_2$ and $C\mathcal{P} = \mathcal{H}_K e_1 + \mathcal{H}_K e_2$, where $e_1, e_2$ are the orthogonal idempotents of $K$. One easily verifies the following relations

$$a_1 e_1 = a_2 e_2, \quad a_1 e_2 = a_2 e_1.$$ 

Define a linear mapping $\varphi: \mathcal{H} \to C_V$ by $\varphi(\xi) = (a_1 a_2 + a_2 a_1)\xi, \xi \in \mathcal{H}$. It is easy to see that $\varphi(\xi^*) = \varphi(\xi)$. Thus the homogeneous components of $\varphi(\xi)$ are symmetric and of odd degree. This implies that $\varphi(\xi) e_1 \in V$ for every $\xi \in \mathcal{H}$. Furthermore, one can easily show that $g(\varphi(\xi)) = g(\varphi(\xi)) = N(\xi) \xi$ for $\xi \in \mathcal{H}$. It follows that $\varphi$ is an isometry of $\mathcal{H}$ onto $V$. Let $\gamma$ be an element of $C\mathcal{P}$.

Proposition 1. Let $V$ be a quaternion space over $k$ which represents 1. Suppose $D(V)$ is not a square in $k$ and put $K = k[D(V)]$. Then

(a) There is a unique quaternion algebra $\mathcal{H}$ over $k$ such that $V$ is isometric to the quaternion space $W = \{\xi \in \mathcal{H} | \gamma \xi = \xi \}$, where $\gamma \xi^{*} \xi$ is the canonical involution, $\gamma \xi^{*} \xi$ is the extension of the conjugation on $K$, and the quadratic mapping on $W$ is the restriction of the reduced norm $N$ of $\mathcal{H}$.

(b) The proper orthogonal transformations of $W$ are all the mappings of the form $\xi \mapsto a \xi \psi^{-1} a^*$, where $a \in \mathcal{H}$ and $N(a) e_1 k$. 

Proof. The uniqueness of $\mathcal{H}$ (up to $k$-isomorphism) follows from the following observation. Let $\mathcal{H}$ be any quaternion algebra over $k$, $\mathcal{H}'$ any quadratic extension of $k$. Then $W' = \{\xi \in \mathcal{H}' | \gamma \xi^{*} \xi = \xi \}$ is a quaternion space with nonsquare discriminant $D(W') = \mathcal{H}' / (D(W')), \mathcal{H}'$ and if $a \in W'$ is any element representing 1, then $\mathcal{H}'$ is $k$-isomorphic to the subalgebra of $C\mathcal{P}$ commuting with $a_1$.

Remarks. 1. From (b) of Proposition 3 we obtain a one-to-one correspondence between quaternion spaces $V$ over $k$ with nonsquare discriminant representing 1 and ordered pairs $(\mathcal{H}, K)$, where $\mathcal{H}$ is a quaternion algebra over $k$ and $K$ is a quadratic extension of $k$. If $k$ is a number field, this may be viewed as a global analogue to the well-known local classification of quadratic spaces by their discriminants and Witt invariants.

2. We note that Proposition 2 is valid in the square discriminant case if we take conjugation to be $(a, \beta) \mapsto (\beta, a)$ on $\mathcal{H}_K \cong \mathcal{H} \otimes_k K$. Then (a), (b) of Proposition 2 correspond to (a), (b) of Proposition 1.

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To determine the proper similitudes of $W$ we proceed as follows. By $K$-linearity and the fact that $W_K = \mathbb{H}_K$, any $\sigma S^+(W)$ can be uniquely extended to an element of $S^+(\mathbb{H}_K)$, which we also denote by $\sigma$. In this way $S^+(W)$ can be identified with the subgroup of $S^+(\mathbb{H}_K)$ consisting of all such that $\sigma(W) = W$. Suppose $\sigma S^+(W)$. Then (c) of Proposition 1 implies that $\sigma(\xi) = a\beta^*\beta$, for all $\xi \in \mathbb{H}_K$, where $\alpha = a^*, \beta \in \mathbb{K}$. Then for every $\xi \in \mathbb{K}$ we must have $a\beta = (a\beta^*)^* = \beta^*a = \beta^*a^*$. Taking $\xi = 1$, we see that $\alpha^{-1} = (\alpha^{-1})^{-1} = \xi = \xi \in \mathbb{K}$. By $K$-linearity, the latter must be true for all $\xi \in \mathbb{H}_K$. Since $\mathbb{H}_K$ is central simple over $K$, we must have $\alpha^* = \beta^* = \beta = \alpha^* \in \mathbb{K}$. Then, for every $\xi \in \mathbb{K}$, we define $\xi = a\beta = a\alpha \beta = a\alpha \beta = a\beta \alpha = \alpha \beta \alpha \beta = \alpha \beta$. It follows that $\alpha \in \mathbb{K}$. We conclude that any proper similitude of $W$ must be of the form $\xi \rightarrow a\alpha \beta^*$, where $\alpha \in \mathbb{K}$, $a \in \mathbb{H}_K$. Conversely, any such mapping is a proper similitude; in fact, if $\eta_{\mathbb{H}_K}(\xi) = k$ is the norm mapping, $N(\alpha \beta) = n_{\mathbb{H}_K}(\eta(\alpha)) = \eta(\xi)$. This proves

**Proposition 3.** Let $V$ be as in Proposition 2.

(a) The proper similitudes of $W$ are all the mappings of the form $\eta \rightarrow a\alpha \beta^*$, where $\alpha \in \mathbb{K}$.

(b) The norm of the proper similitude $\eta \rightarrow a\alpha \beta^*$ is $n_{\mathbb{H}_K}(\eta(\alpha))$.

**Remarks.** A similitude $\eta \rightarrow a\alpha \beta^*$ is an orthogonal transformation if and only if $n_{\mathbb{H}_K}(\eta(\alpha)) = 1$. By Hilbert's Theorem 60, the latter is true if and only if $n(\alpha) = \beta(\beta) = \beta$ for some $\beta \in \mathbb{K}$. Putting $\beta = \beta a$, we see that $N(\beta) = n_{\mathbb{H}_K}(\beta) = \beta^* \beta = \beta^* \beta$. This agrees with (b) of Proposition 2.

2. Proposition 3 is valid when $D(V)$ is a square in $k$ if we take $(a, \beta) \rightarrow (\beta, \alpha)$ as the conjugation on $\mathbb{H}_K$, so that $n_{\mathbb{H}_K}(a, \beta) = ab, a \in \mathbb{K}$. Then (a) of Proposition 3 corresponds to (c) of Proposition 1.

3. Let $\mathbb{H}_K$ be the mapping of the form $\xi \rightarrow a\alpha \beta^*$, where $\alpha \in \mathbb{K}$.

Let us identify $V$ with the subspace of $W_{\mathbb{H}_K} = \mathbb{K}[\mathbb{H}]$ let $\psi(\alpha, \beta)$ be the similitude $\eta \rightarrow a\alpha \beta^*$. Then $\psi: \mathbb{K} \times \mathbb{K} \rightarrow S^+(V)$ is a surjective homomorphism with kernel $\ker(\psi) = \mathbb{K} \times \mathbb{K}$. The graph of $\psi$ is a natural isomorphism $S^+(V) \cong \mathbb{K} \times \mathbb{K} / \ker(\psi)$.

**Corollary.** Let $V$ be a quaternion space over $k$ representing $1$. Let $K = k[D(V)]$ if $D(V)$ is not a square in $k$, $K = k \otimes K$ if $D(V)$ is a square in $k$. Let $\mathbb{H}$ be the quaternion algebra uniquely associated to $V$ as in Proposition 2. Then there is a natural isomorphism

$$S^+(V) \cong \mathbb{K} \times \mathbb{K} / \ker(\psi).$$

§ 2. Local considerations. In this section we assume that $k$ is an algebraic number field. Let $p$ be the ring of integers of $k$, $p$ a prime of $k$ (finite or infinite). If $p$ is finite, we identify it with the prime ideal of $\mathbb{H}$ uniquely associated to it. We denote by $k_{p_1}$ the completion of $k$ with respect to $p$. If $p$ is finite we let $\alpha_p$ denote the ring of integers of $k_{p}$.

Let $V$ be a quaternion space over $k$. Represent the mapping $g$ on $V$ by $g(V)$ for each prime $p$ of $k$. We denote the extended form also by $g$. It is clear that $C_{V_p} = C_p \otimes k_p$ and $C_{V_p} = C_p \otimes k_p$. Furthermore, if $K$ is the center of $C_p$, then $K_p = K \otimes k_p$ is the center of $C_p$. If $k$ is the quaternion algebra associated to $V$ in the manner of § 1, then $k_p = C_p \otimes k_p$ is the quaternion algebra associated to $V_p$ and $C_{V_p} = C_p \otimes k_p = \mathbb{H} \otimes k_p$. The conjugation $\alpha \rightarrow \alpha$ on $k_p$ extends by $k$-linearity to a $k$-automorphism of $\mathbb{H}_p$ which coincides with the one coming from $K_p$.

First suppose $D(V)$ is a square in $k$. Then, according to (a) of Proposition 1, we may identify $V$ with $\mathbb{H}_K$. Since $k$ is a local field, there are only two possibilities for $\mathbb{H}_K$. Either $\mathbb{H}_K = \mathbb{M}(2, k)$ or $\mathbb{H}_K = \mathbb{M}(1, \mathbb{K})$ is the unique division algebra of dimension 4 over $k$.

Now suppose $D(V)$ is not a square in $k$. Then, since $K_p = k_p[V(D(V))]$ is a quaternion extension of $k_p$, it must split $\mathbb{H}_p$, $\mathbb{H}_p = \mathbb{M}(2, k_p)$. We know from § 1 that, for a fixed $K_p$, the quaternion spaces $V_p$ with $k_p[V(D(V))] = K_p$ are in one-to-one correspondence with quaternion algebras $\mathbb{A}_p$ over $k_p$, $V$ being identified with $V_p = \{ \xi \in \mathbb{A}_p | \xi \xi^* - \xi^* \xi = \xi \}$. To obtain a realization of a given $V_p$ as a $k_p$-subspace of $\mathbb{M}(2, k_p)$, it suffices to find a $k_p$-embedding of the corresponding $\mathbb{A}_p$ as a subalgebra of $\mathbb{M}(2, k_p)$, for we would then have $\mathbb{M}(2, k_p) = K_p$, $\mathbb{A}_p = \mathbb{A}_p$.

The fact that $\mathbb{A}_p$ is split by $K_p$ implies that $\mathbb{A}_p$ has a basis $\{1, \mu, v, \}$. Then the corresponding $\mathbb{A}_p$ is a $k_p$-subspace of $\mathbb{M}(2, k_p)$, and it is sufficient to find a $k_p$-embedding of the corresponding $\mathbb{A}_p$ as a subalgebra of $\mathbb{M}(2, k_p)$.

We can then embed $\mathbb{H}_p$ into $\mathbb{M}(2, k_p)$ by

$$\begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \rightarrow \begin{pmatrix} \alpha \mu \beta & \alpha \nu \beta \\ -\beta \nu \alpha & \beta \mu \alpha \end{pmatrix}.$$
We note that \( \mathfrak{N} \) will be split or nonsplit according as the norm residue symbol \( \langle a, D(V) \rangle_p = 1 \) or \(-1\), respectively. One easily checks that the conjugation induced on \( M(2, K_p) \) by this embedding of \( \mathfrak{N} \) is given by

\[
\begin{bmatrix} a & y \\ z & w \end{bmatrix} \mapsto \begin{bmatrix} \bar{w} & \bar{a}y - \bar{z} \\ \bar{c}_p y & \bar{a} \end{bmatrix}, \quad a, y, z, w \in K_p.
\]

It follows that

\[
V_p = \left\{ \begin{bmatrix} a & y \\ -c_p y & \bar{a} \end{bmatrix} \mid a, y \in K_p \right\},
\]

the quadratic mapping being the determinant

\[
\begin{bmatrix} a & y \\ -c_p y & \bar{a} \end{bmatrix} \mapsto ad + c_p y\bar{y}.
\]

In particular, we see that \( V_p \) must be isotropic of index 1 if \( D(V_p) \) is not a square in \( K_p \). We enumerate the various possibilities for \( V_p \).

1. If \( \mathfrak{N}_p \) is split, we may take \( c_p = 1 \) and then

\[
V_p = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & \bar{a} \end{bmatrix} \mid a, y \in R \right\},
\]

2. If \( \mathfrak{N}_p \) is nonsplit and \( p \) is infinite, then \( k_p = R, K_p = C \) and we may take \( c_p = -1 \). Then

\[
V_p = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & \bar{a} \end{bmatrix} \mid a, y \in R \right\},
\]

3. If \( \mathfrak{N}_p \) is nonsplit, \( p \) is finite and \( K_p \) is unramified over \( k_p \), we may take \( c_p = \pi, \) a generator of \( p \). Then

\[
V_p = \left\{ \begin{bmatrix} a & y \\ -\pi \bar{y} & \bar{a} \end{bmatrix} \mid a, y \in k_p \right\},
\]

4. If \( \mathfrak{N}_p \) is nonsplit, \( p \) is finite and \( K_p \) is ramified over \( k_p \), we may take \( c_p = \eta \), any unit of \( k_p \) such that \( \{\eta, D(V)\}_p = -1 \). Then

\[
V_p = \left\{ \begin{bmatrix} a & y \\ -\eta \bar{y} & \bar{a} \end{bmatrix} \mid a, y \in k_p \right\}.
\]

We recall that two quadratic spaces \((V, g), (V', g')\) of the same dimension are said to be similar, written \( V \sim V' \), if there is a linear map \( f: V \rightarrow V' \) and an element \( a \in k^* \) such that \( g'(f(v)) = ag(v) \) for all \( v \in V \).

One can easily show that \( V \sim V' \) implies \( C_p^* \cong C_p^* \) ([1], p. 157, paragraph 12 (a)). For quaternionic spaces over a number field \( k \) the converse is true.

**Proposition 4.** Let \( V, V' \) be quaternionic spaces over a number field \( k \). Then \( V \sim V' \) if and only if \( C_p^* \cong C_p^* \).

Proof. Multiplying each quadratic mapping by a suitable scalar, we may assume that \( V, V' \) both represent 1. If \( C_p^* \cong C_p^* \) are \( k \)-isomorphic, then they must have the same center \( K \). Let \( \mathfrak{N}, \mathfrak{N}' \) be the quaternion algebras uniquely associated to \( V, V' \), respectively. Then \( \mathfrak{N}_K \cong \mathfrak{N}'_K \) implies that \( \mathfrak{N}_p \cong \mathfrak{N}'_p \) for all primes \( p \) which split in \( K \). For such \( p \), \( V_p \) is isometric to \( \mathfrak{N}_p \) and \( V'_p \) is isometric to \( \mathfrak{N}'_p \). Hence \( V_p, V'_p \) are isometric for all \( p \) which split in \( K \). For all other \( p \) it is evident from the preceding discussion that \( V_p \sim V'_p \). We conclude that \( V \sim V' \), by the Hasse principle for similarity ([13], Theorem 1).

**§ 3. Maximal lattices and class numbers.** Let \( k \) be an algebraic number field or a non-archimedean local field of characteristic \( \neq 2 \) and let \( a \) denote its ring of integers. Let \( V \) be a quaternionic space over \( k \). A lattice \( L \) in \( V \) is a finitely generated \( \mathfrak{o} \)-submodule of \( V \) having rank four. Given a lattice \( L \) in \( V \), set

\[
\mathfrak{L}^* = \{ v \in V \mid B(v, v) \in \mathfrak{o} \}.
\]

Then \( \mathfrak{L}^* \) is a lattice in \( V \), called the dual of \( L \), and \( \mathfrak{L}^{**} = L \). The discriminant \( \Delta(L) \) of \( L \) is defined to be the fractional ideal \( [\mathfrak{L}^*: L] \) of \( \mathfrak{o} \) (cf. [2], p. 10). If \( L \) happens to be a free \( \mathfrak{o} \)-module, then

\[
\Delta(L) = \det[B(v_i, v_j)]
\]

where \( \{v_j\} \) is an \( \mathfrak{o} \)-basis of \( L, i, j = 1, 2, 3, 4 \).

Suppose \( L \) is a lattice in \( V \). The fractional ideal of \( \mathfrak{o} \) spanned by the set of all \( g(v), v \in L \), is called the norm of \( L \) and is denoted by \( n(L) \). Set

\[
\mathfrak{L}' = \{ v \in V \mid B(v, v) = n(L) \}.
\]

The reduced discriminant \( \Delta'(L) \) is defined to be \([\mathfrak{L}': L] \). It is clear that \( \Delta'(L) \) is an integral ideal of \( \mathfrak{o} \). If \( L \) is a free \( \mathfrak{o} \)-module, as in the local case for example, then

\[
\Delta'(L) = \det[n(L)^{-1}B(v_i, v_j)]
\]

where \( \{v_j\} \) is an \( \mathfrak{o} \)-basis of \( L \). It is clear from this remark that our definition of reduced discriminant is the same as the one found in [4].

A lattice \( L \) in \( V \) is integral if \( n(L) \) is an integral ideal; it is maximally integral if, in addition, it is not properly contained in another integral lattice. A lattice \( L \) is maximal if it is not properly contained in another lattice having the same norm; \( L \) is \( m \)-maximal if \( L \) is maximal and \( n(L) = m \) is a fractional ideal of \( \mathfrak{o} \). Since \( q \) represents 1 locally, it is clear that a maximally integral lattice is nothing more than an \( m \)-maximal lattice.
We say that two lattices \( L, M \) in \( V \) are similar, and write \( L \sim M \), if \( aL = M \) for some \( a \in S^+(V) \); if \( a \) can be taken from \( O^+(V) \), then we say that \( L, M \) are equivalent, and write \( L \cong M \). The group of all proper similarities, respectively, proper orthogonal transformations, which map a lattice \( L \) onto itself will be denoted by \( S^+(L) \), respectively, \( O^+(L) \). We call the elements of \( S^+(L) \) the units of \( L \) and the elements of \( O^+(L) \) the orthogonal units of \( L \).

From now on we suppose that \( k \) is a number field. If \( L \) is a lattice in \( V \), then \( L_p = L \otimes \mathcal{O}_p \) is a lattice in \( V_p \) for every finite prime \( p \). It is clear that the following relations hold:

\[
(I_p)^p = (I_p)^\#_p, \quad \Delta'(L_p) = \alpha_p \Delta(L), \quad n(L_p) = \alpha_p n(L),
\]

\[
(L'_p) = (L'_p), \quad \alpha'(L_p) = \alpha_p \alpha'(L).
\]

Furthermore, it is clear that \( L \) is integral \( \iff \) \( L_p \) is integral for all finite \( p \), \( L \) is maximal \( \iff \) \( L_p \) is maximal for all finite \( p \), and \( L \) is maximally integral \( \iff \) \( L_p \) is maximally integral for all finite \( p \).

