is

$$o(n(\log n)^{-1}).$$

Now the lemma follows from (4.5).

To complete the proof of Theorem 3, first note that for any $k \geqslant 1$

(4.8) $$N(n) \geqslant N(n, k) - A(n, k).$$

Now the theorem follows immediately from (4.8) and Lemmas 5 and 7.

Without much difficulty we could obtain an asymptotic formula for $N(n)$ even if we only assume

$$B(x) = o\left(\frac{x}{\log x \log\log x}\right).$$

We hope to return to this problem on another occasion.

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Some remarks on L-functions and class numbers

by

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§ 1. Let d denote the discriminant of the quadratic field $K = Q(\sqrt{d})$, and let χ denote the associated real primitive character. c_i will denote a positive computable constant. We simplify matters slightly by assuming $|d| > 4$ so that K contains no complex roots of unity. Dirichlet's formulae now give for the class number $h(d)$,

$$h(d) = \begin{cases} \dfrac{|d|^{1/2}}{\pi} L(1, \chi) & \text{for} \quad d < 0, \\[2mm] \dfrac{d^{1/2} L(1, \chi)}{2\log\varepsilon} & \text{for} \quad d > 0, \end{cases}$$

where ε denotes the fundamental unit of K.

Hecke [5] was the first to connect the magnitude of $L(1, \chi)$ with the question of the existence of real zeros of $L(s, \chi)$ near $s = 1$. For those $d < 0$ for which no such zero exists he was able to give a good effective lower bound for $h(d)$.

Recently, Goldfeld [4] has given a simple proof of the celebrated theorem of Siegel [8]. His argument is easily modified to give a simple proof of Hecke's result. Furthermore, if we let a be fixed with $\frac{1}{2} \leqslant a < 1$, then an affective lower bound for $L(1, \chi)$ (depending on a) can be given under the assumption $L(a, \chi) \geqslant 0$. In particular, we have:

(A) *Let* $\frac{1}{2} < a < 1$ *and assume* $L(a, \chi) \geqslant 0$. *Then, there exists* $c_1(a)$ *such that*

$$L(1, \chi) > c_1(a)\,|d|^{a-1}.$$

(B) *Let* $\delta > 0$ *and assume* $L(\frac{1}{2}, \chi) \geqslant 0$. *Then, there exists* $c_2(\delta)$ *such that*

$$L(1, \chi) > c_2(\delta)(\log|d|)^{2-\delta}|d|^{-1/2}.$$

It is to be noted that the bound gets progressively better as a increases, approaching the Siegel bound as a approaches 1.

For $d<0$, the class number formula combines with (A) and (B) to give lower bounds for $h(d)$. Furthermore, for special classes of real quadratic fields, where the regulator is relatively small, results may also be obtained. For instance, we may consider the following conjecture of S. Chowla:

Let p be a prime of the form x^2+1. For $x>26$, $h(p)>1$.

From (B) we are able to prove:

(C) *There exists c_3 such that if $x>c_3$ and $h(p)=1$, then $L(\frac{1}{2},\chi)<0$.*

Although the case $a=\frac{1}{2}$ is seen to give the worst bound, it is nevertheless of interest for other reasons.

In [7], Selberg and Chowla proved that if there exists a tenth solution of $h(d)=1$ with $d<0$ (a possibility since disproved by Stark [9] and Baker [2]), then $L(\frac{1}{2},\chi)<0$. This type of result can now be extended to larger class numbers. Using a theorem of Tatuzawa [10], we get:

(D) *Let $d<0$ and let h_0 be a positive integer. There exists $c_4(h_0)$ such that there is at most one solution of $h(d)\leqslant h_0$ with $d<-c_4(h_0)$ and that, if such a solution d exists, then for that d $L(\frac{1}{2},\chi)<0$.*

Another interesting point about the case $a=\frac{1}{2}$ is the recent discovery by Armitage [1] that certain Artin L-functions vanish at $\frac{1}{2}$. This leads one to hope that perhaps, by using one of these, a function might be constructed which could give an effective lower bound. This idea has been used recently by Friedlander [3] to give an effective lower bound for the class numbers of totally imaginary quadratic extensions of certain totally real fields (those whose Dedekind zeta-functions vanish at $\frac{1}{2}$).

The proofs of the above results, as will be seen, involve a simple application of Cauchy's theorem. By modifying the integrand it is possible to give easy proofs of other results. By way of example, we give simple proofs of the following well-known results.

(E) $L(1,\chi)>0$.

(F) *If $d>0$, $\sum_{m\leqslant d} m\chi(m)<0$.*

§ 2. We shall have need of:

Lemma. (I) *Let $-\frac{1}{2}\leqslant\sigma<0$.*

$$|\zeta(\sigma+it)L(\sigma+it,\chi)|\leqslant c_5(\sigma)|d|^{1/2-\sigma}(1+|t|)^{1-2\sigma}.$$

(II) *Let $0<\sigma\leqslant\frac{1}{2}$.*

$$|\zeta(\sigma+it)L(\sigma+it,\chi)|\leqslant c_6(\sigma)|d|^{1/2}(2+|t|).$$

(III) $|L(-\frac{1}{4}+it,\chi)|\leqslant c_7|d|^{3/4}(1+|t|)^{3/4}$.

Proof. These are special cases of Theorems 3 and 4 of [6].

