On the zeros of Dirichlet $L$-functions (V)

by

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§ 1. Introduction. Here we will see a comparative study of the zeros of Dirichlet $L$-functions.

Let $L(s, \chi)$ be a Dirichlet $L$-function with a character $\chi$ to modulus $g$. We denote a zero of $L(s, \chi)$ by $e(\chi) = \beta(\chi) + i\gamma(\chi)$. And we assume that the order is given in the set of ordinates of zeros by $0 \leq \gamma_n(\chi) \leq \gamma_{n+1}(\chi)$. In the following we always assume that $\chi_1$ and $\chi_2$ are different primitive characters to the same modulus $g$. Now we are asking the following questions.

(i) Have $L(s, \chi_1)$ and $L(s, \chi_2)$ a coincident zero?

Here we call $\mu$ a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\mu, \chi_1) = L(\mu, \chi_2) = 0$ with the same multiplicity. Also we call $\mu$ a noncoincident zero if it is not a coincident zero.

(ii) Does there exist a zero $e(\chi_2) = \beta(\chi_2) + i\gamma(\chi_2)$ of $L(s, \chi_2)$ in

$$0 \leq \gamma_n(\chi_2) \leq \gamma(\chi_2) \leq \gamma_{n+1}(\chi_2)$$

for almost all $\gamma_n(\chi_2)$?

Here we define $\gamma_n(\chi_2) \leq \gamma_m(\chi_2)$ by $\gamma_n(\chi_2) \leq \gamma_m(\chi_2)$ if $\gamma_n(\chi_2) < \gamma_m(\chi_2)$ and $\gamma_n(\chi_2) \leq \gamma_m(\chi_2) \leq \gamma_{m+1}(\chi_2) \leq \gamma_{m+1}(\chi_2) \leq \ldots$ if $\gamma_n(\chi_2) = \gamma_{m+1}(\chi_2) = \ldots \gamma_n(\chi_2) = \gamma_{m+1}(\chi_2) = \ldots$

(iii) Does it happen

$$\gamma_n(\chi_2) \leq \gamma(\chi_2) \leq \gamma_{n+1}(\chi_2)$$

for almost all $n$?

For later purposes we define $\delta_n(\chi_1, \chi_2)$ by $n - m$ if $\gamma_n(\chi_1) \leq \gamma_m(\chi_2) \leq \gamma_{m+1}(\chi_1)$. Then (iii) asks if $\delta_n(\chi_1, \chi_2) = 0$ for almost all $n$.

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To these questions, we can show

**Theorem 1.** Assume that \(\chi_1\) and \(\chi_2\) are different primitive characters to the same modulus \(q\). Then positive proportion of \(\gamma_n(\chi_1)\) does not have \(\gamma_n(\chi_2)\) in \(\gamma_n(\chi_1) \leq t \leq \gamma_{n+1}(\chi_1)\).

In particular,

**Theorem 1'.** Under the same assumption as above positive proportion of zeros of \(L(s, \chi_1)L(s, \chi_2)\) are non-coincident.

Further,

**Theorem 2.** Under the same hypothesis as above:

(i) For positive proportion of \(n\)
\[A_n(\chi_1, \chi_2) > c_4(\log \log n)^{1/3}\]
and also for positive proportion of \(n\)
\[A_n(\chi_1, \chi_2) < -c_4(\log \log n)^{1/3},\]
where \(c_4\) is some positive absolute constant.

(ii) For any positive increasing function \(\Phi(n)\) which tends to \(\infty\) as \(n\) tends to \(\infty\),
\[|A_n(\chi_1, \chi_2)| > \frac{2n\log \log n}{\Phi(n)}
\]
for almost all \(n\).

Hence we see \(\gamma_n(\chi_2)\) almost never lies in \(\gamma_n(\chi_1) \leq t \leq \gamma_{n+1}(\chi_1)\). Theorem 1 comes from mean value theorem of
\[\mathcal{S}(t+h, \chi_1) - \mathcal{S}(t+h, \chi_2) - \mathcal{S}(t, \chi_1) - \mathcal{S}(t, \chi_2)\]
(cf. Lemma 1 in §2), where
\[\mathcal{S}(t, \chi) = \frac{1}{\pi} \arg L\left(\frac{1}{2} + it, \chi\right)\]
as usual. Theorem 2 comes from mean value theorem of \(\mathcal{S}(t+h, \chi_1) - \mathcal{S}(t, \chi_1)\) (cf. Lemma 2 in §2).

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**§ 2. Lemmas.**

2.1. For simplicity we write
\[\Delta_h(\mathcal{S}(t+h, \chi_1) - \mathcal{S}(t+h, \chi_2)) = (\mathcal{S}(t+h, \chi_1) - \mathcal{S}(t, \chi_1)) - (\mathcal{S}(t+h, \chi_2) - \mathcal{S}(t, \chi_2)).\]
We will prove the following two lemmas.