Two lattices \( L, M \) in \( V \) are in the same class if \( L \cong M \), in the same similarity class if \( L \sim M \); they are in the same genus if \( L_p \cong M_p \) for all finite \( p \), in the same ideal class if \( L_p \cong M_p \) for all finite \( p \); if \( L \) is in the genus of \( M \) for some \( \sigma \in S^+(V) \), then \( L, M \) are in the same similarity genus. The basic finiteness theorems for quadratic spaces state that each genus decomposes into a finite number of classes and each ideal class decomposes into a finite number of similarity classes ([14], p. 79).

Let \( \mathcal{G} \) be a genus of lattices in \( V \) and \( \mathcal{G}_p \), the similarity genus containing \( \mathcal{G} \). Then the number of similarity classes represented in \( \mathcal{G}_p \) is the same as the number of similarity classes in \( \mathcal{G}_p \). However, the number of classes in \( \mathcal{G}_p \) may be greater than the number of similarity classes represented in \( \mathcal{G} \). The precise relationship between these two numbers is given in [14], § 2. In particular, if \( k = \mathbb{Q} \), the field of rational numbers, then these two numbers coincide if \( q \) has signature \( \neq 0 \).

The collection of maximally integral lattices in \( V \) forms a genus which we denote by \( \mathcal{M} \) (cf. [12], p. 240). We denote the common discriminant of all the lattices in \( \mathcal{M} \) by \( \Delta \). Then \( \mathcal{M} \) can also be described as the set of all integral lattices in \( V \) having discriminant \( \Delta \). Let \( \mathfrak{D} \) denote the ideal-complex containing \( \mathcal{M} \). We can also describe \( \mathfrak{D} \) as the set of all maximal lattices having reduced discriminant \( \Delta \) (cf. [4], p. 87). The class numbers of interest to us will be \( H_0 \), the number of classes in \( \mathcal{M} \), and \( H_1 \), the number of similarity classes in \( \mathfrak{D} \). The number of similarity genera in \( \mathfrak{D} \) can be expressed in terms of the number of ambiguous ideal classes in \( K \) (cf. [14], § 2, Satz 6). In particular, if \( k = \mathbb{Q} \) and \( q \) has signature \( \neq 0 \), then \( \mathcal{D} \) decomposes into \( q^2 \) similarity genera, where \( q^2 \) is the number of strict genera of \( K \). Unfortunately, if \( q \) is definite, the various similarity genera need not have the same number of similitude classes, so that the most we can say about the relation of \( H_0 \) to \( H \) in the case \( k = \mathbb{Q} \), \( g \) definite, is that \( H_0 \leq H \), with equality \( \iff q^2 - 1 = K \) has prime discriminant.

We proceed now to classify the maximal lattices in \( V \) locally. We may assume that \( V \) represents 1. Our first step is to classify the maximally integral lattices in \( V \). Since the maximally integral lattices are all locally equivalent, it is enough to exhibit one maximally integral lattice \( M_p \) for each finite prime \( p \) of \( K \). As usual, we must consider various cases.

**Notation.** Let \( \mathcal{D} \) denote the ring of integers in \( K \); for a finite prime \( p \) of \( k \), let \( \mathcal{D}_p = \mathcal{D} \otimes \mathcal{O}_p \) denote the ring of integers in \( K_p \).

1. \( D(V) \) is a square in \( k_p \).
   - (a) If \( V_p \) is isotropic, then \( V_p = \mathcal{D}_p = \mathbb{M}(2, k_p) \) and we may take \( M_p = \mathbb{M}(2, \mathcal{O}_p) \).
   - (b) If \( V_p \) is anisotropic, then \( V_p = \mathcal{D}_p = \mathbb{D}_p \) the unique quaternion division algebra over \( k_p \) and we take \( M_p = \mathfrak{e}_p \), the maximal order of \( \mathfrak{D}_p \).

2. \( D(V) \) is not a square in \( k_p \), \( K_p = k_p(V/\mathfrak{D}(V)) \).
   - (a) If \( \mathcal{D}_p \) is split, then
     \[
     V_p = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & \bar{d} \end{bmatrix} \mid a, \bar{d} \in k_p, y \in K_p \right\}.
     \]
   - We claim
     \[
     \mathbb{M}_p = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & \bar{d} \end{bmatrix} \mid a, \bar{d} \in \mathcal{O}_p, y \in \mathcal{D}_p \right\}
     \]
   - is maximally integral in \( V_p \). Suppose not. Then there is an integral lattice \( L_p \supset \mathbb{M}_p = L_p \neq \mathbb{M}_p \). Take \( v \in L_p \setminus \mathbb{M}_p \). Then
     \[
     v = \begin{bmatrix} a & y \\ -\bar{y} & \bar{d} \end{bmatrix}, \quad a, \bar{d} \in k_p, y \in K_p.
     \]
   - Suppose either \( a \) or \( d \) is not in \( \mathcal{O}_p \). By symmetry, we may assume \( a \notin \mathcal{O}_p \).
     - Put
       \[
       \omega = \begin{bmatrix} a & y \\ -\bar{y} & \bar{d} + 1 \end{bmatrix}.
       \]
   - Then \( \omega \in L_p \) but \( N(\omega) = N(v) \neq a \notin \mathcal{O}_p \), a contradiction. Hence \( a, d \in \mathcal{O}_p \), which implies \( y \notin \mathcal{D}_p \) since \( \mathcal{D}_p = \{y \in K_p \mid y \in \mathcal{O}_p \} \).
   - (b) If \( \mathcal{D}_p \) is nonsplit, then
     \[
     V_p = \left\{ \begin{bmatrix} a & y \\ -\bar{y} & \bar{d} \end{bmatrix} \mid a, \bar{d} \in k_p, y \in K_p \right\}
     \]
if \( K_p \) is unramified over \( k_p \), where \( \pi \) is a generator of \( p \), and

\[
V_p = \begin{pmatrix} a & y \\ \bar{u}_p & \bar{d} \end{pmatrix} \quad \text{if} \quad d \in k_p, \quad y \in K_p
\]

if \( K_p \) is ramified over \( k_p \), where \( u_p \) is a unit of \( k_p \) such that \( \{u_p, D(V)\}_p = -1 \).

Accordingly, reasoning exactly as in 2(a), we may take

\[
(M_p)_p = \begin{pmatrix} a & y \\ -\pi y & \bar{d} \end{pmatrix} \quad \text{if} \quad d \in \mathcal{O}_p, \quad y \in \mathcal{O}_p
\]

\[
(M_p)_p = \begin{pmatrix} a & y \\ -u_p y & \bar{d} \end{pmatrix} \quad \text{if} \quad d \in \mathcal{O}_p, \quad y \in \mathcal{O}_p
\]

Suppose that \( \sigma \) is an order of a quaternion algebra over a local or global number field. The level (Stufe) of \( \sigma \) is defined to be \( \sigma(n\sigma^{-1}) \). By an \textit{Eichler order} we mean an order of a quaternion algebra over a local or global number field having square-free level (cf. [6], p. 136). Suppose \( D(V) \) is not a square in \( k \), so that \( K \) is a quadratic extension of \( k \). Let \( p \) be a finite prime of \( k \) which splits in \( K \). We say that an \( \mathcal{O}_p \)-order \( \mathcal{O}(p) \) of \( \mathbb{H}_K = \mathbb{H}_p \oplus \mathbb{H}_q \) is an \textit{Eichler order} if it is \( \mathcal{O}_p \)-isomorphic to an order of the form \( \mathcal{O}_1 \oplus \mathcal{O}_2 \), where \( \mathcal{O}_1, \mathcal{O}_2 \) are Eichler orders of \( \mathbb{H}_p \).

Let \( A \) be an \( \mathcal{O}_p \)-lattice of \( \mathbb{H}_K \). For any finite prime \( \mathfrak{p} \) of \( k \) we put \( A_p = \mathcal{O}_p \cap A \). Then if \( A \) is an \( \mathcal{O}_p \)-lattice of \( \mathbb{H}_K \), we say that a lattice \( A \) of \( \mathbb{H}_K \) is \textit{symmetric} if \( \mathcal{A} = A \); similarly, a lattice \( A(p) \) of \( \mathbb{H}_p \) is symmetric if \( (A(p))^* = A(p) \). It is clear that a lattice \( A \) of \( \mathbb{H}_k \) is symmetric \( \Leftrightarrow A_p \) is symmetric for every finite prime \( p \) of \( k \).

\textsc{Lemma.} For any finite prime \( p \) of \( k \) there exists an Eichler order \( \mathcal{O}(p) \) of \( \mathbb{H}_p \) such that \( \mathcal{O}(p) \) is symmetric and \( \mathcal{O}(p) \cap V_p \) is a maximally integral lattice in \( V_p \). The order \( \mathcal{O}(p) \) is maximal except if \( p \) ramifies in \( \mathbb{H} \) and remains prime in \( K \).

\textbf{Proof.} \( D(V) \) is a square in \( K_p \). Then \( \mathbb{H}_p = \mathbb{H}_p \oplus \mathbb{H}_p, \ V_p = \{ (\xi, \bar{\xi}) \mid \xi \in \mathbb{H}_p \} \). We take \( \mathcal{O}(p) = \mathbb{H}(2, a_p) \oplus \mathbb{H}(2, a_p) \) for case (a) and \( \mathcal{O}(p) = \mathcal{O}_p \oplus \mathcal{O}_p \) for case (b). For cases 2(a) and 2(b) (ii) we take \( \mathcal{O}(p) = \mathbb{H}(2, \mathcal{O}_p) \).

For case 2(b)(i) we have

\[
\mathcal{O}(p) = \begin{pmatrix} a & y \\ \bar{u}_p y & \bar{d} \end{pmatrix} \quad \text{if} \quad d \in \mathcal{O}_p, \quad y \in \mathcal{O}_p
\]

where \( u_p \) is a unit of \( \mathcal{O}_p \). Thus an \( \mathcal{O}_p \)-basis of \( M_p \) is given by

\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & \bar{a}_1 \\ 0 & 0 & 0 & \bar{a}_2 \end{pmatrix}
\]

The dual of this basis with respect to the bilinear form \( B \) is given by

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & \bar{a}_2 \\ 0 & 0 & 0 & \bar{a}_1 \end{pmatrix}
\]

It follows directly that \( A(M_p) = A_K \mathcal{O}_p = \mathcal{O}_p \). For case 2(b)(i) we have

\[
M_p = \begin{pmatrix} a & y \\ -\pi y & \bar{d} \end{pmatrix} \quad \text{if} \quad d \in \mathcal{O}_p, \quad y \in \mathcal{O}_p
\]

where \( \pi \) is a generator of \( p \).
Then
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & a_0 \\
0 & 0 \\
-\pi \bar{a}_1 & 0 \\
-\pi a_0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & a_0 \\
0 & 0 \\
-\pi \bar{a}_2 & 0 \\
-\pi a_0 & 0
\end{bmatrix}
\]
is an \( p \)-basis for \( M_p \) and
\[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & \pi^{-1} \bar{a}_1 \\
0 & 0 \\
-\bar{a}_1 & 0 \\
-\bar{a}_0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & \pi^{-1} \bar{a}_2 \\
0 & 0 \\
-\bar{a}_2 & 0 \\
-\bar{a}_0 & 0
\end{bmatrix}
\]
is its dual basis. Hence \( d(M_p) = p^4 A_K = p^5 \). We have proved the following:

**Proposition 6.** Let \( V \) be a quaternion space over \( k \) representing \( 1 \). Let \( \Delta \) be the discriminant of the genus \( \mathcal{M} \) of maximal integral lattices in \( V \).

(a) If \( D(V) \) is a square in \( k \), then \( \Delta = (a_1 \ldots a_4)^2 \), where \( a, \ldots, a_4 \) are the anisotropic finite primes of \( V \).

(b) If \( D(V) \) is not a square in \( k \), \( K = k(\sqrt{D(V)}) \) is the quaternion algebra over \( k \) associated to \( V \), then \( \Delta = \Delta_K \delta \), where \( \Delta_K \) is the discriminant of \( K \) and \( \delta \) is the product of all finite primes of \( K \) which ramify in \( \mathcal{M} \) but not in \( K \).

We now complete the local classification of maximal lattices in \( V \) by exhibiting a representative \( M_p \) for each local similarity class of maximal lattices in \( V_p \), where \( p \) is a finite prime.

If \( D(V) \) is a square in \( k_p \), then every element of \( k_p \) is of norm a proper square. Hence every maximal lattice in \( V_p \) is similar to a maximally integral lattice. Thus we may take \( M_p = \) a maximally integral lattice in \( V_p \). \( D(V) \) is not a square in \( k_p \), then we must distinguish between the cases \( k_p \), ramified over \( k_p \), and \( k_p \), unramified over \( k_p \). If \( k_p \), ramified over \( k_p \), then (b) of Proposition 3 shows that some prime element \( \pi \) of \( k_p \) is the norm of a similitude. It follows that every maximal lattice in \( V_p \) is similar to a maximally integral lattice \( M_p \). If \( k_p \), unramified over \( k_p \), then the elements of \( k_p^* \) which are the norms of similitudes are those having even \( p \)-order. Hence every maximal lattice is similar either to a maximally integral lattice or to a maximal lattice of norm \( p \).

If \( \mathcal{M}_p \) is split, then it is easy to verify that
\begin{equation}
M_p = \begin{bmatrix}
a & \pi \bar{y} \\
-\pi \bar{y} & \bar{a}
\end{bmatrix}, \quad \begin{bmatrix}
a, \bar{a} & y, \bar{a} \\
\bar{a} & y \end{bmatrix}, \quad \begin{bmatrix}
a & \pi \bar{y} \\
-\pi \bar{y} & \bar{a}
\end{bmatrix}
\end{equation}
is \( p \)-maximal, where \( \pi \) is a prime element of \( k_p \). It is clear that \( \Delta(M_p) = p^4 \).

If \( \mathcal{M}_p \) is nonsplit, then we take
\begin{equation}
M_p = \begin{bmatrix}
\pi a & y \\
-\pi \bar{y} & \bar{a}
\end{bmatrix}, \quad \begin{bmatrix}
a, \bar{a} & y, \bar{a} \\
\bar{a} & y \end{bmatrix}, \quad \begin{bmatrix}
\pi a & y \\
-\pi \bar{y} & \bar{a}
\end{bmatrix}
\end{equation}
From \( \Delta(M_p) = p^4 \) it follows that \( M_p \) must be \( p \)-maximal ([4], p. 50).

Summarizing, we have shown:

**Proposition 7.** Let \( V \) be a quaternion space over \( k \) representing \( 1 \). Let \( a_1, \ldots, a_4 \) be the anisotropic finite primes of \( V \).

(a) If \( D(V) \) is a square in \( k \), then \( V \) has a unique ideal complex of maximal lattices \( \mathcal{M} \), the set of all maximal lattices of reduced discriminant \( (a_1 \ldots a_4)^2 \).

(b) If \( D(V) \) is not a square in \( k \), then for every finite set \( \mathcal{P}_1, \ldots, \mathcal{P}_s \) of primes of \( k \) which remain prime in \( K = k(\sqrt{D(V)}) \) and we have a unique ideal complex of maximal lattices \( V \) having reduced discriminant
\[
\Delta_{K}(a_1 \ldots a_4)^2(\mathcal{P}_1 \ldots \mathcal{P}_s).
\]

Every maximal lattice of \( V \) lies in one of these ideal complexes.

We see that \( \Delta_K(a_1 \ldots a_4)^2 \) is the “smallest” reduced discriminant of maximal lattices in \( V \), hence the “smallest” reduced discriminant of all lattices in \( V \). Accordingly, it is natural to call \( \Delta_K(a_1 \ldots a_4)^2 \) the fundamental discriminant of \( V \), denoted by \( \mathcal{A} \). Then Proposition 4 can be rephrased as: \( V_1 \sim V \) if and only if \( D(V) = D(V_1) \), \( A_V = A_{V_1} \), and \( V_1, V \) have the same Witt index for every infinite prime \( p \) of \( k \).

If \( V, V \) are quaternion spaces which are similar by a mapping \( f: V \rightarrow V \), then \( f \) gives a one-to-one correspondence between similarity classes and ideal complexes of \( V \) and those of \( V \). In particular, since \( f \) preserves maximal lattices and reduced discriminants, \( f \) must take an ideal complex of maximal lattices in \( V \) to the ideal complex of maximal lattices in \( V \) of the same reduced discriminant. Suppose \( \mathcal{A} \) is an ideal complex of maximal lattices of \( V \). Let \( \mathcal{A}' \) be the reduced discriminant of \( \mathcal{A} \). We can choose \( L, \mathcal{S} \) such that \( L_p \) is either maximally integral or \( p \)-maximal for every finite prime \( p \). Let \( S \) denote the set of finite primes \( p \) for which \( L_p \) is not maximally integral. If \( S \) is an even number of elements, let \( \mathcal{A}' \) be the quaternion algebra over \( k \) obtained by taking \( \mathcal{A}_p' \) different from \( \mathcal{A}_p \) for all \( p \in S \); if \( S \) has an odd number of elements and at least one prime \( k \) (finite or infinite) ramifies in \( K \), then let \( \mathcal{A}_p' \) be the one obtained by changing \( \mathcal{A}_p \) at all \( p \in S \cup \{ r \} \). Then the quaternion space \( V' \) associated to \( \mathcal{A}' \) is similar to \( V \) and its maximally integral lattices have discriminant \( \mathcal{A}' \). Thus if at least one prime of \( k \) ramifies in \( K \), then for purposes of studying similarity classes of maximal lattices, there is no loss of generality in restricting ourselves to the ideal complex \( \mathcal{A} \) of \( V \); provided we let \( V \) vary. However, if every prime of \( k \) is unramified in \( K \) (i.e. \( K \) is a subfield of the Hilbert class field of \( k \)), then there exist ideal complexes of maximal lattices which are never mapped to \( \mathcal{A} \) by any similarity mapping. Such an ideal complex will be called intransigent. We note that intransigent ideal complexes exist only if \( k \) has even class number. To study similarity classes of intransigent ideal complexes we need only consider those having
reduced discriminant \( \Delta p \), where \( p \) is a finite prime of \( k \) which remains prime in \( K \) and ramifies in \( \frak{A} \). Such an ideal complex contains a lattice \( \frak{M} \) such that \( \frak{M} \) is maximally integral for \( p \neq p_p \), but

\[
\frak{M}_p = \left[ \begin{array}{cc} \pi \phi_p & y \\ -\pi \phi_p & \delta \end{array} \right] \quad \text{for} \quad a, \delta \in \frak{O}_p, \quad y \in \frak{O}_p \setminus \frak{O}_p^0,
\]

where \( (x) = p_p \). Then \( \frak{M}_p = \frak{M} \cap \frak{V}_p \), where

\[
\Delta_p = \left[ \begin{array}{cc} \pi \frak{O}_p \setminus \frak{O}_p^0 & \frak{O}_p \setminus \frak{O}_p^0 \\ \pi \frak{O}_p \setminus \frak{O}_p^0 & \frak{O}_p \setminus \frak{O}_p^0 \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ \pi & 0 \end{array} \right],
\]

a left ideal of \( \frak{M}(2, \frak{O}_p) \). We note that \( \Delta_p \) is symmetric. Then Proposition 5 can be reformulated for intransitive ideal complexes to read: There exists a symmetric lattice \( A \) with left order of level \( \delta \frak{O}_p \), such that \( \frak{M} = A \cap \frak{V} \) is a \( p \)-maximal lattice of \( \frak{V} \). Because of this formal resemblance, and in order to simplify our notation, we extend the use of \( \Omega, H \) by permitting them to denote, respectively, an intransitive ideal complex, and the number of nilpotent classes therein. Consistent with Proposition 5, we let \( \Omega \) denote the left order of \( A \). Note that, although \( A \) is symmetric, \( \Omega \) is not.