Proof of (A). We consider the integral

$$I=\frac{1}{2\pi i}\int_{(2)}\zeta(s+a)L(s+a,\chi)\frac{x^s}{s(s+1)(s+2)}\,ds=\frac{1}{2}\sum_{m\leqslant x}\frac{b(m)}{m^a}\left(1-\frac{m}{x}\right)^2$$

where $b(m)$ denotes the number of integral ideals of $Q(\sqrt{d})$ having norm m.

Since $b(1)=1$ and $b(m)\geqslant 0$, we have, for $x\geqslant 2$, $I\geqslant 1/8$.

On shifting the line of integration to $\sigma=-\left(\frac{1+a}{2}\right)$, we get

$$I=\frac{L(1,\chi)x^{1-a}}{(1-a)(2-a)(3-a)}+\frac{\zeta(a)L(a,\chi)}{2}+\\+\frac{1}{2\pi i}\int_{\left(\frac{-1-a}{2}\right)}\zeta(s+a)L(s+a,\chi)\frac{x^s}{s(s+1)(s+2)}\,ds.$$

Using the first part of the lemma we can bound the absolute value of this latter integral as

$$\leqslant c_8(a)|d|^{1-\frac{a}{2}}x^{\frac{-1-a}{2}}.$$

Choosing $x=|d|$, this is dominated by $1/16$ for $|d|>d_0(a)$. Using the hypothesis $L(a,\chi)\geqslant 0$, the result follows for $|d|>d_0(a)$ and adjusting the constant gives the result for all d.

Proof of (B). We use the above argument with the following differences. Since $b(m^2)\geqslant 1$, we have

$$I\geqslant\frac{1}{2}\sum_{m\leqslant x^{1/2}}\frac{1}{m}\left(1-\frac{m^2}{x}\right)^2$$
$$\geqslant\frac{1}{2}\sum_{m\leqslant x^{1/4}}\frac{1}{m}\left(1-\frac{1}{x^{1/2}}\right)^2$$
$$\geqslant\frac{1}{4}\sum_{m\leqslant x^{1/4}}\frac{1}{m}\qquad(\text{for } x\geqslant 16)$$
$$\geqslant\frac{1}{16}\log x.$$

Assume, without loss of generality that $\delta\leqslant\frac{1}{2}$. Choosing

$$x=\frac{|d|}{(\log|d|)^{2(1-\delta)}},$$

and shifting the line of integration to $\sigma=-\frac{1}{2}(1+\delta)$, the result follows as before.

Proof of (C). It is easily seen that in $Q(\sqrt{p})$, $x+\sqrt{p}$ is the fundamental unit, whence

$$\log\varepsilon < \log 2\sqrt{p} < \log p \quad (\text{for } p>2).$$

From the class number formula it follows that

$$h(p) > \tfrac{1}{2}\sqrt{p}L(1,\chi)/\log p$$

and, applying (B), with any fixed $\delta<1$, gives the result.

Proof of (D). By Theorem 3 of [10], there exists $c_9(h_0)$ such that $d<-c_9(h_0)$, $h(d)\leqslant h_0$, has at most one solution. Applying (B) with, say, $\delta=1$, there exists $c_{10}(h_0)$ such that $d<-c_{10}(h_0)$, $h(d)\leqslant h_0$, implies $L(\frac{1}{2},\chi)<0$. Choosing $c_4=\max\{c_9,c_{10}\}$, the result follows immediately.

Proof of (E). We consider the integral,

$$I=\frac{1}{2\pi i}\int_{(2)}\zeta(s)L(s,\chi)\frac{x^s}{s(s+1)(s+2)}\,ds=\frac{1}{2}\sum_{m\leqslant x}b(m)\left(1-\frac{m}{x}\right)^2\geqslant c_{11}x^{1/2}$$

$$(\text{since } b(m^2)\geqslant 1).$$

On shifting the contour to $\sigma=\delta$ where $\delta>0$ is small, and applying the second part of the lemma with $x=c_{12}(\delta)|d|^{1/(1-2\delta)}$, for suitable c_{12},

$$L(1,\chi)>c_{13}(\delta)|d|^{1/(4\delta-2)}.$$

Proof of (F). We consider the integral,

$$I=\frac{1}{2\pi i}\int_{(2)}L(s,\chi)\frac{x^s}{s(s+1)}\,ds=\sum_{m\leqslant x}\chi(m)\left(1-\frac{m}{x}\right).$$

Shifting the contour to $\sigma=-\frac{1}{4}$ and applying the third part of the lemma,

$$I=L(0,\chi)+O(x^{-1/4}d^{3/4}).$$

From (E) and from the functional equation for $L(s,\chi)$, it follows immediately that $L(0,\chi)$ is positive and, moreover, for $x>c_{14}d^4$, $I>0$.

We can choose x to be at least this large and still demand that x be a multiple of d, say $x=dr$. Then,

$$0<\sum_{m\leqslant dr}\chi(m)\left(1-\frac{m}{x}\right)=-\frac{1}{x}\sum_{m\leqslant dr}m\chi(m)$$

whence $\sum_{m\leqslant dr}m\chi(m)<0$.

Since for any integer s,

$$\sum_{sd<m\leqslant(s+1)d}m\chi(m)=\sum_{m\leqslant d}(m+sd)\chi(m+sd)=\sum_{m\leqslant d}m\chi(m),$$

the result follows.

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