**Lemma 1.** Let \(a_i, a'_i\) be fixed, \(0 < a_i \leq \frac{1}{2}\) for \(i = 1, 2\). Let \(\chi_i\) be a primitive character to modulus \(q_i\) for \(i = 1, 2\), and suppose that \(\chi_1 \neq \chi_2\), \(q_1, q_2 \leq T^{1/4-a_1}, T^{1/2-a_2} < H < T\), and that \(0 < h < \frac{1}{\log \log T}\). Then
\[
\int_{\frac{1}{2}}^{1} \left| A_h(\mathcal{S}(t, \chi_1) - \mathcal{S}(t, \chi_2)) \right|^2 dt
\]
\[
\mathcal{O}\left(\frac{2H^k}{T} + \mathcal{O}\left(\frac{H^k}{\log \log T}\right)^{\frac{k}{k-1}}\right)
\]
and
\[
\mathcal{O}\left(\frac{H^k}{\log \log T}\right)^{\frac{k}{k-1}}
\]
if \(k = 2k - 1\).

**Lemma 2.** Under the same hypothesis as above excluding the hypothesis to \(h\),
\[
\int_{\frac{1}{2}}^{1} \left| A_h(\mathcal{S}(t, \chi_1) - \mathcal{S}(t, \chi_2)) \right|^2 dt
\]
\[
\mathcal{O}\left(\frac{2H^k}{T} + \mathcal{O}\left(\frac{H^k}{\log \log T}\right)^{\frac{k}{k-1}}\right)
\]
if \(k = 2k - 1\).

2.2. We will prove these only for \(k = 2k\). Odd cases come similarly. For simplicity we write
\[|f| = \left(\int_{\frac{1}{2}}^{1} f(t)^2 dt\right)^{1/2k}.
\]
We saw in [1] for \(x\) in \(T^{a_2/k} < x < T^{1/2}\)
(1)
\[\mathcal{S}(t, \chi_1) - \frac{1}{\pi} \text{Im} \sum_{p < x^2} \frac{\chi_1(p)}{p^{1/2+i(x+h)}} \leq \kappa H T^{1/2k}\]
and also
(2)
\[\mathcal{S}(t+h, \chi_1) - \frac{1}{\pi} \text{Im} \sum_{p < x^2} \frac{\chi_1(p)}{p^{1/2+i(x+h)}} \leq \kappa H T^{1/2k}\]
for each \(i = 1, 2\) and for \(h\) in \(0 < h < H - (H/\sqrt{T})^{1/4}\). Also we saw in [1] that for complex numbers \(a(p)\) if \(F_a(x) = \sum_{p < x^2} \frac{a(p)p^{\alpha}}{p^{1/2+i(x+h)}} \leq 1\) for \(\alpha \geq 2\) and \(F_{12}(x) \leq x^c\) with some \(c > 0\), then for \(x = T^{(k+1)/2k}\)
(3)
\[\mathcal{Im} \sum_{p < x^2} \frac{a(p)}{p^{1/2+i(x+h)}} \leq \mathcal{O}\left(\frac{2H^k}{T} + \mathcal{O}\left(\frac{H^k}{\log \log T}\right)^{\frac{k}{k-1}}\right)\]
if \(F_a(x) \rightarrow \infty\) as \(x \rightarrow \infty\), where \(0 < (k-2) = \max\{0, k-2\}\).
2.3. Proof of Lemma 1. We take \( s = T^{1+\alpha} \). Now
\[ \Delta_h[S(t, \chi) - S(t, \chi_0)] = \Delta_h[S(t, \chi) - S(t, \chi_0)] - \frac{1}{\pi} \sum_{p < x} \frac{a(p)}{p^2 - \sigma + it} + \frac{1}{\pi} \sum_{p < x} \frac{a(p)}{p^{\sigma + it}}, \]
where
\[ a(p) = \left( \chi(p) - \chi_0(p) \right) \left( e^{-a \log p} - 1 \right). \]
Hence by (1) and (2)
\[ \left\| \Delta_h[S(t, \chi) - S(t, \chi_0)] \right\| = \left\| \frac{1}{\pi} \sum_{p < x} \frac{a(p)}{p^{\sigma + it}} \right\| + O(\sqrt{h^2 H^2}). \]
Now
\[ \sum_{p < x} \frac{|a(p)|^2}{p} = \sum_{p < x} \frac{|\chi(p) - \chi_0(p)|^2}{p} \left( e^{-a \log p} - 1 \right)^2 \]
\[ = 4 \sum_{p < x} \frac{1 - \cos(k \log p)}{p} - 2 \sum_{p < x} \frac{\chi(p) \chi_0(p)}{p} \left( 1 - \cos(k \log p) \right) - \]
\[ - 2 \sum_{p < x} \frac{\chi(p) \chi_0(p)}{p} \left( 1 - \cos(k \log p) \right) \]
\[ = 4 \sum_{p < x} \frac{1 - \cos(k \log p)}{p} - 4 \sum_{p < x} \frac{1 - \cos(k \log p)}{p} + O(1) \]
by Mertens' Theorem
\[ = 4 \log(k \log x) + O(1) \]
under our assumption on \( k \). Hence by (3) we get
\[ \left\| \Delta_h[S(t, \chi) - S(t, \chi_0)] \right\|^2 = \frac{2k!}{(2\pi)^{2k}} \frac{(\log(3 + k \log T))^k}{H^2} + O(\Delta_h^{4k} H (\log(3 + k \log T))^{2k}). \]