\section{The relation of \( H \) to a generalized type number.}

In this section we determine the local groups of units \( S^+(\frak{M}_p) \) for a maximal lattice \( \frak{M} \) in \( \frak{V} \) and use this information to relate \( H \) to a certain generalized type number associated to the quaternion algebra \( \frak{K} \).

According to our discussion in \( \S \, 3 \), we may assume that either (1) \( \frak{M} \) is maximally integral, \( \frak{M} = \frak{V} \cap \frak{V} \), where \( \Omega \) is a symmetric Eichler order of \( \frak{K} \) of level \( \delta \), or (2) \( \frak{M} \) is \( p \)-maximal, \( \frak{M} = \frak{A} \cap \frak{V} \), where \( \frak{A} \) is a symmetric lattice whose left order \( \Omega \) is an Eichler order of level \( \delta \frak{O}_p \). Furthermore, by throwing the primes which ramify in \( K \) or \( \frak{K} \) into the exceptional set mentioned in the proof of Proposition 5, we may assume that \( \frak{M}_p, \frak{O}_p, \frak{A}_p \) for these \( p \) are in the standard forms given in (17), (18), and in the proof of the lemma preceding Proposition 5.

Let \( p \) be a prime of \( k \). By the corollary to Proposition 3, we have

\[
S^+(\frak{V}_p) = k_p^+ \times \frak{K}_p^+ / \Gamma(k_p^+ \times \frak{K}_p^+).
\]

If \( p \) is a prime, let \( U_p, U(\frak{O}_p), U(\frak{O}_p) \) denote the unit groups of \( \frak{O}_p, \frak{O}_p, \frak{O}_p \), respectively. If \( p \) is infinite, put \( \frak{O}_p = k_p, \frak{O}_p = K_p, \frak{O}_p = \frak{K}_p \), and let \( U, U(\frak{O}_p), U(\frak{O}_p) \) denote \( k_p^+, K_p, \frak{K}_p \), respectively. Then

\[
U \cap \Gamma(k_p^+ \times \frak{K}_p^+) = U \cap \Gamma(\frak{O}_p) = \Gamma(U_p \times U(\frak{O}_p)) \setminus \Gamma(\frak{O}_p^0, \frak{O}_p^0)(a) = e^{-1}.
\]

Then

\[
U_p \times U(\frak{O}_p) \cap \Gamma(k_p^+ \times \frak{K}_p^+) = U_p \times U(\frak{O}_p).
\]

Hence we have a natural embedding

\[
U_p \times U(\frak{O}_p) / \Gamma(U_p \times U(\frak{O}_p)) \subset S^+(\frak{V}_p).
\]

**Lemma.** Let \( \Omega, \frak{M} \) be chosen as in Proposition 5 or its analogue for intransitive ideal complexes. Then

\[
U_p \times U(\frak{O}_p) / \Gamma(U_p \times U(\frak{O}_p)) \subset S^+(\frak{M}_p).
\]

**Proof.** Let \( u e U_p, v e U(\frak{O}_p) \). Suppose \( \frak{M}_p \) is maximally integral. Then

\[
\frak{M}_p = \frak{O}_p \cap \frak{V}_p = \frak{O}_p \times \frak{O}_p \cap \frak{V}_p = \frak{O}_p \times \frak{O}_p \times \frak{O}_p \times \frak{V}_p = \frak{O}_p \times \frak{O}_p \times \frak{O}_p \times \frak{V}_p = \frak{O}_p \times \frak{O}_p \times \frak{O}_p \times \frak{V}_p.
\]

Now suppose \( \frak{M}_p \) is \( p \)-maximal. Then \( \frak{M}_p = \frak{O}_p \cap \frak{V}_p \), where

\[
\frak{V}_p = \left[ \begin{array}{cc} 0 & 1 \\ \pi & 0 \end{array} \right].
\]

We note that \( \frak{K}_p = \frak{K}_p \cap \frak{V}_p \). Then \( e \in U(\frak{K}_p) \) and

\[
M_p = \frak{O}_p \times \frak{K}_p \cap \frak{V}_p = \frak{O}_p \times \frak{K}_p \cap \frak{V}_p = \frak{O}_p \times \frak{K}_p \cap \frak{V}_p = \frak{O}_p \times \frak{K}_p \cap \frak{V}_p.
\]

Let \( \frak{O}_p \) be an Eichler order in a quaternion algebra \( \frak{K} \) over a local or global number field \( k \). The normator \( N(\frak{O}_p) \) is defined to be the group of all \( \alpha \in \frak{K} \) such that \( \alpha \frak{O}_p^{-1} = \frak{O}_p \). If \( k \) is a global number field and \( p \) is a finite prime of \( k \), then

\[
[N(\frak{O}_p) : k_p^+ U(\frak{O}_p)] = \left\{ \begin{array}{cc} 1 & \text{if } \frak{O}_p \text{ is of level } \frak{O}_p, \\ 2 & \text{if } \frak{O}_p \text{ is of level } p. \end{array} \right.
\]

In the case where \( \frak{O}_p \) is of level \( p \), the non-trivial coset mod \( k_p^+ U(\frak{O}_p) \) can be represented by a generator \( H \) of the unique two-sided ideal prime of \( \frak{O}_p \) (cf. [5], § 2).

We proceed to determine coset representatives for

\[
(U_p \times U(\frak{O}_p) / \Gamma(U_p \times U(\frak{O}_p))) \setminus S^+(\frak{M}_p).
\]

**Notation.** Suppose \( a, b \in k_p^+, a, b \in \frak{K}_p \). Then \( (a, b) \mapsto (\overline{b}, \overline{b}) \) will mean

\[
(ab^{-1}, ab^{-1}) \in \Gamma(k_p^+ \times \frak{K}_p^+).
\]

1. \( D(\frak{V}) \) is a square in \( k_p \). Then

\[
V_p = \{ (\xi, \xi^*) \mid \xi \in \frak{K}_p \}, \quad M_p = \{ (\xi, \xi^*) \mid \xi \in \frak{O}_p \},
\]

where \( \frak{O}_p \) is a maximal order of \( \frak{K} \). From \( \frak{K}_p = \frak{O}_p \), and \( \frak{O}_p \cap \frak{V}_p = \frak{M}_p \), we deduce \( \frak{O}_p = \frak{O}_p \times \frak{O}_p \). Suppose \( (a, b) \in \Gamma(k_p^+ \times \frak{K}_p^+) \), and \( \frak{O}_p \times \frak{K}_p \subset \frak{V}_p \) (cf. Corollary, Proposition 3). Then \( a = (b, a), b \in \frak{K}_p, \gamma \in \frak{K}_p^* \), and

\[
M_p = ca \frak{M}_p \frak{A}^* = (ca \frak{A}_p, \gamma) \cap \frak{V}_p.
\]
Thus \( \psi \gamma^* = \psi \), which implies \( \beta, \gamma^* \in \mathbb{H}(\mathbb{P}) \). According to (19), if \( \mathfrak{A} \) is split we may assume \( \beta, \gamma^* \in \mathbb{P}^* \), i.e., \( \beta, \gamma \in \mathbb{P}^* \). Then \( \psi \gamma^* \mathfrak{A} \gamma = \psi \gamma \mathfrak{A} \psi \), which implies \( \psi = (\beta \gamma)^{-1} \psi \), \( \psi, \gamma \in \mathbb{P} \). Hence \( (c, a) = (u, 0) \in \mathbb{P} \times U \). We conclude that

\[
S^+(M_p) = U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p))
\]

if \( \mathfrak{A} \) is split. If \( \mathfrak{A} \) is nonsplit we have the additional possibility \( a = (\Pi, \Pi) \), \( c = N(\Pi)^{-1} \), which gives

\[
\left[ S^+(M_p); U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p)) \right] = 2.
\]

2. \( D(V) \) is a square in \( k_p \). Suppose that \( \sigma \mathfrak{A} \mathfrak{A}^* = M_p \), \( (c, a) \in k_p^* \times X_{\mathbb{P}} \). Multiplying \( (c, a) \) by a suitable element of \( \Gamma(\mathbb{P}^* \times \mathbb{P}^*) \) we may take \( a \in \mathbb{P}_p \). Let \( \pi \) be a prime element of \( \mathbb{P}_p \). If \( \mathbb{P}_p = M(2, \mathbb{P}_p) \), then, multiplying \( a \) on the left by a suitable unit of \( \mathbb{P}_p \) (cf. p. 133), we may assume that

\[
(20) \quad a = \begin{bmatrix} x^i & x^j \\ 0 & x^j \end{bmatrix}, \quad i, j \geq 0
\]

where \( x \in \mathbb{P}_p \) is reduced mod.\( \pi^i \). If, on the other hand,

\[
\mathbb{P}_p = \begin{bmatrix} \mathbb{P}_p & \mathbb{P}_p \\ \pi \mathbb{P}_p & \pi \mathbb{P}_p \end{bmatrix},
\]

then we have the additional possibility

\[
(21) \quad a = \begin{bmatrix} x^i & x^{i+1} \\ 0 & x^{i+1} \end{bmatrix}, \quad i, j \geq 0
\]

where \( x \) is reduced mod.\( \pi^{i+1} \). For the sake of simplicity we write \( n = n_{k_p} \) in the ensuing discussion.

(a) Suppose \( \mathfrak{A} \) is split and \( k_p \) is totally real over \( k \). Then

\[
M_p = \begin{bmatrix} a + y & a - y \\ -y & a \end{bmatrix}, \quad a, \bar{a} \in \mathbb{P}_p, \quad y \in \mathbb{P}_p
\]

and \( \mathbb{P}_p \) is a maximal order of \( \mathfrak{A} \mathfrak{A}^* = M(2, \mathbb{P}_p) \). We claim that \( \mathbb{P}_p = M(2, \mathbb{P}_p) \). Put \( A_p = \mathbb{P}_p \mathfrak{A}_p \), an \( \mathbb{P}_p \)-lattice in \( \mathfrak{A} \mathfrak{A}^* \). Then \( \mathbb{P}_p = A_p \), \( M(2, \mathbb{P}_p) = A_p \), \( A_p = \mathbb{P}_p \mathfrak{A}_p \). Furthermore,

\[
\Delta(A_p) = \mathbb{P}_p \\
\Delta(M_p) = \mathbb{P}_p = \Delta(\mathbb{P}_p) = \Delta(M(2, \mathbb{P}_p)).
\]

Hence \( \mathbb{P}_p = A_p = M(2, \mathbb{P}_p) \). Taking \( \pi \in k_p \), we have

\[
(32) \quad e^* = \begin{bmatrix} x^i & x^j \\ 0 & x^j \end{bmatrix} \begin{bmatrix} a & y \\ -y & a \end{bmatrix} \begin{bmatrix} x^i & x^j \\ 0 & x^j \end{bmatrix} = e^* \begin{bmatrix} ax^i - dxn(a) - \tau(x^i y^j + \bar{y}x^j) & dx^i x + \pi^{i+1} y \\
-x(\pi x^i + \pi^{i+1} \bar{y}) & dx^i x + \pi^{i+1} \bar{y} \end{bmatrix}.
\]

It follows immediately that \( \sigma \mathfrak{A}^* \mathfrak{A} = u \in U_p \). Taking \( \gamma = 0, a = 0, d = 1 \), we see that \( \sigma \mathfrak{A}^* \mathfrak{A} = u \). Since \( x \) is reduced mod.\( \pi \), this implies \( x = 0 \). Then \( \sigma \mathfrak{A} = U_p \), which shows that \( i = j \) and \( (c, a) = (x, a) \in \mathfrak{A}^* \mathfrak{A} = U_p \). We conclude that

\[
S^+(M_p) = U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p)).
\]

(b) Suppose \( \mathbb{P}_p \) is ramified over \( k_p \). We may assume that \( M_p = \mathbb{P}_p \), \( \mathbb{P}_p \) are in the standard form

\[
M_p = \begin{bmatrix} a & y \\ -y & a \end{bmatrix}, \quad a, \bar{a} \in \mathbb{P}_p, \quad y \in \mathbb{P}_p,
\]

and \( \mathbb{P}_p = M(2, \mathbb{P}_p) \). Following the same argument as in 2 (a) (but using \( n(\pi)^i \), \( n(y)^i \) instead of \( \pi^{i+1} \), \( \pi^{i+1} \)), we deduce once again that

\[
S^+(M_p) = U_p \times U(\mathbb{P}_p) / \Gamma(U_p \times U(\mathbb{P}_p)).
\]

(c) Suppose \( \mathfrak{A} \) is nonsplit and \( k_p \) is totally real over \( k \). Let \( \pi \) be a prime element of \( k_p \).

(i) \( M_p \) is maximally integral. We take \( M_p = \mathbb{P}_p \), \( \mathbb{P}_p \) in the standard form

\[
M_p = \begin{bmatrix} a & y \\ -y & a \end{bmatrix}, \quad a, \bar{a} \in \mathbb{P}_p, \quad y \in \mathbb{P}_p,
\]

and \( \mathbb{P}_p = M(2, \mathbb{P}_p) \). Taking \( \pi \in k_p \), we have

\[
(33) \quad \begin{bmatrix} x^i & x^j \\ 0 & x^j \end{bmatrix} \begin{bmatrix} a & y \\ -y & a \end{bmatrix} \begin{bmatrix} x^i & x^j \\ 0 & x^j \end{bmatrix} = \begin{bmatrix} ax^i - dxn(a) - \tau(x^i y^j + \bar{y}x^j) & dx^i x + \pi^{i+1} y \\
-x(\pi x^i + \pi^{i+1} \bar{y}) & dx^i x + \pi^{i+1} \bar{y} \end{bmatrix}.
\]

As before, we have \( \sigma \mathfrak{A}^* \mathfrak{A} = u \in U_p \). This implies \( \sigma \mathfrak{A}^* \mathfrak{A} = u \mathfrak{A} \mathfrak{A}^* \), i.e., \( \sigma \mathfrak{A}^* \mathfrak{A} = u \mathfrak{A} \mathfrak{A}^* \mathfrak{A} \). However, \( \mathfrak{A} \mathfrak{A}^* \mathfrak{A} \) has even p-order. Hence we must have \( \sigma \mathfrak{A}^* \mathfrak{A} = u \mathfrak{A} \mathfrak{A}^* \mathfrak{A} \), which shows that \( x = 0, i = j \), and \( (c, a) = (u, 0, 1) \).
Then $\varpi t_{i+1} = u \varpi U_p$ and $\pi^{2i+1} \upsilon (\omega)$, which implies $\pi^{2(i+1)} \upsilon (\omega)$. Hence $\varpi = 0, i = j$, and $(c, a) = (u \varpi^{-1}, \Gamma)$, where 

$$
\Pi = \begin{bmatrix}
0 & 1 \\
\varpi & 0
\end{bmatrix}.
$$

We conclude that

$$
[S^+(M_p); U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p))] = 2.
$$

(ii) $M_p$ is $p$-maximal. Here we take $M_p, \Omega_p$ in the standard forms

$$
M_p = \begin{bmatrix}
\pi a & \gamma \\
-\gamma \delta & \delta
\end{bmatrix}, \quad \Omega_p = \mathbb{M}(2, \mathbb{Q}_p).
$$

We need only consider

$$
a = \begin{bmatrix}
\pi^x & x \\
0 & \pi^y
\end{bmatrix}, \quad x \text{ reduced mod } \varpi.
$$

This case can be treated by assuming that $\pi \mid a$ in (22). We note that this assumption does not affect the argument following (23). Hence the same conclusion is valid, and we must have

$$
S^+(M_p) = U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p)).
$$

We summarize the preceding discussion in Proposition 8. Let $\Omega, M$ be chosen as in Proposition 5 or its analogue for intransitive ideal complexes.

(a) If $\Omega$ is of level $\mathbb{Q}_p$, then

$$
S^+(M_p) = U_p \times U(\Omega_p) / \Gamma(U_p \times U(\Omega_p)).
$$

(b) If $\Omega$ is of level $p$, then

$$
[S^+(M_p); U_p \times (\Omega_p) / \Gamma(U_p \times U(\Omega_p))] = 2,
$$

the non-trivial core being represented by $\psi[N(II)^{-1}, (II, II)]$ if $p$ splits in $K$, and by $\psi[N(II)^{-1}, (II)]$ if $p$ remains prime in $K$.

Remark. It is clear that Proposition 8 remains valid if we replace $M$ by an arbitrary lattice $L \subseteq \mathbb{Q}$ and $\Omega$ by the left order of $L$.

Let $J_k, J_K, J_{\mathbb{Q}_k}$ denote the ideal groups of $k, K, \mathbb{Q}_k$, respectively. We have natural inclusions $J_p \subseteq J_k \subseteq J_{\mathbb{Q}_k}$ which are compatible with the natural inclusions $k^* \subseteq K^* \subseteq \mathbb{Q}_k^*$. The norm mapping $\nu_{\mathbb{Q}_k^*}: K^* \rightarrow k^*$ extends in the usual way to a mapping $\nu_{\mathbb{Q}_k^*}: J_K \rightarrow J_k$. We put

$$
\Gamma(J_K \times J_K) = \{ (c, c) : J_K \times J_K | \eta_{\mathbb{Q}_k^*}(c) = c^{-1} \},
$$

$$
S^+(V)_\Lambda = J_K \times J_{\mathbb{Q}_k} / \Gamma(J_K \times J_K).
$$

(25)

Then

$$
k^* \times \mathbb{Q}_k^* \cap \Gamma(J_k \times J_K) = \Gamma(k^* \times K^*)
$$

implies that we have a natural embedding

$$
S^+(V) \subseteq S^+(V)_\Lambda.
$$

(26)

Let $V_\Lambda$ denote the adelicization of $V$. For each lattice $L$ in $V$ put $\bar{L} = \prod L_p$, where $L_p = V_p$ for infinite $p$. Then $\bar{L}$ is an open subgroup of $V_\Lambda$. Conversely, if $L(p)$ is a lattice in $V_p$ for each finite prime $p$ and $L(p) = \mathbb{Z}_p$ for each infinite $p$, and if $\prod L(p)$ is an open subgroup of $V_\Lambda$, then there exists a lattice $L$ in $V$ such that $L_p = L(p)$ for every $p$. Furthermore, $L$ is uniquely determined by $\bar{L}$, since $L = \bar{L} \cap V$. For $a \in J_{\mathbb{Q}_k}$, $a = (a_p)$, put $a^* = (a_p^*)$. Then $S^+(V)_\Lambda$ acts transitively on $\mathcal{A}$ by

$$
L \rightarrow a \bar{L} a^* \cap V, \quad a \in J_k, a \in J_{\mathbb{Q}_k}.
$$

This action is compatible with the action of $S^+(V)$ on $\mathcal{A}$ already defined and the embedding (28). Thus $L_1, L_2 \subseteq \mathcal{A}$ are isomorphic if and only if they are in the same orbit under the action of the subgroup $S^+(V)$. For any lattice $L \subseteq \mathcal{A}$ let $S^+(\bar{L})$ denote the isotropy group of $L$ under the action of $S^+(V)_\Lambda$. Then

$$
H = \text{card}(S^+(V) \setminus S^+(V)_\Lambda / S^+(\bar{L})).
$$

(27)

Choose $\Omega, \mathbb{M}$ as in Proposition 8. Set

$$
U_p = \prod U_p, \quad U_K = \prod U(\Omega_p), \quad U(\bar{L}) = \prod U(\Omega_p),
$$

$$
R(\bar{L}) = \prod R(\Omega_p),
$$

where $R(\Omega_p) = R_p^{\mathbb{Q}_k}$ for infinite $p$. Define $R(\bar{L})$ to be the group of all $\nu \in J_{\mathbb{Q}_k}$ such that $\nu \bar{L} \nu^* = n \bar{L}$ for some $n \in J_k$. We call $n$ a multiplier of $\nu$ and denote the set of all multipliers of $\nu$ by $m(\nu)$. Then $m(\nu) = n U_\mathbb{Q}$, and we may regard $m$ as a homomorphism $m: R(\bar{L}) \rightarrow J_K / U_K$. It is clear that $R(\bar{L}) = R(\Omega_p)$, where $R(\Omega_p) = \{ \eta_p \in \mathbb{Q}_k^* | \eta_p \Omega_p^{\nu_p} = n_p \Omega_p, n_p \in \mathbb{Z}_p \}$.

If $\nu \in J_{\mathbb{Q}_k}(\Omega_p)$, then $\nu \nu_\mathbb{Q} \Omega_p^{\nu_\mathbb{Q}} = \Omega_p$, since $\nu_\mathbb{Q} \Omega_p^{\nu_\mathbb{Q}}$ is the left order of $\nu_\mathbb{Q} \Omega_p^{\nu_\mathbb{Q}}$ and $\Omega_p$ is the left order of $\nu_\mathbb{Q} \Omega_p$. Thus we have

$$
K^* \times U(\Omega_p) = R(\Omega_p) \subseteq R(\Omega_p)
$$

and

$$
J_K \times U(\bar{L}) = R(\bar{L}) \subseteq R(\bar{L}).
$$

(28)

(29)
It follows from (28) that $\mathfrak{N}(\mathcal{O}_p) = K^*_p U(\mathcal{O}_p)$ if $\mathcal{O}_p$ has level $\mathcal{H}_p$. If $\mathcal{O}_p$ has level $p$ and $p$ remains prime in $K$, then $\mathcal{H}(\mathcal{O}_p)^* = N(\mathcal{H}) \mathcal{O}_p$, where

$$\mathcal{H} = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}.$$ 

Hence $\mathfrak{N}(\mathcal{O}_p) = \mathfrak{M}(\mathcal{O}_p)$ in that case. If, however, $p$ splits in $K$, then

$$[\mathfrak{N}(\mathcal{O}_p):K^*_p U(\mathcal{O}_p)] = 4 \quad \text{but} \quad [\mathfrak{N}(\mathcal{O}_p):K^*_p U(\mathcal{O}_p)] = 2,$$

because

$$(\mathcal{H}, \mathcal{H}) \mathcal{O}_p (\mathcal{H}, \mathcal{H})^* = N(\mathcal{H}) \mathcal{O}_p,$$

$$(\mathcal{H}, \mathcal{H}) \mathcal{O}_p (\mathcal{H}, \mathcal{H}) = (1, \mathcal{H}) \mathcal{O}_p = (\mathcal{H}, \mathcal{H}) \mathcal{O}_p = (\mathcal{H}, \mathcal{H}) \mathcal{O}_p.$$ 

In particular, $\mathfrak{N}(\mathcal{O})|\mathfrak{M}(\mathcal{O})$ is an elementary abelian 2-group of order $2^r$.

The usual type number $t(\mathcal{O})$ is defined by

$$t(\mathcal{O}) = \text{card}(\mathfrak{N}_K^* / K^* \mathcal{J}_K[\mathfrak{N}(\mathcal{O})]/[\mathfrak{N}(\mathcal{O})]).$$

We generalize this notion slightly by introducing

$$t_k(\mathcal{O}) = \text{card}(\mathfrak{N}_K^*/K^* \mathcal{J}_K[\mathfrak{M}(\mathcal{O})]/[\mathfrak{M}(\mathcal{O})]).$$

Then $t_k(\mathcal{O}) = t(\mathcal{O})$ if $f = 0$, that is, if $V_\gamma$ is isotropic for every finite prime $p$ of $k$.

The mapping $(a, a) \mapsto a \mathcal{J}_K[a, a] \mathcal{J}_K$, induces a homomorphism

$$\varphi: S^+(V) \rightarrow \mathcal{J}_K[\mathfrak{M}(\mathcal{O})]/[\mathfrak{M}(\mathcal{O})].$$

Then $\varphi(S^+(V)) = \mathfrak{N}_K^*/K^* \mathcal{J}_K[\mathfrak{M}(\mathcal{O})]$ and Proposition 8 shows that

$$\varphi(S^+(\mathcal{M})) = \mathfrak{N}(\mathcal{O})/[\mathfrak{N}(\mathcal{O})].$$

This implies that $t_k(\mathcal{O}) \leq H$. Suppose $(a, a), (b, b) \in J_\mathcal{K} \times J_\mathcal{N}_\mathcal{K}$ and $\beta = \gamma a \gamma$ where $\gamma \in \mathfrak{N}_K^*, \gamma \in \mathfrak{N}(\mathcal{O})$ with multiplier $\mathcal{M}$.

Let $k$ be the class number of $K$ and let $\epsilon_1, \ldots, \epsilon_k$ be a complete set of representatives for $J_\mathcal{K} / K^* U_\mathcal{K}$.

Then $ba^{-1} = \delta \epsilon_i$ for some $i = 1, 2, \ldots, k$, where $\delta \in K^*$, $\epsilon_i \in U_\mathcal{K}$, and we have $(b, b) = (a, a)(\delta, \gamma)(\epsilon_i, a^{-1} u, v)$. We conclude that $t_k(\mathcal{O}) \leq H \leq k_\mathcal{O}(\mathcal{O})$. In particular, $H = t_k(\mathcal{O})$ if $\mathcal{O}$ has class number 1. If $H$ does not have class number 1 it seems difficult to give the precise relation between $t_k(\mathcal{O})$ and $H$, the major obstacle being the fact that elements of $\mathfrak{N}(\mathcal{O})$ need not have principal multipliers. However, we can say a little bit in case $h = 2$.

Let $\gamma \in \mathfrak{N}(\mathcal{O})$, $\delta = 1, \ldots, t_k(\mathcal{O})$, represent all the (generalized) type classes for $\mathcal{O}$. Suppose $\gamma, \delta \in J_\mathcal{K}$ and $(\gamma, \gamma), (\delta, \delta)$ represent the same similitude class. Then $i = j$ and $ba^{-1} = k^* m(\mathfrak{N}(\mathcal{O}))$. Putting $h' = [J_\mathcal{K}: k^* m(\mathfrak{N}(\mathcal{O}))]$, we conclude that $h' < H$. Let $\nu_1, \ldots, \nu_h$ be all the finite primes $p$ of $k$ which remain prime in $K$ and for which $\mathcal{O}_p$ is of level $p$. For each $i = 1, \ldots, h$ let $\pi_i$ be an ideal of $K$ whose $p_i$-th component is a prime element and whose other components are 1. Let $(\nu_1, \ldots, \nu_h)$ be the subgroup of $J_\mathcal{K}$ generated by $\pi_1, \ldots, \pi_h$. It is easy to see that

$$n(\mathfrak{N}(\mathcal{O})) = U_\mathcal{K} \mathcal{O}_K(\mathcal{J}_K)(\pi_1, \ldots, \pi_h).$$

We know by class field theory that $[J_k: k^* \mathcal{O}_K(\mathcal{J}_K)] = 2$, $\pi_i \in k^* \mathcal{O}_K(\mathcal{J}_K)$, $i = 1, \ldots, h$. Hence $h' = 1$ except if $e = 0$ and $U_\mathcal{K} \leq k^* \mathcal{O}_K(\mathcal{J}_K)$, in which case $h' = 2$. If we assume, in addition, that $h' = 2$, then $K$ is the Hilbert class field of $k$ and $H = 2t_k(\mathcal{O})$. We have proved:

**Proposition 9.** Let $\mathcal{O}$ be chosen as in Proposition 5 or its analogue for nonisotropic ideal complexes:

(a) If $k$ has class number 1, then $H = t_k(\mathcal{O})$.

(b) If $K$ is the Hilbert class field of $k$ and $\mathcal{O}$ is a maximal order, then $H = 2t_k(\mathcal{O})$.

Remark. Part (a) is valid even if $D(V)$ is a square in $k$, $K = k \mathcal{O}$.

In that case $t_k(\mathcal{O})$ can be interpreted as the number of classes of normal ideals of $\mathfrak{M}$ (cf. [17], Proposition 1).

§ 5. Some weight computations. We assume from now on that $k = \mathcal{O}$, the field of rational numbers. In general, we will make the convention that a fractional ideal of $\mathcal{O}$ will be identified with the unique positive rational number generating it. This will be the case, in particular, for the norm $\mathcal{N}(L)$ of a lattice $L$ and the level $\delta$ of an order $\mathcal{O}$. We will make exceptions for $\mathcal{J}_K$ and the discriminant and reduced discriminant of a lattice $L$, defining the latter two by

$$\Delta(L) = \text{det}[B(v_i, v_j)], \quad \Delta^*(L) = \text{det}[\mathcal{N}(L)^{-1} B(v_i, v_j)],$$

where $\{v_i\}$ is a $Z$-basis of $L$. It is clear from these definitions that $\Delta(L), \Delta^*(L)$ will be negative if $q$ is indefinite and of signature $\neq 0$. By taking into account the signs of the discriminants, it is easy to verify that (b) of Proposition 6 is still valid.

Throughout this section we assume $q$ is positive definite and $D(V)$ is not a square in $\mathcal{O}$. Thus $K$ is a real quadratic extension of $\mathcal{O}$ and $\mathfrak{M}$ is a definite quaternion algebra over $\mathcal{O}$. We choose $\mathcal{O}, \mathfrak{M}$ as in Proposition 5.

For each prime $p$ of $\mathcal{O}$, $p$ of $K$, let the corresponding normalized valuation be denoted by $|_{\mathcal{O}}$, $|_p$, respectively. Put

$$J_\mathcal{O}^p = \{a_p \mathcal{J}_\mathcal{O} \bigg| \prod_p \frac{\mathcal{N}(a_p)}{p} = 1\},$$

$$J_\mathcal{K}^p = \{a_p \mathcal{J}_\mathcal{K} \bigg| \prod_p \frac{\mathcal{N}(a_p)}{p} = 1\},$$

$$J^p = \{a_p \mathcal{J}_\mathcal{K} \bigg| \prod_p \frac{\mathcal{N}(a_p)}{p} = 1\}.$$
By the product formula, \( Q^* \subset J_Q, K^* = J_K, \mathbb{R}_K^* = J_K^* \). Set

\[
U_Q = U_Q \cap J_Q^*, \quad U_K = U_K \cap J_K^*, \quad U(Q) = U(Q) \cap J_K^*.
\]

Then

\[
\mathbb{R}(Q) = \mathbb{R}(Q) \cap J_K^*, \quad \mathbb{R}(Q) = \mathbb{R}(Q) \cap J_K^*.
\]

Put

\[
G = \mathbb{R}(Q), \quad G = J_K^* / J_K^*, \quad G^*(Q) = J_K^* / J_K^*, \quad G^*(Q) = J_K^* / J_K^*.
\]

Then \( G^* \) is a locally compact group, \( G \) is a discrete subgroup of \( G^* \), and \( G^*(Q) \) is a compact space. It follows, in particular, that \( G^* \) is unimodular. Similarly, if we put

\[
J(Q) = J(Q) \cap J_K^*, \quad S^*(Q) = J(Q) \cap J_K^* = J(Q) \cap J_K^*, \quad S^*(Q) = J(Q) \cap J_K^*,
\]

then \( S^*(Q) \) is a locally compact unimodular group, \( S^*(Q) \) is a discrete subgroup of \( S^*(Q) \), and \( S^*(Q) \) is a compact space. The homomorphism \( g \mapsto S^*(Q) \) is a one-to-one correspondence between \( S^*(Q) \) and \( S^*(Q) \).

Furthermore, we have

\[
\lambda(G \setminus G^*) = \frac{1}{\text{card}(G \setminus G^*)}.
\]

Let \( \lambda \) be the Haar measure on \( G^* \) such that \( \lambda(G^*) = 1 \). For simplicity, let \( \lambda \) denote the right invariant measure on \( G^* \) which “lifts” \( \lambda \) by means of the local homeomorphism: \( G^* \to G \).

Let \( M_1, \ldots, M_H \) represent the similitude classes in \( S^* \). We may assume

\[
M_j = q_j M_j^*, \quad q_j e^{\mathbb{R}(Q)}, \quad j = 1, \ldots, H.
\]

Then we have a disjoint double coset decomposition

\[
G^* = \bigcup_{j=1}^H g_j G_j^* G_j^*(Q)
\]

which shows that

\[
\lambda(G \setminus G^*) = \sum_{j=1}^H \lambda(G \setminus G_j^* G_j^*(Q)) = \sum_{j=1}^H \lambda(g_j G_j^*(Q)).
\]

Hence

\[
\lambda(G \setminus G^*) = \lambda(G \setminus G_j^* G_j^*(Q)).
\]

Since \( q \) is definite, \( S^*(M_j) = S^*(M_j^*), j = 1, \ldots, H \). Let \( q_j \) be the restriction of \( \lambda \) to \( S^*(M_j) \). Then

\[
S^*(M_j) / \text{Ker} q_j \cong G \setminus G_j^* G_j^*(Q) g_j^{-1}, \quad j = 1, \ldots, H.
\]

To determine \( \text{Ker} q_j \), suppose \( \alpha \in Q^*, \alpha \in K^*, \) and \( \text{Ker} q_j \alpha^2 = M_j \). Then \( \text{Ker} q_j \alpha^2 = M_j \) and we must have \( \alpha \equiv 0 \) or \( \alpha \in \mathbb{R}(Q) \). It follows that \( \text{Ker} q_j = \{-1\} \) and

\[
\lambda(G \setminus G_j^*) = \sum_{j=1}^H \frac{1}{\text{card}(G \setminus G_j^*)}.
\]

We write

\[
\sum_{j=1}^H \frac{1}{\text{card}(G \setminus G_j^*)} = M(3), \quad \text{the Minkowski–Siegel weight (Mass)}
\]

of \( S^* \).

On the other hand, we choose the Haar measure \( \lambda \) on \( G_j^* \) such that \( \lambda(G^*(Q))^2 = 1 \) and then by means of showing that

\[
\lambda(G \setminus G_j^*) = \sum_{j=1}^H \frac{1}{\text{card}(G \setminus G_j^*)}.
\]

where the \( \Omega_n, \nu = 1, \ldots, \Omega(\nu) \) are representatives for the types of Eichler lattices, \( \lambda(\Omega) \) are representatives of the discriminants of \( \mathbb{R}(Q) \). We know by a result of Eichler (16), (18), (19) that

\[
\lambda(G \setminus G_j^*) = \sum_{n=1}^H \frac{1}{\text{card}(G \setminus G_j^*)}.
\]

where \( \zeta(2) \) is the Dedekind zeta function of \( K, p_1, \ldots, p_e \) are the isotropic primes of \( V \) dividing \( \delta \), and \( q_1, \ldots, q_r \) are the anisotropic primes of \( V \) dividing \( \delta \). The value \( \zeta(2) \) can be expressed in terms of generalized Bernoulli numbers (11), p. 135) as

\[
\zeta(2) = \frac{\pi^2}{12} B^2_2,
\]

where \( B^2_2 = 1/6 \) and \( B^2_2 \) is the generalized Bernoulli number associated to the Kronecker symbol \( \chi(m) = \zeta(2) \).

Using the fact that \( \lambda(G \setminus G_j^*) = 2 \lambda(G \setminus G_j^*) \), we conclude
Proposition 10. Let $\lambda$ be the Haar measure on $G_\lambda$ such that $\lambda(G_\lambda^2) = 1$. Then

$$\lambda(G \setminus G_\lambda^2) = 2M(3) = \frac{\prod_{j=1}^e (p_j^k + 1) \prod_{j=1}^f (q_k^\gamma - 1)^2 \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m^2}{3 \cdot 2^{2e+4}}.$$  

Remarks. 1. The character sum appearing in (37) can be further simplified as follows (cf. [10], § 6)

$$\sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m^2 = \frac{4 \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m}{\left( \frac{A_K}{2} \right)}.$$

(38)

$$= \begin{cases} 4 \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m & \text{if } A_K \equiv 1 \pmod{4}, \\ \left( \frac{A_K}{2} \right) \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m & \text{if } A_K \equiv 4 \pmod{3}, \\ \left( \frac{A_K}{4} \right) \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m & \text{if } A_K \equiv 0 \pmod{3}, \\ 2 \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m & \text{if } A_K > 8 \\ 2 & \text{if } A_K = 8. \end{cases}$$

2. Exactly the same sort of reasoning can be used to evaluate $M(3)$ in the square discriminant case (cf. [17], § 7).

Let $G_\gamma, \gamma = 1, \ldots, g^\gamma$, be the similitude genera contained in $G_\lambda$, where $g^\gamma$ is the number of strict genera of $K$. Let $M(G_\gamma)$ denote the Minkowski–Siegel weight of $G_\gamma$, $\gamma = 1, \ldots, g^\gamma$, and $M(G)$ the Minkowski–Siegel weight of $G$, the genus of maximally integral lattices. Then

$$M(3) = \sum_{\gamma = 1}^{g^\gamma} M(G_\gamma).$$

We claim that all the $G_\gamma$ have the same weight, so that $M(3) = g^\gamma M(G)$. Let $Q_\lambda^\gamma$ denote the subgroup of $Q_\lambda$ of positive rationals. We extend the notion of the norm of a similitude by setting $n(\sigma) = n(\sigma M(3))$, $\sigma \in S^+(V)_\lambda$. It is clear that $n(S^+(V)_\lambda) \simeq Q_\lambda^\gamma$ is a homomorphism. We recall that two maximal lattices $L_1, L_2$ are in the same similitude genus if and only if $n(L_1)/n(L_2) = n_{\lambda}(\sigma), \sigma \in K^\gamma$ ([14], p. 338). This implies that $N = n^{-1}(Q_\lambda^\gamma \cap n_{\lambda}(K^\gamma))$ is a normal subgroup of $S^+(V)_\lambda$ of index $g^\gamma$ which contains both $S^+(V)$ and $S^+(M)$. In particular, $N$ must be an open normal subgroup of $S^+(V)_\lambda$. The similitude genera $G_\gamma, \gamma = 1, \ldots, g^\gamma$, are in one-to-one correspondence with the cosets of $S^+(V)_\lambda \mod N$. Let $\sigma_\gamma S^+(V)_\lambda$ be a representative for the coset corresponding to $G_\gamma, \gamma = 1, \ldots, g^\gamma$. Then the similitude classes in $G_\lambda$ are in one-to-one correspondence with the double cosets of $S^+(V)_\lambda \setminus N \sigma_\gamma S^+(M)$. If we choose the Haar measure $\mu$ on $S^+(V)_\lambda$ such that $\mu(S^+(M)) = 1$, then

$$M(G_\lambda) = \mu(S^+(V)_\lambda \setminus N \sigma_\gamma S^+(M)) = M(G), \quad \gamma = 1, \ldots, g^\gamma.$$ 

Let $t$ denote the number of distinct primes dividing $A_K$. Then $g^\gamma = 2^{2t}$, which shows

**Corollary.** Let $\mathfrak{M}$ be the genus of maximally integral lattices in $V$. Let $t$ be the number of distinct primes dividing $A_K$. Then

$$M(\mathfrak{M}) = \prod_{\gamma = 1}^e (p_j^k - 1) \prod_{\gamma = 1}^f (q_k^\gamma - 1)^2 \left( \sum_{m=1}^{d_k} \left( \frac{A_K}{m} \right) m^2 \right)$$

(42)

$$\frac{3 \cdot 2^{2e+4}}{A_K}.$$

Let $O^+(V)_\lambda$ be the adelization of $O^+(V)$. Let $\tau = \prod_{\gamma} \tau_{\gamma}$ be the Tamagawa measure on $O^+(V)_\lambda$. The Minkowski–Siegel theorem states that

$$M(\mathfrak{M}) \left( \prod_{\gamma} \tau_{\gamma}(O^+(M)) \right) = 2 \tau(O^+(V) \setminus O^+(V)_\lambda).$$

(43)

It turns out that the product of local measures appearing in (43) can be computed in an elementary fashion. This computation, together with the preceding corollary, gives an elementary proof of the Minkowski–Siegel theorem in this special case. Having already done this for the square discriminant case in [16], § 5, we obtain in this manner an elementary proof that the Tamagawa number of $O^+(V)$ is 2 for any definite quaternion space $V$ over $Q$. As this particular aspect is not of primary importance to us here, we will only sketch the means by which the local measures $\tau_{\gamma}(O^+(M))$ can be computed.

For this purpose it is convenient to take the description of $O^+(V)$ provided by (b) of Proposition 2. Put

$$\tilde{\mathfrak{M}}^\gamma = \{ \mathfrak{a} \in \mathfrak{M}^\gamma \mid N(\mathfrak{a}) = O^+(V) \}, \quad \tilde{\mathfrak{M}}^\gamma_{\mathfrak{p}} = \{ \mathfrak{a}_{\mathfrak{p}} \in \mathfrak{M}^\gamma_{\mathfrak{p}} \mid N(\mathfrak{a}_{\mathfrak{p}}) = O^+(V_{\mathfrak{p}}) \}.$$

Then (b) of Proposition 2 shows that we have canonical isomorphisms

$$\tilde{\mathfrak{M}}^\gamma/O^+(V) \cong O^+(V), \quad \tilde{\mathfrak{M}}^\gamma/O^+(V_{\mathfrak{p}}) \cong O^+(V_{\mathfrak{p}}).$$

Put $\tilde{U}(O_{\mathfrak{p}}) = U(O_{\mathfrak{p}}) \cap \tilde{\mathfrak{M}}^\gamma_{\mathfrak{p}}$. Since $\tilde{U}(O_{\mathfrak{p}}) \cap \tilde{\mathfrak{M}}^\gamma_{\mathfrak{p}} = U_{\mathfrak{p}}$, we may regard $\tilde{U}(O_{\mathfrak{p}})/U_{\mathfrak{p}}$ as a subgroup of $O^+(V_{\mathfrak{p}})$. In fact, we have

$$\tilde{U}(O_{\mathfrak{p}})/U_{\mathfrak{p}} \subseteq O^+(M_{\mathfrak{p}}).$$
One can show that $O^+(M_p) = \tilde{U}(\mathcal{O}_p)/U_p$ except if $p \mid A_K$, in which case

$$[O^+(M_p) : \tilde{U}(\mathcal{O}_p)/U_p] = 2.$$  

Let $K_\lambda$, $(\mathfrak{A}_K)_\lambda$ be the adele rings of $K$, $\mathfrak{A}_K$, respectively. The Tamagawa measure $\sigma = \prod_p \sigma_p$ on $K_\lambda$ is given by: $\sigma_p = 1$ for $p < \infty$, $\sigma_p = \frac{1}{A_K^2}$ times ordinary Lebesgue measure for $p$ infinite. For each infinite prime $p$ of $K$ we have $\mathfrak{A}_K^p = H$, the Hamilton quaternions. Any $\alpha \in H$ can be written $\alpha = x_i + y_j + z_k + w_k$, where $1, i, j, k$ is the standard basis of $H$. We define a measure $\omega = \prod_p \omega_p$ on $(\mathfrak{A}_K)_\lambda$ by setting $\omega_p = 1$ for $p < \infty$, $\omega_p = m_\lambda d\omega dx dy dz$, $m_\lambda$ for $p$ infinite. The Tamagawa measure on $(\mathfrak{A}_K)_\lambda$ is $c\omega$, where $c = 16(2n)^2$.

Put $\mathfrak{N}^H = \{\alpha \in \mathfrak{A}_K^p | N(\alpha) = 1\}$, $\mathfrak{N}_\alpha = \{\alpha \in \mathfrak{N}_\alpha | N(\alpha) = 1\}$, $\mathfrak{Q}_\alpha = \mathfrak{Q}_\alpha \cap \mathfrak{N}_\alpha$. Let $\nu = \prod_p \nu_p$ be the Haar measure on $(\mathfrak{A}_K)_\lambda$, the adelicization of $\mathfrak{N}_\alpha$, obtained from $\omega$ and $\nu$ by the usual limiting procedure. Then $\nu$ is the Tamagawa measure on $(\mathfrak{A}_K)_\lambda$. For each prime $p$ of $\mathfrak{Q}$ let

$$\nu_p = \prod_{\alpha \in \mathfrak{N}_\alpha}.$$  

If $p \nmid \delta$, then $\mathfrak{Q}_\alpha^p = SL(2, \mathfrak{O}_p)$ and it is well known that

$$\nu_p(\mathfrak{N}_\alpha^p) = (1 - p^{-1})(1 - \left(\frac{A_K}{p}\right)p^{-1}),$$  

where $\left(\frac{A_K}{p}\right)$ is the Kronecker symbol.

If $p = p_i$, $i = 1, \ldots, \delta$, then it is easy to show that

$$\nu_p(\mathfrak{N}_\alpha^p) = \frac{p^2}{p + 1}(1 - p^{-2})(1 + p^{-2}).$$  

If $p = q_k$, $k = 1, \ldots, f$, then

$$\nu_p(\mathfrak{N}_\alpha^p) = \frac{p^2}{(p - 1)^2}(1 - p^{-2}).$$  

Finally,

$$\nu_p(\mathfrak{N}_\alpha) = \frac{A_K^2}{\pi}.$$  

The inclusion $\mathfrak{N}_\alpha \subset \mathfrak{A}_K^p$ induces a natural mapping $\mathfrak{N}_\alpha^p \to \mathfrak{A}_K^p/\mathfrak{Q}^p$ with kernel $\{\pm 1\}$ which is an isogeny at the local level: $\mathfrak{N}_\alpha^p \to \mathfrak{A}_K^p/\mathfrak{Q}^p$. The restriction: $\mathfrak{Q}_\alpha^p \to \tilde{U}(\mathcal{O}_p)/U_p$ is an isogeny with cokernel of order $[U_p : (U_p)^2]$ for $p < \infty$ and order 1 for $p = \infty$. Since $[U_p : (U_p)^2] = 2$ for $p \neq 2$, but $[U_p : (U_p)^2] = 4$, we have

$$\prod_p \tau_p(\tilde{U}(\mathcal{O}_p)/U_p) = c \prod_p \nu_p(\mathfrak{N}_\alpha^p).$$  

Hence

$$\prod_p \tau_p(\mathfrak{Q}_\alpha^p) = c^2 2^{\delta + \delta - 1} \prod_p \nu_p(\mathfrak{N}_\alpha^p)$$

$$= \left(\frac{A_K^2}{\pi} \prod_{k = 1}^c (p_k^2 + 1) \prod_{k = 1}^l (q_k - 1)^2 \right) = \left(\frac{A_K^2}{\pi} \prod_{k = 1}^c (p_k^2 + 1) \prod_{k = 1}^l (q_k - 1)^2 \right) = 2^M \left(\frac{\delta}{M}\right)^{-1}.$$  

§ 6. Class number formulas. Let us first dispose of the indefinite case.

**Proposition 11.** Suppose $V$ is indefinite. Let $t$ be the number of distinct primes dividing $A_K$.

(a) If $q$ has signature $\neq 0$, or if $K$ has an element of norm $-1$, then $H = 2^{t-1}$.

(b) If $q$ has signature $0$ and $K$ does not have an element of norm $-1$, then $H = 2^{t-2}$.

**Proof.** By the theorem of Kronecker ([9], p. 330), every similitude genus of maximal lattices contains just one similitude class. It follows that $H$ is the number of similitude classes in $\mathfrak{N}$. We noted in §3 that $\mathfrak{N}$ contains $g^+ = 2^{-1}$ similitude classes if $q$ has signature $\neq 0$. Suppose $q$ has signature 0. Then $K$ is a real quadratic extension of $\mathfrak{Q}$, and, since $\mathfrak{A}_K$ is totally indefinite,

$$n(S^+(V)) = \nu_{\mathfrak{A}_K}(\mathfrak{N}(\mathfrak{A}_K)) = \nu_{\mathfrak{A}_K}(\mathfrak{A}_K).$$  

It follows that $L_1, L_2 \in \mathfrak{N}$ are in the same similitude genus if $n(L_1)/n(L_2) = \nu_{\mathfrak{A}_K}(\mathfrak{A}_K)$. We conclude that $H = g$, the number of genera of $K$. It is well known that $g = 2^{t-1}$ if $\nu_{\mathfrak{A}_K}$ represents $-1$, $g = 2^{t-2}$ if $H$ does not.

Now suppose that $V$ is definite. We impose the following additional conditions:

(i) $V_p$ is isotropic for every finite prime $p$.

(ii) The fundamental unit of $K$ has norm $-1$.

Condition (i) implies that $\mathfrak{A}_K$ is split at every finite prime of $K$. Hence $\delta = p_1 \ldots p_6$ and $H = t$, the type number of Bichler orders of level $\delta$ in $\mathfrak{A}_K$. We could then apply the general formula for the type number of Bichler orders in a totally definite quaternion algebra ([6], [14], [15]) to obtain a formula for $H$. Unfortunately, this formula is rather complicated and not very explicit. However, the imposition of condition (ii) greatly simplifies the computation of the terms appearing in the Selberg trace formula expression for $H$, and results in explicit formulas for $H$ of a very elementary sort (cf. Theorems 1 and 2). In this section we will only state
those formulas. Their actual derivation will be left to the remaining sections. We note that condition (ii) implies that $A_K$ only has prime divisors $p$ of the form $p \equiv 1 \pmod{4}$ or $p = 2$.

**Notation.** For any positive integer $m$, let $\lambda(m)$ denote the number of primes dividing $m$; let $h(-m)$ denote the class number of $\mathbb{Q}(\sqrt{-m})$.

**Theorem 1.** Suppose $V$ is a definite quaternionic space over $\mathbb{Q}$ satisfying conditions (i) and (ii). Let $\mathfrak{I}$ be the ideal complex containing the maximally integral lattices of $V$. Let $H$ be the number of isomorphisms classes in $\mathfrak{I}$ and $\mathfrak{A}$ the reduced discriminants of $\mathfrak{I}$. Let $\delta$ be defined by $\delta = A_K\delta$, where $K = \mathbb{Q}(\sqrt{\Delta})$ (cf. § 3, Proposition 6). Denote the square-free kernel of $\mathfrak{A}$ by $D$. If $D$ is odd and $\Delta > 5$, then

$$H = 2M(3) + c_1 h(-D) + c_2 h(-3D) +$$

$$+ \sum_{\nu = 2, h(-nD)h(-nD)} 2^{-\nu(n)\nu} h(-nD),$$

where $nd \neq 1, 3, d < \sqrt{D}$ and

$$c_1 = \begin{cases} \frac{1}{2} & \text{if } 2 \nmid \delta, \\
\frac{3}{10} & \text{if } 2 | \delta, \end{cases}$$

$$c_2 = \begin{cases} \frac{1}{8} & \text{if } 3 \nmid \delta, \\
\frac{1}{2} & \text{if } 3 | \delta, D \equiv 1 \pmod{8}, \\
\frac{3}{8} & \text{if } 3 | \delta, D \equiv 5 \pmod{8} \end{cases}$$

and if $D \equiv 1 \pmod{8},$

$$\sigma(m) = \begin{cases} -2 & \text{if } m \equiv 3 \pmod{8}, \\
0 & \text{if } m \equiv 7 \pmod{8}, \\
2 & \text{if } m \equiv 1 \pmod{4} \end{cases}$$

while if $D \equiv 5 \pmod{8},$

$$\sigma(m) = \begin{cases} 0 & \text{if } m \equiv 3 \pmod{4}, \\
2 & \text{if } m \equiv 2 \pmod{4}, 2 | \delta, \\
2 & \text{if } m \equiv 1 \pmod{4}, 2 | \delta, \\
3 & \text{if } m \equiv 1 \pmod{4}, 2 \nmid \delta, \end{cases}$$

Furthermore,

$$M(3) = \prod_{p \leq 2} \frac{(p^2 + 1)}{3 \cdot 2^{p-3}} \left( \frac{D}{2} \right)$$

$$\sum_{m=1}^{D} \left( \frac{D}{m} \right) \sigma(m),$$

where $\left( \frac{D}{m} \right)$ is the Kronecker symbol.

**Corollary.** Suppose that $\Delta = p$, a prime greater than 5. Then

$$H = \frac{\sum_{m=1}^{(p-1)/2} \left( \frac{p}{m} \right)m}{12 \left( \frac{p}{2} \right)},$$

$$+ \frac{h(-p)}{8} + \frac{h(-3p)}{6}.$$

**Remarks.** 1. If $\Delta = p$, a prime, then $A_K = p, p \equiv 1 \pmod{4}$. It is well known in this case that the fundamental unit has norm $-1$ ([3], p. 185).

2. Tamagawa has shown, under the assumption of the Corollary, that $H = h(\mathfrak{I}_K)/h(K)$, where $h(\mathfrak{I}_K)$ is the ideal class number of $\mathfrak{I}_K$ and $h(K)$ is the class number of $K$. Combining this with Peters' formula for $h(\mathfrak{I}_K)$ ([14], p. 363), we obtain another proof of (52). Still another proof of (52) can be found in [8].

**Theorem 2.** Suppose $V$ is a definite quaternionic space over $\mathbb{Q}$ satisfying conditions (i) and (ii). Let $\mathfrak{I}, H, \mathfrak{A}, \delta, D$ be defined as in Theorem 1. If $D$ is even, then

$$H = 2M(3) + \frac{5}{8} h(-D) + c_2 h(-3D) +$$

$$+ \sum_{\nu = 2, h(-nD)h(-nD)} 2^{-\nu(n)\nu} h(-nD),$$

where $nd > 3, d$ is odd and

$$c_2 = \begin{cases} \frac{1}{4} & \text{if } 3 \nmid \delta, \\
\frac{7}{10} & \text{if } 3 | \delta \end{cases}$$

and for $m > 3,$

$$\sigma(m) = \begin{cases} 5 & \text{if } m \equiv 3 \pmod{8}, \\
1 & \text{if } m \equiv 7 \pmod{8}, \\
3 & \text{if } m \equiv 1 \pmod{4}, \end{cases}$$

Furthermore,

$$M(3) = \prod_{p \leq 2} \frac{(p^2 + 1)}{3 \cdot 2^{p-3}} \left( \frac{D}{2} \right)$$

$$\sum_{m=1}^{D} \left( \frac{D}{m} \right)(D + ((-1)^{(m-1)/2} - 1)m) \quad \text{if } D \neq 2,$$

$$M(3) = \prod_{p \leq 2} \frac{(p^2 + 1)}{3 \cdot 2^{p-3}} \quad \text{if } D = 2.$$
At this point it seems appropriate to interpret our lattice-theoretic results in the traditional language of quadratic forms. A \( \text{quaternary (quadratic)} \) form over \( Q \) is a homogeneous polynomial \( f = f(X_1, X_2, X_3, X_4) \) of degree 2 with coefficients in \( Q \). The \textit{discriminant} \( \Delta(f) \) of \( f \) is defined by

\[
\Delta(f) = \det \left[ \frac{\partial^2 f}{\partial X_i \partial X_j} \right], \quad i, j = 1, 2, 3, 4.
\]

We always assume \( \Delta(f) \neq 0 \). Given a quaternary form \( f \) over \( Q \) we can define a quaternary space \( V_f \) over \( Q \) by evaluating \( f \) at the elements of \( Q^4 \). We assume that \( V_f \) is not negative definite.

A quaternary form \( f \) over \( Q \) is said to be \textit{integral} if all of its coefficients are integers. An integral form is \textit{primitive} if the g.o.d. of its coefficients is 1. Two quaternary forms \( f, f' \) over \( Q \) are \textit{equivalent}, written \( f \cong f' \), if there is an element \( \sigma \in \text{GL}(4, Q) \) such that \( f'(X_1, X_2, X_3, X_4) = f(\sigma X_1, \sigma X_2, \sigma X_3, \sigma X_4) \), where \( \sigma X_i = \sum a_{ij} X_j \) if \( \sigma = (a_{ij}) \). If \( f \cong f' \), then \( \Delta(f) = \Delta(f') \), \( f \) is integral \( \Rightarrow f' \) is integral, and \( f \) is primitive \( \Leftrightarrow f' \) is primitive. Classical reduction theory shows that the number of equivalence classes of integral quaternary forms having a fixed discriminant is finite. It is this notion of class number which is of primary interest from the classical point of view. We proceed now to relate it to the lattice-theoretic notion.

Let \( V \) be a quaternary space over \( Q \) with quadratic mapping \( q: V \rightarrow Q \). We fix an ordered basis \( v_1, v_2, v_3, v_4 \) for \( V \) and call any other ordered basis \( w_1, w_2, w_3, w_4 \) of \( V \) \textit{positively oriented} if the linear automorphism defined by \( v_i \mapsto w_i \), \( i = 1, 2, 3, 4 \), has positive determinant. Let \( L \) be a lattice in \( V \) with positively oriented \( Z \)-basis \( v_1, v_2, v_3, v_4 \). Define \( f_L \) to be the unique quaternary form such that

\[
f_L(a_1, a_2, a_3, a_4) = n(L)^{-1}q\left(\sum a_i v_i\right)
\]

for all \( a_i \in Q, \ i = 1, 2, 3, 4 \). Then \( f_L \) is a primitive integral form and \( \Delta(f_L) = \Delta(L) \). It is clear that a different choice of positively oriented \( Z \)-basis of \( L \) will yield an equivalent form. Hence the equivalence class \( \{f_L\} \) is uniquely determined by \( L \).

Two lattices \( L, M \) are said to be \textit{strictly similar}, written \( L \approx L \), if \( \sigma L = M \) for some \( \sigma \in \text{SL}(4, Q) \) with \( n(\sigma) > 0 \). Strict similarity can differ from ordinary similarity only if \( q \) has signature 0, as all proper similitudes have positive norms when \( q \) has signature \( \neq 0 \). Suppose \( L, M \) are lattices in \( V \) and \( L \approx M \). Then \( \sigma L = M \), where \( \sigma \in \text{SL}(4, Q) \), \( n(\sigma) > 0 \). It follows that \( n(L) = n(M) = n(\sigma)n(L) \). Let \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) be a positively oriented \( Z \)-basis of \( L \). Then \( \sigma(\sigma_1), \sigma(\sigma_2), \sigma(\sigma_3), \sigma(\sigma_4) \) is a positively oriented \( Z \)-basis of \( M \) and we have

\[
f_M(a_1, a_2, a_3, a_4) = n(M)^{-1}q\left(\sum a_i \sigma(\sigma_i)\right)
\]

for all \( a_i \in Q, \ i = 1, 2, 3, 4 \). Hence \( f_L = f_M \), and we have a well-defined mapping \( \{L\} \rightarrow \{f_L\} \) from strict similitude classes of lattices in \( V \) to equivalence classes of primitive quaternary forms. Suppose \( L, M \) are lattices in \( V \) with \( f_L \approx f_M \). We may assume, without loss of generality, that \( f_L = f_M \). Then there exist positively oriented \( Z \)-bases \( v_1, v_2, v_3, v_4 \) and \( y_1, y_2, y_3, y_4 \) of \( L, M \), respectively, such that

\[
n(L)^{-1}q\left(\sum a_i v_i\right) = n(M)^{-1}q\left(\sum a_i y_i\right)
\]

for all \( a_i \in Q, \ i = 1, 2, 3, 4 \). Define \( \sigma \in \text{GL}(V) \) by \( \sigma(v_i) = y_i, \ i = 1, 2, 3, 4 \). Then (58) implies that \( \sigma \in \text{SL}(V) \) and \( n(\sigma) = n(M)n(L)^{-1} > 0 \), which shows \( L \approx M \). Thus \( \{L\} \rightarrow \{f_L\} \) defines a one-to-one correspondence between strict similitude classes of lattices in \( V \) and equivalence classes of primitive quaternary forms. This correspondence is "discriminant preserving" in the sense that \( \Delta(L) = \Delta(f_L) \). Furthermore, it is clear that any primitive quaternary form \( f \) arises in this manner (simply take \( V = V_f, L = Z^4 \)).

If two quaternary spaces over \( Q \) are similar by a positive factor of similarity, they yield the same classes of primitive forms. Therefore, in order to obtain all classes of primitive (non-negative definite) quaternary forms, it suffices to fix one quaternary space \( (V, q) \) for each possible fundamental discriminant when \( q \) is positive definite or the signature of \( q \) is 0; when \( q \) is indefinite with signature \( \neq 0 \), we must also take \( (V, -q) \). Notation. For any integer \( m \) let \( H^+(m), H^-(m) \) denote, respectively, the number of classes of positive definite integral quaternary forms with discriminant \( m \), the number of classes of indefinite integral quaternary forms with discriminant \( m \).

**Theorem 3.** Suppose \( K \) is a quadratic extension of \( Q \) and \( V \) is a quaternary space with \( \Delta = \Delta_K \).

(a) If \( V \) is positive definite, then \( H^+(\Delta_K) = H \).

(b) If \( V \) is indefinite with signature \( \neq 0 \), then \( H^-(\Delta_K) = 2H \).

(c) If \( V \) has signature 0 and the fundamental unit of \( K \) has norm \( -1 \), then \( H^-(\Delta_K) = H \).

(d) If \( V \) has signature 0 and the fundamental unit of \( K \) has norm 1, then \( H^-(\Delta_K) = 2H \).
Proof. If \( L \subseteq 3 \), then \( f_L \) is an integral quaternion form with \( d(f_L) = d'(L) = \Delta_L \). Conversely, suppose \( f \) is an integral quaternion form with \( d(f) = \Delta_K \). Then \( f \) must be primitive since \( \Delta_K \) is not divisible by the fourth power of a prime. If \( V \) is positive definite, then \( f = f_L \) for some lattice \( L \) in \( V \), and \( d'(L) = d(f_L) = \Delta_K \) implies that \( L \subseteq 3 \). If \( V \) is indefinite of signature \( \phi = 0 \), then \( f = f_L \) for some lattice \( L \) in \( (V, q) \) or \( (V, -q) \). The classes of forms coming from \( (V, q) \) are disjoint from those coming from \( (V, -q) \), as they have different signatures. Hence their total number is \( 2H \). If \( V \) has signature \( 0 \), then \( f = f_L \) for some \( L \subseteq 3 \) and \( H'(\Delta_K) \) is the number of strict similitude classes in \( 3 \). We must compare the latter number with \( N \). Suppose \( I, M \subseteq 3 \) and \( cL = M \), where \( a \in S^+(V) \) and \( n(a) < 0 \). Then \( L \approx M \iff N \) if and only if there is a \( \tau \in S^+(L) \) with \( n(\tau) = -1 \). From the remark following Proposition 8 it is evident that \( \tau \in S^+(L) \) implies \( n(\tau) = \eta_{\Omega_K}(\tau) \) for some unit \( \tau \) of \( \mathfrak{O} \). On the other hand, since \( \mathfrak{A}_K \) is totally indefinite, the strong approximation theorem of Eichler ([5], p. 239) shows that any unit \( \tau \) of \( \mathfrak{O} \) can be expressed as \( \tau = n(\tau) \), where \( \tau \) is a unit of the left order of \( L \); thus \( \eta_{\Omega_K}(\tau) = n(\tau) \), where \( \tau = n(1, a) \in S^+(L) \). We conclude that \( L \) has a unit of norm \( 1 \) if and only if \( \mathfrak{O} \) has a unit of norm \( 1 \). Assertions (c), (d) follow immediately from this observation.

Corollary 1. Let \( K \) be a real quadratic extension of \( \mathbb{Q} \) whose fundamental unit has norm \( -1 \). Let \( D \) denote the square-free kernel of \( \Delta_K \).

(a) If \( D \) is odd, \( D > 3 \), then

\[
H'(\Delta_K) = \frac{1}{4} \sum_{m=-2}^{D-1} \left( \frac{D}{m} \right) m \left( \frac{h(-D)}{8} + \frac{h(-3D)}{6} + \frac{1}{4} \sum_{1 < d < D/3} \frac{h(-d) h(-D/d)}{12 \left( \frac{D}{2} - d \right)} \right)
\]

(b) If \( D \) is even, \( D \neq 2 \), then

\[
H'(\Delta_K) = \frac{1}{24} \sum_{m=-2}^D \left( \frac{D}{m} \right) \left( D + (1 - m^2)/2 \right) m \left( \frac{h(-D)}{8} + \frac{h(-3D)}{6} + \frac{3}{4} \sum_{1 < d < 3D/4} \frac{h(-d) h(-D/d)}{12 \left( \frac{D}{2} - d \right)} \right)
\]

Corollary 2. Let \( K \) be a quadratic extension of \( \mathbb{Q} \). Let \( t \) be the number of distinct primes dividing \( \Delta_K \).

(a) If \( \Delta_K > 0 \) and \( t = 1 \), then \( H'(\Delta_K) = 0 \).

(b) If \( \Delta_K > 0 \), \( t > 1 \), and the norm of the fundamental unit of \( K \) is \( -1 \), then \( H'(\Delta_K) = 3^{t-1} \).

(c) If \( \Delta_K > 0 \) and \( \eta_{\Omega_K} \) does not represent \( -1 \), then \( H'(\Delta_K) = 3^{t-1} \).

(d) If \( \Delta_K < 0 \), or if \( \Delta_K > 0 \), \( \eta_{\Omega_K} \) represents \( -1 \), but the fundamental unit of \( K \) has norm \( 1 \), then \( H'(\Delta_K) = 2^t \).

Proof. If \( \Delta_K > 0 \) and \( t = 1 \), then \( \mathfrak{O} \) is nonsplit at \( \infty \). Hence \( \Gamma' \) must be definite and \( H'(\Delta_K) = 0 \). The remaining assertions (b), (c), (d) follow from Proposition 11. Note that \( t > 1 \) if \( \Delta_K > 0 \) and the fundamental unit of \( K \) has norm \( 1 \).

§ 7. Normalizers of Eichler orders. The remaining two sections are devoted to the derivation of formulas (50) and (53) for \( H \). We have shown that \( H = \text{t}_K \), the type number of Eichler orders of level \( \delta \) in \( \mathfrak{A}_K \). In order to compute \( t_K \) we must first determine which \( \mathfrak{a} \in \mathfrak{A}_K \) lie in the normalizer \( \mathfrak{N}(\Omega) \) (cf. § 4) of some Eichler order \( \Omega \) of level \( \delta \). To do this we fix \( \Omega \) and use the structure of ambiguous ideals of \( \mathfrak{O} \) to put the minimal polynomial of \( \alpha \) over \( \mathfrak{O} \) into a standard form. From this we obtain necessary conditions on \( \alpha \) which we then show to be sufficient (Proposition 14).

Suppose \( \mathfrak{a} \in \mathfrak{N}(\Omega), \mathfrak{a} \neq \mathfrak{K}^x \). We may assume that \( \alpha \) is integral over \( \mathfrak{O} \). From our local discussion in § 4 it is evident that the principal ideal \( (N(\alpha)) \) is \( \eta_1N \), where \( \eta_1 \) is a rational integer dividing \( \delta \), and \( \eta_1 \) is an integral ideal of \( \mathfrak{O} \). Then \( \eta_1^2 = (N(\alpha)) \), which implies that \( \eta_1 \) lies in an ambiguous ideal class of \( \mathfrak{O} \). Since the fundamental unit of \( K \) has negative norm, \( \eta_1 \) is equivalent to an ambiguous ideal \( \mathfrak{j} \) of \( \mathfrak{O} \), i.e., there is an element \( \mathfrak{a} \in \mathfrak{K}^x \) such that \( \exists \mathfrak{k} \in \mathfrak{K} \), where \( \mathfrak{k} \) is a primitive integral ideal of \( \mathfrak{O} \) satisfying the condition \( \mathfrak{j} = \mathfrak{k} \) ([3], p. 189, Ex. 6). It follows that \( (N(\alpha)) = \eta_1^2 \), where \( \eta_1 \) is ambiguous. Then \( \eta_1^2 = (d) \), where \( d \) is a rational integer dividing \( D ([3], p. 190, Ex. 11) \). Hence we may assume, without loss of generality, that \( (N(\alpha)) = (m) \), where \( m \) is a rational integer dividing \( \delta D \). From the description of the local normalizers \( \mathfrak{N}(\mathfrak{O}_l) \) it is clear that an element of \( \mathfrak{N}(\Omega) \) having integral norm must be integral over \( \mathfrak{O} \). Thus the minimal polynomial of \( \alpha \) over \( \mathfrak{O} \) must be of the form \( \mathfrak{O}^2 + b \mathfrak{O} + c \), where \( b \mathfrak{O} \) and \( c \) are units of \( \mathfrak{O} \). As \( N(\alpha) = cm \) is totally positive and the fundamental unit of \( K \) has negative norm, \( m \) must be the square of another unit of \( \mathfrak{O} \). Hence we may assume that \( m = 1 \). The minimal polynomial of \( \alpha \) then has the form \( \mathfrak{O}^2 + b \mathfrak{O} + c \), where \( b \mathfrak{O} \) and \( c \mathfrak{O} \).

Lemma. Suppose \( \omega \) is a unit of \( \mathfrak{O} \) and \( N(\omega) = 1 \). Then \( \omega \) is a root of unity.

Proof. This is an immediate consequence of the fact that the group of all units of \( \mathfrak{O} \) having norm \( 1 \) is a finite group ([3], p. 193, Satz 2).

Proposition 12. If \( \mathfrak{a} \in \mathfrak{N}(\Omega) \), then \( \mathfrak{a}^* = \mathfrak{a} \omega \), where \( \omega \) is a root of unity which commutes with \( \mathfrak{a} \). If \( \omega \neq -1 \), then \( K(\mathfrak{a}) = K(\omega) \).
Proof. Put $\omega = a^\sigma a^{-1}$. It is clear that $\omega$ commutes with $a$. Furthermore, since $\mathcal{O}^\ell = \mathcal{O}$, $a^\sigma$ must be an element of $\mathfrak{O}(\mathcal{O})$. From the description of local normalizers we see that $a^\sigma a^{-1}$ is a unit of $\mathcal{O}$ if $a \in \mathfrak{O}(\mathcal{O})$. Then $\omega$ is a unit of $\mathcal{O}$ and $N(\omega) = 1$. The preceding lemma shows that $\omega$ must be a root of unity. If $\omega \neq -1$, then $K(\omega)$, $K(\omega)$ are both proper extensions of $K$ contained in $\mathfrak{X}_K$, and $\omega$ commutes with $a$. Hence $K(\omega) = K(a)$.

Corollary. If $a \in \mathfrak{O}(\mathcal{O})$, $a \in \mathfrak{X}_K$, then $a^\sigma = \omega a$, where $\omega \in \mathfrak{O}(\mathcal{O})$ and $\omega$ is a primitive $n$-th root of unity for one of the following values of $n$: $2, 3, 4, 5, 6, 8, 10$.

Proof. If $\omega \neq -1$, then $[K(\omega) : \mathcal{O}] = [K(a) : \mathcal{O}] = 4$, which implies $[K(\omega) : \mathfrak{O}] = 4$. The only possible values of $n$ for which this is true are $2, 3, 4, 6, 8, 10, 12$. However, $n = 12$ is impossible since $K$ would then have to be $Q(\sqrt{3})$, whose fundamental unit has norm 1.

Notation. If $a \in \mathfrak{X}_K$ and $g$ is an algebraic number, then $a \simeq g$ will mean that there exist $a, y \in \mathfrak{X}_K$ such that $ax$ and $yg$ have the same minimal polynomial over $K$. For $a, b \in \mathfrak{X}_K$ the condition $a \simeq b$ is equivalent to $a, b \mod K^*$ being conjugate in $\mathfrak{X}_K/K^*$.

Let $a$ be an element of $\mathfrak{O}(\mathcal{O})$ with minimal polynomial $X^2 + bX + m$, where $b \in \mathcal{O}$ and $m \in \mathfrak{O}(\mathcal{O})$. Taking $\omega$ as in the corollary, we see that $b = -(a + a^\sigma) = -(1 + \omega)a$, $m = ax^2 = \sigma x^2$. Thus $c^2 = \sigma x^2$. If $\omega \neq -1$ then $c \simeq \sqrt{-m}$. If $\omega = -1$ our approach will be to find solutions, if any, of the equation $c^2 = \sqrt{-m}$ in the ring of integers of $K(\omega)$.

We first dispose of the exceptional case where $\omega$ is a primitive fifth or tenth root of 1. Let $\omega$ be a primitive fifth root of 1, then $-\omega$ is a primitive tenth root of 1. Since $[Q(\omega) : \mathcal{O}] = 4$, we must have $K(\omega) = Q(\omega)$, $K = Q(\sqrt{5})$. The equation $c^2 = \pm \sqrt{-m}$ has a solution only if the $\pm$ sign holds, and the solution is $a = \pm c \simeq c^\sigma$. Now consider the equation $a^2 = \pm \sqrt{m}$, $m > 1$. Then $(\sigma a)^2 = \sigma a^2 = \pm m$, and since $Q(\sqrt{5})$ is the only quadratic field of $Q(a)$, we must have $(\sigma a)^2 = 5$. We have $a \simeq c^2 \simeq c^\sigma$ once again. As $a$ is a unit, $a \in \mathfrak{O}(\mathcal{O})$ if and only if $\omega \in \mathfrak{O}$. Since $\omega$ generates the ring of integers in $Q(a)$, the criterion of Eichler ([8], p. 133) shows that $a \in \mathfrak{O}$ for some $\mathfrak{O}$ of level $d$ if and only if no prime $p$ divides $d$ remains prime in $Q(a)$. Suppose $a > 5$, that is, $d > 1$. Let $p \mid b$. Since $p$ remains prime in $Q(\sqrt{5})$, we must have $p : 2, 3 (\mod 5)$; all such primes remain prime in $Q(\omega)$, as their residue classes are of order 4 in $\mathbb{Z}/5)([3], p. 87).$ This shows

Lemma 1. Suppose $a \in \mathfrak{O}(\mathcal{O})$ and $a^\sigma = \omega a$, where $\omega$ is a primitive fifth or tenth root of unity. If $d > 5$ then $\omega \simeq 1$.

Remark. If $a = 5$, then $a \simeq \omega \simeq -\omega$ certainly does occur in an Eichler order of level 1 in $\mathfrak{X}_K$ (i.e. a maximal order of $\mathfrak{X}_K$).

We are left with the cases where $\omega$ is a primitive third, fourth, sixth, or eighth root of 1.

Let $\zeta$ be a primitive cube root of 1. Then $-\zeta$ is a primitive sixth root of 1. If $\omega$ is a primitive eighth root of 1, then $K = Q(\sqrt{2})$ and $K(\omega) = K(\sqrt{2})$. Hence matters are reduced to studying solutions of $a^2 = \omega^{-1}m$ in the ring of integers of $K(\sqrt{2})$ or $K(\sqrt{-2})$. We know that $\sqrt{-2}, \zeta$ generate the rings of integers of $K(\sqrt{-2}), K(\sqrt{-2})$, respectively. To determine the ring of integers of $K(\sqrt{-2}), K(\sqrt{-2})$ over $K$ and comparing them to the ideals $(3), (3)$, respectively, of $K$. We do this by means of the next proposition. For any positive integer $m$ let $A(-m)$ denote the discriminant of the imaginary quadratic extension $Q(\sqrt{-m})$.

Proposition 13. Let $m$ be a positive integer and $K = Q(\sqrt{D})$ a real quadratic extension of $Q$. Put $L = K(\sqrt{-m})$ and denote by $A_LK$ the relative discriminant of $L$ over $K$. Then

$$A_LK = A(-m) A(-mD) / A_K$$

Proof. By the conductor-discriminant product formula ([2], p. 160), the discriminant of $L$ over $Q$ is given by $A(-m) A(-mD) A_K$. On the other hand, it is also given by $\mathfrak{n}_{KQ}(A_LK) A_K([2], p. 17)$, hence the result.

Notation. For any finite extension $L$ of $K$ let $\mathfrak{c}_L$ denote the ring of integers of $L$.

Corollary. Let $K = Q(\sqrt{D})$ be a real quadratic extension of $Q$, where $D$ is square-free and $D = 1$ or $2(\mod 4)$.

(a) If $L = K(\sqrt{-1})$, then $\mathfrak{c}_L = \mathfrak{n}(D + \sqrt{-1})$.

(b) If $L = K(\sqrt{-3})$, then $\mathfrak{c}_L = D + D\sqrt{3}$.

Proof. First suppose $L = K(\sqrt{-D})$. If $D = 1(\mod 4)$, then

$$A(-D) = -4A_K$$

$$\mathfrak{n}_{KQ}(A_LK) = 16.$$ Hence $A_LK = (4)$, which shows that $\mathfrak{c}_L = D + \sqrt{-D}$.

If $D$ is even, then $A(-D) = -2A_K$, $\mathfrak{n}_{KQ}(A_LK) = 4$. This shows that $A_LK = (2)$ and the conductor of $D + \sqrt{-D}$ is $\sqrt{p}$, where $p^2 = (2)$. It follows that $\mathfrak{c}_L = \mathfrak{n}(D + \sqrt{-1})$.

Now suppose $L = K(\sqrt{-3})$. Then $A(-3D) = -3A_K$, $\mathfrak{n}_{KQ}(A_LK) = 9$. Hence $A_LK = (3)$ and $\mathfrak{c}_L = D + D\sqrt{3}$.

We now complete the determination of all possible $a \in \mathfrak{O}(\mathcal{O})$, where $a$ is assumed to have a minimal polynomial of the form $X^2 + bX + m$, with $b \in \mathcal{O}$ and $m \in \mathfrak{O}(\mathcal{O})$. We shall consider the cases $b \geq 0$ and $b < 0$ separately.

Case $b \geq 0$. In this case we have $\omega = a^\sigma a^{-1} = \omega a$, where $\omega \in \mathfrak{O}(\mathcal{O})$ and $\omega$ is a primitive $n$-th root of unity for one of the following values of $n$: $2, 3, 4, 5, 6, 8, 10$.
Suppose \( m = 1 \). Then \( a \) must be a root of unity. If \( \Delta > 5 \), then \( a \simeq \sqrt{-1} \), \( \zeta \) or \( \eta \), a primitive eighth root of \( 1 \). We see that if \( \eta \) is a primitive eighth root of \( 1 \), then \( K = \mathbb{Q}(\sqrt{2}) \) and \( \eta \simeq 1 + \sqrt{-1} \). Now suppose \( m > 1 \).

(i) If \( \alpha = \pm \sqrt{-1} \), then, according to (a) of the corollary,
\[
a = \frac{1}{2}(x + y\sqrt{-1}), \quad \text{where } x, y \in \mathbb{O}, \quad \text{and } a^2 = \pm m\sqrt{-1}.
\]
Taking \( a^* \) instead of \( a \), if necessary, we may assume that \( a^* = m\sqrt{-1} \). Then
\[
4a^2 = (x + y\sqrt{-1})^2 = x^2 - y^2 + 2xy\sqrt{-1} = 4m\sqrt{-1}.
\]
This implies \( x = \pm y \), \( \pm x\sqrt{-1} = 2m\sqrt{-1} \). Hence \( a^2 = 2m \). If \( m \) is odd we must have \( 2m = D \), \( a = \pm \sqrt{D} \). If \( m \) is even, then \( (a^2)^2 = m^2 \) implies \( m/2 = 1 \) or \( D \), from which it follows that \( x = \pm 2 \) or \( x = \pm 2\sqrt{-1} \).

In all cases we have \( a \simeq 1 + \sqrt{-1} \).

(ii) If \( \alpha = \pm \zeta \), then, according to (b) of the corollary,
\[
a = x + y\zeta, \quad \text{where } x, y \in \mathbb{O}, \quad \text{and } a^2 = \pm m\zeta.
\]
Thus
\[
a^2 = a^2 + (x^2 + y^2)\zeta + 2xy = a^2 + y^2(\zeta - 1) + 2xy
\]
\[
= a^2 + 2xy - (x^2 - y^2)\zeta = \pm m\zeta,
\]
which gives \( x = \pm y \). If \( x = y \), we must have \( y^2 = m \); if \( x = -y \), we must have \( y^2 = m \). The equation \( y^2 = m \) implies \( y = \pm 1 \) or \( y = \pm \sqrt{D} \), in which case \( a \simeq \zeta - 1 \). The equation \( y^2 = m \) implies \( m = D \), \( y = \pm \sqrt{D} \), in which case \( a \simeq \zeta - 1 \).

(iii) If \( \alpha = \eta \), then
\[
K = \mathbb{Q}(\sqrt{2}), \quad K(\eta) = K(\sqrt{-1}), \quad \eta = \pm (\sqrt{2} \mp \sqrt{-2}).
\]
We shall show that the equation \( a^2 = m\eta^{-1} \) is impossible. Taking \( a^* \) instead of \( a \), if necessary, we may assume
\[
a^2 = \pm \frac{m}{2}(\sqrt{2} \mp \sqrt{-2}) = \pm \frac{m\sqrt{2}}{2}(1 \pm \sqrt{-1}).
\]
Furthermore, multiplying \( a \) by \( \sqrt{-1} \), if necessary, we may assume
\[
a^2 = \frac{m\sqrt{2}}{2}(1 + \sqrt{-1}).
\]
As before, we can write \( a = \frac{1}{2}(x + y\sqrt{-1}), x, y \in \mathbb{O} \). Then
\[
4a^2 = a^2 + 2xy(\sqrt{-1}) = 2m\sqrt{2}(1 + \sqrt{-1}).
\]
Hence \( a^2 = y^2 = 2m\sqrt{2}, xy = m\sqrt{2} \), which implies \( a^2 - y^2 = 2xy \). Then \( x = y(1 \pm \sqrt{2}), xy = y^2(1 \pm \sqrt{2}) = m\sqrt{2} \). Taking norms, we obtain \( n_{K_p}(y^2) = 2m^2 \), an impossibility.

The following lemma is well known (cf. [14], p. 354; [15], p. 35).

**Lemma 2.** Let \( k \) be a local field with prime ideal \( p \). Let \( \mathfrak{O}_k \) be an Eichler order of level \( \mathfrak{p} \) in \( \mathfrak{M}(2, k) \). Suppose \( a \in \mathfrak{O}_k \) and \( N(a) \) has \( p \)-order equal to \( 1 \). If \( p \) divides \( T(a) \), the reduced trace of \( a \), then \( a \in \mathfrak{R}(\mathfrak{O}_k) \).

**Lemma 3.** Suppose \( a \in \mathfrak{O} \). In order that \( a \in \mathfrak{R}(\mathfrak{O}) \) it is sufficient that \( (a) \simeq -m, m \mid \mathfrak{D}, a = \zeta, a = 1 + \sqrt{-1} \) if \( 2 \mid \mathfrak{D}, a = \zeta \sqrt{-3} \) if \( 3 \mid \mathfrak{D} \).

(b) \( a \in \mathfrak{Q}_p \) for all primes \( p \) of \( K \) dividing \( N(a) \).

Proof. It is enough to show that \( a \in \mathfrak{R}(\mathfrak{Q}_p) \) for every finite prime \( p \) of \( K \). Since \( N(a) \) is square-free and \( N(a) \mid N(a) \), in each case, Lemma 2 implies that \( a \in \mathfrak{R}(\mathfrak{Q}_p) \) for every \( p \parallel N(a) \). If \( p \nmid N(a) \), then \( a \in \mathfrak{U}(\mathfrak{Q}_p) \). If \( p \mid N(a), D \), let \( \pi \) be a generator of \( p \). Then \( (b) \) implies that \( a \simeq \pi \), where \( \pi \in \mathfrak{U}(\mathfrak{Q}_p) \). Hence \( a \in \mathfrak{R}(\mathfrak{Q}_p) \).

**Proposition 1A.** Suppose \( a \notin \mathfrak{K}^K \), \( a \in \mathfrak{K}^K \). If \( \Delta > 5 \), then in order that \( a \in \mathfrak{R}(\mathfrak{Q}) \) for some Eichler order \( \mathfrak{Q} \) of level \( \Delta \) in \( \mathfrak{K} \) it is necessary and sufficient that one of the following holds: \( a \simeq -m, m \mid \mathfrak{D} \); \( a \simeq \zeta, a = 1 + \sqrt{-1} \) if \( 2 \mid \mathfrak{D} \); \( a \simeq \zeta \sqrt{-3} \) if \( 3 \mid \mathfrak{D} \).

Proof. The necessity follows from Lemma 1 and the discussion (i), (ii), (iii). To prove sufficiency there is no loss of generality in replacing each \( a \) by \( a \). Then, according to Lemma 3, it is enough to choose \( \mathfrak{O} \) so that \( a \in \mathfrak{O} \) and (b) holds. Let \( p \mid N(a), D \) and let \( \pi \) be a generator of \( p \). Since \( a \in \mathfrak{O} \) is integral over \( \mathfrak{O} \) in all cases, we can assume (b) of Lemma 3 by insisting that \( \mathfrak{O} \) be a maximal order containing \( a \pi \). Hence it is enough to find an Eichler order \( \mathfrak{Q} \) of level \( \Delta \) such that \( a \in \mathfrak{Q} \). According to the criterion of Eichler ([8], p. 133), we need only show that no prime \( p \) dividing \( \Delta \) remains prime in \( K(a) \). The only way that this could happen is if \( p \) remains prime in \( K(a) \), a quadratic extension of \( K \). But then we would have \( K(a) = K_p \), the unique unramified extension of degree 2 over \( K_p \). This would imply \( a \in K_p, K(a)_p = K(a) \otimes K_p \cong K_p \otimes K_p \). Hence \( p \) would split in \( K(a) \).

**Remark.** If \( A = A_p = 5 \), then \( a \) normalizes an Eichler order of level 1 if and only if \( a \simeq \sqrt{-1}, \zeta \) or \( \omega \), a fifth root of unity (note that \( \sqrt{-5} \simeq \sqrt{-1} \)).

**§ 8. The Selberg Trace Formula.** In this section we complete the derivation of formulas (50) and (53). We regard the type number \( t \) as the trace of a certain convolution operator and then use the Selberg trace formula to express this trace as a finite sum of integrals over adele homo-
genuine spaces associated to $G = \mathfrak{G}_K$. This sum is indexed by the conjugacy classes in $G$ of elements which normalize Eichler orders of level $\delta$. Explicit representatives for these conjugacy classes were found in §7. To determine the contribution of each representative, we apply the Chevalley–Hasse–Noether Theorem. As a result, the evaluation of each contribution is reduced to the computation of a sum of local unit indices for certain “admissible orders” coming from a fixed imaginary quadratic extension of $K$ (Proposition 13). The admissible orders and their corresponding unit indices are then readily determined (cf. Proposition 16) to yield our formulas.

We use the notation of §5, allowing $\Omega$ to denote an arbitrary Eichler order of level $\delta$ in $\mathfrak{M}_K$. In addition, we introduce $G(\Omega) = \mathfrak{R}(\Omega)/G^K = G^\delta(\hat{\mathcal{O}})$ as a set of global elements with $\gamma$ constant on double cosets $G\gamma G^\delta(\hat{\mathcal{O}})$, $\gamma \in G^\delta(\hat{\mathcal{O}})$. Then $G(\Omega) = G^\delta(\hat{\mathcal{O}})/G^\delta(\hat{\mathcal{O}})$ is a complex vector space of dimension $t_\delta$. Let $G^\delta(\hat{\mathcal{O}})$ denote the characteristic function of $G^\delta(\hat{\mathcal{O}})$. Then, with respect to the measure $\lambda$, the convolution operator $f \mapsto F_B * f$, $f \mapsto I_\delta(\gamma \cdot G^\delta(\hat{\mathcal{O}}))$ is the identity mapping. Hence its trace $Tr(F_B)$ is equal to $t_\delta$. Fix a representative $s$ from each conjugacy class of $G$ and denote the centralizer of $s$ in $G$ by $G(s)$. Applying the Selberg trace formula (cf. [19], [20]), we obtain

$$t_\delta = \text{Tr}(F_B) = \sum_{s \in G^\delta(\hat{\mathcal{O}})/G^\delta(\hat{\mathcal{O}})} \int_{G^\delta(\hat{\mathcal{O}})} \psi_s(g') d\lambda(g')$$

where $\psi_s(g') = F_B(s^{-1}g')$, $g' \in G^\delta(\hat{\mathcal{O}})$.

If $\gamma \in J^\delta_K$, we put $\gamma \cdot G^\delta(\hat{\mathcal{O}}) = \gamma \cdot G^\delta(\hat{\mathcal{O}}) \cap \mathfrak{M}_K$. A representative $s$ makes a non-zero contribution to the trace sum (62) if and only if $g^{-1}sg \in G^\delta(\hat{\mathcal{O}})$ for some $g \in G^\delta(\hat{\mathcal{O}})$. Let $a \in \mathfrak{M}_K$ represent $s$, and $\gamma \in J^\delta_K$ represent $g$. Then $g^{-1}sg \in G^\delta(\hat{\mathcal{O}}) \Rightarrow s \in G^\delta(\gamma \cdot G^\delta(\hat{\mathcal{O}})) \Rightarrow s \in G(\gamma \cdot G^\delta(\hat{\mathcal{O}}))$, since $s \in G$. We conclude that $s$ will make a non-zero contribution to (62) if and only if $a$ normalizes some Eichler order of $\mathfrak{M}_K$ of level $\delta$. Thus, to evaluate (62), we only need to have a range over the finite set of values consisting of 1 and the ones specified in Proposition 14. If $a = 1$, then $s$ contributes $\lambda(G^\delta(\hat{\mathcal{O}}))$ to the trace sum. This contribution can be evaluated by means of (38), (40), (41), and accounts for the leading term in (50) and (53).

Now suppose $a \neq K^K$. Put $K_a = K(a)$, $\theta_a = \text{the ring of integers in } K_a$. Suppose $g^{-1}sg \in G^\delta(\hat{\mathcal{O}})$ for $g \in G^\delta(\hat{\mathcal{O}})$. If $\gamma \in J^\delta_K$ represents $g$, then we have seen above that $a \in \mathfrak{R}(\gamma \cdot G^\delta(\hat{\mathcal{O}}))$. Since $a$ is integral, this implies $a \in \mathfrak{R}(\gamma \cdot G^\delta(\hat{\mathcal{O}}))$, hence $a \in \gamma \cdot G^\delta(\hat{\mathcal{O}})$, an order of $K_a$. Any order of $K_a$ arising in this manner will be called admissible for $a$. Equivalently, an order $\mathfrak{O}$ of $K_a$ is admissible for $a$ if $a \in \mathfrak{O}$ and, for some Eichler order $\mathfrak{O}$ of level $\delta$, $\mathfrak{O} = \mathfrak{O} \cap K_a$ and $a \in \mathfrak{O}(\mathfrak{O})$. Let $\mathfrak{O}$ be admissible for $a$. We denote by $\langle \Omega | \mathfrak{O} \rangle$ the set of all $\gamma \in J^\delta_K$ such that $\gamma \cdot G^\delta(\hat{\mathcal{O}})$, $\mathfrak{O} = \mathfrak{O} \cap K_a$. Then the contribution of $s$ to the trace sum is

$$\sum_{\gamma \in \mathfrak{O}} \lambda\{G(s) \cap \langle \Omega | \mathfrak{O} \rangle / J^\delta_K\},$$

the sum being taken over all orders $\mathfrak{O}$ which are admissible for $a$. It is easy to see that

$$[G(s); K_a / K^K] = \begin{cases} 2 & \text{if } a \text{ is pure}, \\ 1 & \text{if } a \text{ is not pure}. \end{cases}$$

Let $J_{K_a}$ be the ideal group of $K_a$. We regard $J_{K_a}$ as a subgroup of $J^\delta_K$ in the usual way, and put $J_{K_a} = J_{K_a} \cap J^\delta_K$. Then the Chevalley–Hasse–Noether Theorem shows that

$$\langle \Omega | \mathfrak{O} \rangle = J_{K_a} \gamma \cdot G(\Delta),$$

where $\gamma_0$ is a fixed element of $\langle \Omega | \mathfrak{O} \rangle$ (cf. [17], §8, Proposition 8). Let $t_a$ denote the number of primes $p \mid \delta$ which ramify in $K_a$ but do not divide the conductor of $\mathfrak{O}$. Putting $\Omega_0 = \gamma \cdot G(\Delta)$ and arguing as in §8 of [17], we see that $\langle \Omega | \mathfrak{O} \rangle$ is equal to $2^{t_a}$ disjoint right translates of $J_{K_a} \cdot U^1(\hat{\mathcal{O}})$. Therefore, we have

$$\lambda\{K_a / K^K \cap \langle \Omega | \mathfrak{O} \rangle / J_{K_a} \cdot U^1(\hat{\mathcal{O}})\} = 2^{t_a-\delta} \lambda\{K_a / K^K \cap J_{K_a} \cdot U^1(\hat{\mathcal{O}})\}$$

$$= 2^{t_a-\delta} \lambda\{K_a \cap J_{K_a} \cdot U^1(\hat{\mathcal{O}})\}$$

$$= \frac{2^{t_a-\delta}}{h(K)} \lambda\{K_a \cap J_{K_a} \cdot U^1(\hat{\mathcal{O}})\}.$$
Let $E_K, E_a, E(\mathfrak{c})$, denote the unit groups of $\mathcal{O}, \mathfrak{c}_a, \mathfrak{c}$, respectively, and put $U_1 = U^1(\mathfrak{c})$. Then
\begin{align*}
E_K &= U_K \cap K^x, \quad E_a = U_1 \cap K_a^x, \quad E(\mathfrak{c}) = U^1(\mathfrak{c}) \cap K_a^x, \\
\text{and} \quad (68) \quad &\frac{2^{-l}h(\mathfrak{c})}{h(K)} \frac{h(U^1(\mathfrak{c}))}{h(K)} = \frac{2^{-l}h(\mathfrak{c})}{h(K)} \frac{h(U^1(\mathfrak{c}))}{h(K)} \\
&= \frac{2^{-l}h(\mathfrak{c})}{h(K)[E(\mathfrak{c}) : E_K]}.
\end{align*}
Let $h(K_a)$ denote the class number of $K_a$. It is easy to see that
\begin{align*}
(69) \quad &\frac{h(\mathfrak{c})}{h(K_a)} = \frac{[U_1^* : U_1^*(\mathfrak{c})]}{[E(\mathfrak{c}) : E_K]}, \\
\text{Hence} \quad (70) \quad &\frac{h(K_a)}{h(K)} = \frac{[U_1^* : U_1^*(\mathfrak{c})]}{[E(\mathfrak{c}) : E_K]}.
\end{align*}
Applying (64), we obtain
\begin{align*}
(71) \quad &\frac{h(K_a)}{2h(K)[E(\mathfrak{c}) : E_K]} \sum_{\mathfrak{c}} 2^{-l}g[U_1^* : U_1^*(\mathfrak{c})] \quad \text{if} \quad a \text{ is pure}, \\
(72) \quad &\frac{h(K_a)}{h(K)[E(\mathfrak{c}) : E_K]} \sum_{\mathfrak{c}} 2^{-l}g[U_1^* : U_1^*(\mathfrak{c})] \quad \text{if} \quad a \text{ is not pure},
\end{align*}
where $\mathfrak{c}$ ranges over all orders of $K_a$ which are admissible for $a$.

We denote by $W_a$ the group of roots of unity contained in $K_a$.

**Proposition 15.** Suppose $s$ is represented by a $a$ as in Proposition 14. Then $K_a = K(V^2 - m)$, where $m = 3$ or $m \mid \delta D$, and the contribution of $s$ to the trace sum (62) is
\begin{align*}
(73) \quad &\frac{s(a)h(-m)}{2 \text{card}(W_a)} h(-mD) \sum_{\mathfrak{c}} 2^{-l}g[U_1^* : U_1^*(\mathfrak{c})] \quad \text{if} \quad a \text{ is pure}, \\
(74) \quad &\frac{s(a)h(-m)}{\text{card}(W_a)} h(-mD) \sum_{\mathfrak{c}} 2^{-l}g[U_1^* : U_1^*(\mathfrak{c})] \quad \text{if} \quad a \text{ is not pure},
\end{align*}
where $s(a) = 1$ if $K_a = Q(V^2 - 1, V^2 - 2)$, $s(a) = 2$ if $K_a = Q(V^2 - 1, V^2 - 2)$, and $\mathfrak{c}$ ranges over all orders of $K_a$ which are admissible for $a$.

**Proof.** Proposition 14 shows that $K_a = K(V^2 - m) = Q(V^2 - m, V^2 - mD)$, where $m = 3$ or $m \mid \delta D$. The classical formula of Bachmann ([17], p. 74) for the class number of an imaginary bicyclic biquadratic number field shows that
\begin{equation}
(75) \quad \frac{h(K_a)}{h(K)} = \frac{s(a)}{2} \frac{[E_a : W_a E_K]}{[E(\mathfrak{c}) : E_K]} h(-mD).
\end{equation}
Noting that
\begin{equation*}
\frac{[E_a : W_a E_K]}{[E(\mathfrak{c}) : E_K]} = \frac{2}{\text{card}(W_a)}
\end{equation*}
and applying the lemma, we obtain the result.

Remark. In our final class number formulas no special consideration will be necessary for the case where $K_a = Q(V^2 - 1, V^2 - 2)$. This is because $Q(V^2 - 1, V^2 - 2)$ has twice as many roots of unity as $Q(V^2 - 1, V^2 - m)$ if $m \neq 2, m \mid \delta D$, which cancels the doubling of the factor $s(a)$.

To explicitly determine the contributions (73, 74) we must carry out two steps. First, we must determine all admissible orders $\mathfrak{c}$ for each $a$ as given in Proposition 14. Second, we must evaluate the unit index $[U_1^* : U_1^*(\mathfrak{c})]$ for each such $\mathfrak{c}$. The first step is an application of Proposition 13. Since $[U_1^* : U_1^*(\mathfrak{c})] = \prod_{\mathfrak{c} \neq \mathfrak{c}_0} [U(\mathfrak{c}_0 : U(\mathfrak{c})]$, the second step is reduced to the computation of local indices $[U(\mathfrak{c}_0 : U(\mathfrak{c})]$ for finite $p$. Such a computation is a special case of the following elementary proposition, whose proof we omit.

**Proposition 16.** Let $k$ be a local field of characteristic $\neq 2$, with ring of integers $\mathfrak{o}$ and prime ideal $p$. Put $q = \text{card}(\mathfrak{o}/p)$. Let $k_a = k + \mathfrak{k}_a$, where $a \in k, a = k$. Denote the maximal order of $k_a$ by $\mathfrak{c}_a$ and the unique order of $k_a$ of conductor $q^i, i \geq 1$, by $\mathfrak{c}_a$. Then $[U(\mathfrak{c}_a) : U(\mathfrak{c}_0)] =$
\begin{align*}
(\text{a}) \quad &q^{i-1}(q+1) \text{ if } \mathfrak{k}_a \text{ is an unramified field extension of } k, \\
(\text{b}) \quad &q^i \text{ if } \mathfrak{k}_a \text{ is a ramified field extension of } k, \\
(\text{c}) \quad &q^{i-1}(q-1) \text{ if } \mathfrak{k}_a \text{ is a split extension of } k.
\end{align*}

In order to compute $l_0$, we note that $p_i$ ramifies in $K_a$ if and only if $p_i$ ramifies in $Q(a), i = 1, \ldots, e$. It follows that $l_0$ is the number of $p_i, i = 1, \ldots, e$, which ramify in $Q(a)$ but do not divide the conductor of $\mathfrak{c}$.

Remark. It should be noted that the maximal order $\mathfrak{c}_a$ is always admissible for $a$. This is essentially what we showed when we applied Siegel's criterion ([16], p. 133) in the proof of Proposition 14.

Put $(1, a) = \mathfrak{c} + \mathfrak{c}_a$. Assume first that $(1, a)$ is odd. Then $D = r_1 \ldots r_t$, where $r_j$ is a prime, $r_j \equiv 1 \pmod{4}$, $j = 1, \ldots, t$. Let $r_1, \ldots, r_t$ be the prime ideals of $K$ such that $r_j^2 = (r_j)$. 

(1) Suppose \( a \) is not pure, so that \( a = \zeta, 1 + V_1, \) or \( \zeta V -3. \) Since \( D \) is odd, the proof of the corollary to Proposition 13 shows that \( (1, a) = \phi_a \) in each of these cases. Thus \( \phi_a \) is the only admissible order in each case.

(i) \( a = \zeta. \) Then \( I_q = 0 \) if \( 3 \nmid \delta, I_q = 1 \) if \( 3 \mid \delta. \) Applying (74), we see that \( a \) contributes

\[
\frac{h(-3D)}{6} \text{ if } 3 \nmid \delta, \\
\frac{h(-3D)}{12} \text{ if } 3 \mid \delta.
\]

(ii) \( a = 1 + V_1. \) Then \( 2 \mid \delta, I_q = 1, \) and the contribution of \( a \) is

\[
\frac{h(-D)}{8}.
\]

(iii) \( a = \zeta V -3. \) Then \( 3 \mid \delta, I_q = 1, \) and the contribution is

\[
\frac{h(-3D)}{12}.
\]

Now suppose that \( a \) is pure, \( a = V_1 - m, \) where \( m \mid D. \) Let us write \( m = \pi d, \) where \( n \mid \delta, d \mid D. \) We must consider two separate cases:

(2) If \( m = 1, 2 \pmod{4}, \) then, using the notation of Proposition 13, we have \( A(-m) = -4\pi d, A(-mD) = -4\pi D + d, \) \( n_{\pi d} \circ (A_{\pi D}) \equiv 10n. \) Hence \( A_{\pi D} = (-4n), \) which implies that \( (1, a) \) has conductor \( \prod r \mid D. \) Suppose \( r \) is a prime ideal dividing \( d. \) Let \( x \) be a local generator for \( r. \) Then, for any \( \mathcal{E} \) of \( \delta \) of level \( \delta, \) we have \( \alpha \in \mathcal{R}(\mathcal{O}_D), \) and if only if \( \alpha / \mathcal{E} \in \mathcal{R}(\mathcal{O}_D) \).

It follows that \( \phi_a \) is the only admissible order for \( a. \)

(i) \( m = 1 \pmod{4}. \) Then \( I_q = \lambda(n) \) if \( 2 \nmid \delta, I_q = \lambda(n) + 1 \) if \( 2 \mid \delta. \) Hence, according to (73), the contribution of \( a \) is

\[
\frac{h(-D)}{8} \text{ if } m = 1, 2 \nmid \delta, \\
\frac{h(-D)}{16} \text{ if } m = 1, 2 \mid \delta.
\]

(3) If \( m = 3 \pmod{4}, \) then \( A(-m) = -\pi d, A(-mD) = -nD + d, \) \( n_{\pi d} \circ (A_{\pi D}) \equiv 10n. \) Hence \( A_{\pi D} = (-n), \) from which it follows that \( (1, a) \) has conductor \( \prod r \mid D. \) We conclude that the conductor of an admissible order for \( a \) must divide (2). In particular, it follows that \( I_q = \lambda(n) \) for any admissible \( \phi. \) Since \( a \) is a unit in \( K_{a,p}, \) for any \( p \) dividing (2), any order of conductor dividing (2) must in fact be admissible for \( a. \)

(i) \( D = 5 \pmod{8}. \) Then \( 2 \) remains prime in \( K, K_2 = Q(a)_2 \) is the unique unramified quadratic extension of \( Q. \) It follows that \( K_{a,p} \) is a split extension of \( K_2. \) Let \( \phi_a \) be the order of \( K_2 \) of conductor (2). Applying (c) of Proposition 16, we see that \( [U^2_2; U^2_2(\phi_a)] = [2^2 - 1 = 3. \) Thus \( a \) contributes

\[
\frac{h(-3D)}{6} \text{ if } m = 3, \\
\frac{h(-3D)}{12} \text{ if } m > 3.
\]

(ii) \( D = 4 \pmod{8}. \) Then \( 2 \) splits in \( K, (2) = p_1, p_2, K_2 = K_{p_1} \circ K_{p_2} \equiv Q \circ Q_2. \) Let \( \phi_1, \phi_2, \phi_3, \phi_4 \) denote the orders of \( K_2 \) of conductor \( p_1, p_2, p_3, p_4, \) respectively.

(ii) Suppose \( m = 3 \pmod{8}. \) Then \( K_{a,p_1}, K_{a,p_2} \) are unramified quadratic extensions of \( K_{a_1}, K_{a_2}, \) respectively. Applying (a) of Proposition 16, we see that

\[
[U^2_2; U^2_2(\phi_a)] = [U^2_2; U^2_2(\phi_a)] = 3, \quad [U^2_2; U^2_2(\phi_a)] = 9.
\]

Hence the contribution of \( a \) is

\[
\frac{h(-3D)}{6} \text{ if } m = 3, \\
\frac{h(-3D)}{12} \text{ if } m > 3.
\]

(3) Suppose \( m = 7 \pmod{8}. \) Then \( K_{a,p_1}, K_{a,p_2} \) are split extensions of \( K_{a_1}, K_{a_2}, \) respectively. Applying (c) of Proposition 16, we see that

\[
[U^2_2; U^2_2(\phi_a)] = [U^2_2; U^2_2(\phi_a)] = [U^2_2; U^2_2(\phi_a)] = 1.
\]

Hence the contribution of \( a \) is

\[
2^{-\pi a} h(-nD/d) \text{ if } m > 3.
\]

Adding the contributions of \( \sqrt{-1} \) and \( 1 + \sqrt{-1} \) (if any), we obtain the term \( q_1 h(-D) \) of (50). Adding the contributions of \( \zeta, \zeta \sqrt{-3}, \sqrt{-3}, \) we obtain the term \( q_1 h(-3D) \) of (50). Note that \( \sqrt{-nD} \cong \sqrt{-1}. \) Hence we may assume that \( a = \sqrt{-nD} \) with \( d < \sqrt{-D}. \) Then the remaining terms in (50) are accounted for by (82), (83), (84), (85), (86), (88) and (89).
Now assume that $D$ is even. Then $D = 2 \tau_1 \ldots \tau_t$, where $\tau_j$ is a prime, $\tau_j \equiv 1 \pmod{4}$, $j = 1, \ldots, t$. Let $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the prime ideals of $K$ such that $\mathfrak{p}^2 = (2), \mathfrak{p}_j^2 = (\tau_j), j = 1, \ldots, t$. Let $\pi$ denote a local generator of $\mathfrak{p}^2$.

(1) Suppose $a$ is not pure.

(i) $a = \xi$. Then $\mathfrak{c}_a$ is the only admissible order, with contribution

$$
\frac{h(-3D)}{6} \quad \text{if} \quad 3 \mid \delta,
$$

$$
\frac{h(-3D)}{12} \quad \text{if} \quad 3 \not\mid \delta.
$$

(ii) $a = \xi \sqrt{-3}$. Once again $\mathfrak{c}_a$ is the only admissible order, with contribution

$$
\frac{h(-3D)}{12}.
$$

(iii) $a = 1 + \sqrt{-1}$. The proof of the corollary to Proposition 13 shows that $(1, a)$ has conductor $\mathfrak{p}$. Then $1, a/\pi$ generate $\mathfrak{c}_{a/\pi}$, which shows that $\mathfrak{c}_a$ is the only admissible order. Hence the contribution is

$$
\frac{h(-D)}{4}.
$$

Suppose $a$ is pure, $a = \sqrt{-m}$, $m = nd$, $n \mid \delta$, $d \mid D$. Since $\sqrt{-nd} \approx \sqrt{-nD/d}$ and $D$ is even, we may assume that $d$ is odd. It follows, in particular, that $m$ is odd.

(2) If $m \equiv 1 \pmod{4}$, then $\Delta(-m) = -4nd$, $\Delta(-mD) = -4nd/d$, which implies $\Delta_{L/K}(-2\omega)$. Hence $\mathfrak{p}$ ramifies in $K$, and the conductor of $(1, a)$ is $\mathfrak{p} \prod \mathfrak{p}_j$. The fact that $a$ is a unit in $K$, implies that the admissible orders for $\mathfrak{a}$ are those whose conductor divides $\mathfrak{p}$. In particular, $l_0 = \lambda(n)$ for any admissible $\mathfrak{c}$. Let $\mathfrak{c}_a$ be the order of conductor $\mathfrak{p}$ in $K$. Applying (b) of Proposition 16, we see that $[U_{l_0}^1 : U^1(\mathfrak{c}_a)] = 2$. We conclude that the contribution of $a$ is

$$
\frac{\sqrt{2} h(-D)}{6} \quad \text{if} \quad m = 1,
$$

$$
3 \cdot 2^{-\lambda(n) - 1} h(-nd) h(-nD/d) \quad \text{if} \quad m \geq 5.
$$

(3) If $m \equiv 3 \pmod{4}$, then $\Delta(-m) = -nd$, $\Delta(-mD) = -nd/d$, $\Delta_{L/K} = (n)$. It follows that $(1, (1 + a)/2)$ has conductor $\mathfrak{p} \prod \mathfrak{p}_j$ and $(1, a)$ has conductor $2 \prod \mathfrak{p}_j$. Reasoning as before, we see that the admissible orders are those whose conductor divides $(2) = \mathfrak{p}^2$, and $l_0 = \lambda(n)$ for any admissible $\mathfrak{c}$. Let $\mathfrak{c}_a, \mathfrak{c}_b$ be the orders of $K$ of conductor $\mathfrak{p}^2$, $\mathfrak{p}^2$, respectively.

(i) $m \equiv 3 \pmod{8}$. Then $\mathfrak{p}$ remains prime in $K$, and (a) of Proposition 16 shows that

$$
[U_{l_0}^1 : U^1(\mathfrak{c}_a)] = 3, \quad [U_{l_0}^1 : U^1(\mathfrak{c}_b)] = 6.
$$

Hence the contribution of $a$ is

$$
\frac{5}{12} h(-3D) \quad \text{if} \quad m = 3,
$$

$$
5 \cdot 2^{-\lambda(n) - 1} h(-nd) h(-nD/d) \quad \text{if} \quad m > 3.
$$

(ii) $m \equiv 7 \pmod{8}$. Then $\mathfrak{p}$ splits in $K$, and (e) of Proposition 16 shows that

$$
[U_{l_0}^1 : U^1(\mathfrak{c}_a)] = 1, \quad [U_{l_0}^1 : U^1(\mathfrak{c}_b)] = 2.
$$

We conclude that $a$ contributes

$$
2^{-\lambda(n) - 1} h(-nd) h(-nD/d).
$$

Compiling the contributions (90)–(98), we account for all the terms appearing in (53).

Remark. The exceptional case $A = A_L = 5$ can also be handled by the Selberg trace formula. In this case a complete set of $a$ with non-zero contribution to the trace sum is given by $1, \sqrt{-1}, \xi, \omega, \omega'$, where $\omega, \omega'$ are two primitive fifth roots of unity which are not conjugate over $Q(\sqrt{5})$. The corresponding contributions are $2M(S) = 1/60, 1/4, 1/3, 1/5, 1/5$, respectively. Hence $H = 1/60 + 1/4 + 1/3 + 2/5 = 1$, as it should be.

We conclude with the following small table, which bears comparison with the table in [31], p. 148–148.
On the zeros of Dirichlet $L$-functions (VI)

by

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§ 1. Here we will see a $q$-analogue of the author's previous work [4]. We will quote this by (V).

Let $L(s, \chi)$ be a Dirichlet $L$-function with a character $\chi$ to modulus $q$. We write a nontrivial zero of $L(s, \chi)$ by $\gamma_q = \beta(s) + i\gamma(s)$. As before for given two Dirichlet $L$-functions $L(s, \chi_0)$ and $L(s, \chi_1)$, we call $\gamma_q$ a coincident zero of $L(s, \chi_0)$ and $L(s, \chi_1)$ if $\gamma_q \neq 0$ with the same multiplicity. We call $\gamma_q$ a noncoincident zero of $L(s, \chi_0)$ and $L(s, \chi_1)$ if $\gamma_q$ is not a coincident zero. We assume the order is given in the set of ordinates of zeros of $L(s, \chi)$ by $0 < \gamma_q(x) < \gamma_{n+1}(x)$. Also in the set $\gamma_q(x_1), \gamma_q(x_2), \ldots, x = 1, 2, \ldots, m = 1, 2, \ldots$ the order is given by

$$\gamma_q(x_1) < \gamma_q(x_2), \quad \text{if} \quad \gamma_q(x_1) < \gamma_q(x_2)$$

and

$$\gamma_q(x_1) \leq \gamma_q(x_2) \leq \gamma_{n+1}(x_1) \leq \gamma_{n+1}(x_2) \leq \cdots$$

if

$$\gamma_q(x_1) = \gamma_{n+1}(x_1) = \cdots = \gamma_q(x_2) = \gamma_{n+1}(x_2) = \cdots$$

Now, we are concerned with the following problems, which are similar to problems (i), (ii) and (iii) in (V).

(i) *Have different primitive $L$-functions $L(s, \chi_0)$ and $L(s, \chi_1)$ a coincident zero?

(ii) *For given positive real numbers $t_1$ and $t_2$, and for almost all pairs of primitive characters $(\chi_1, \chi_2)$ does there exist a zero of $L(s, \chi_2)$ in

$$\gamma_q(x_1) \leq \beta(x_1) \leq \gamma_{n+1}(x_1)$$

for each $\gamma_q(x_1)$ in $t_1 < \gamma_q(x_1) < t_2$?

(iii) *For some $\gamma_q(x_1)$, does it happen that $\gamma_q(x_1) < \gamma_q(x_1) < \gamma_{n+1}(x_1)$ for almost all primitive characters $\chi_1$?

Our answers to these are the following theorems.

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