2.4. Proof of Lemma 2. In this case \( a(p) = \chi_0(p) \) and
\[ \sum_{p < x} \frac{|a(p)|^2}{p} = 2 \sum_{p < x} \frac{1}{p} - 2 \sum_{p < x} \frac{1}{p^2} + O(1) \]
by Mertens' Theorem. This is equal to
\[ 2 \log \log x + O(\log \log x) \]
Hence under our assumption on \( x \), taking \( x = T^{1/2 + \epsilon} \), we get our conclusion in the same way as above.

§ 3. Proof of Theorems.

3.1. Let \( N(t, \chi) \) be the number of zeros \( z = \beta + iy \) of \( L(s, \chi) \) in \( 0 < \beta < 1, 0 < y < t \), possible zeros on \( \text{Im} s = 0 \) or \( t \) counted one-half only. As is well known
\[ N(t, \chi) = \frac{1}{2\pi} \log t - \frac{1 + \log 2\pi}{2\pi} - \frac{1}{3} - S(t, \chi) - S(0, \chi) + O\left( \frac{1}{1 + t} \right) \]
for \( t > 0 \).

Hence
\[ \Delta_h[N(t, \chi) - N(t, \chi_0)] = \left( N(t + h, \chi) - N(t, \chi) \right) - \left( N(t + h, \chi_0) - N(t, \chi_0) \right) \]
is essentially \( \Delta_h[S(t, \chi) - S(t, \chi_0)] \). From Lemma 1 we see the following

**Corollary.** (i) Under the same hypothesis as Lemma 2, for \( h = 2\pi C \log T \),
\[ \Delta_h[N(t, \chi) - N(t, \chi_0)] > 4 \log (\log C)^{\frac{1}{2}} (\log \log C)^{\frac{1}{2}} \]
for positive proportion of \( t \) in \( (T, T + H) \) if \( C > C_0 \). Also for \( h = 2\pi C \log T \),
\[ \Delta_h[N(t, \chi) - N(t, \chi_0)] < -4 \log (\log C)^{\frac{1}{2}} (\log \log C)^{\frac{1}{2}} \]
for positive proportion of \( t \) in \( (T, T + H) \) if \( C > C_0 \), where the Lebesgue measure of such \( t \) is at least \( c_0 H \exp\left( - (\log \log C)^{-1} \right) \). \( c_0 \) and \( c_0 \) are some positive absolute constants, and \( c_0 \) is a suitable small positive number.

(ii) Under the same situation as (i) the Lebesgue measure of \( t \) for which \( t \in (T, T + H) \) and
\[ \Delta_h[N(t, \chi) - N(t, \chi_0)] > 4 \log (\log C)^{\frac{1}{2}} \]
is at least \( c_0 H \), where \( c_0 \) is some positive absolute constant which does not depend on \( C \). Same is true for
\[ \Delta_h[N(t, \chi) - N(t, \chi_0)] < -4 \log (\log C)^{\frac{1}{2}} \]

**Proof.** We write \( f(t) = \Delta_h[N(t, \chi) - N(t, \chi_0)] \). From Lemma 1
\[ \int_{-H}^{H} f^{2k-1}(t) dt = \frac{2k!}{(2\pi)^{2k}} \frac{1}{H} (4 \log (3 + k \log T))^k + O\left( (\Delta_h)^{4k} H (\log(3 + k \log T))^{2k-1} \right) \]
and\[ \int_{-H}^{H} f^{2k}(t) dt = O(\Delta_h^{4k} H (\log(3 + k \log T))^{2k-1}). \]
We write \( E_M = \{ t \in (T, T+H) : f(t) > M \} \) for \( M > 0 \). And let \( \varphi_M(t) \) be the characteristic function of \( E_M \). Now
\[
\int_{T}^{T+H} f^{2k+1}(t) \varphi_0(t) \frac{dt}{t} = \int_{T}^{T+H} f^{2k+1}(t) \varphi_M(t) \frac{dt}{t} + \int_{T}^{T+H} f^{2k+1}(t)(1 - \varphi_M(t)) \varphi_0(t) \frac{dt}{t}
\leq \sqrt{|E_M|} \left( \int_{T}^{T+H} f^{2k+1}(t) \frac{dt}{t} \right)^{1/2} + M^{2k+1} H,
\]
where \( |E_M| \) is the Lebesgue measure of \( E_M \). On the other hand, by Hölder inequality
\[
\int_{T}^{T+H} f^{2k+1}(t) \varphi_0(t) \frac{dt}{t} = \frac{1}{2} \int_{T}^{T+H} \left( \int_{T}^{T+H} f^{(2k+1)(2k-1)}(t) \frac{dt}{t} \right)^{1/2} \cdot \left( \int_{T}^{T+H} f^{(2k+1)(2k-1)}(t) \frac{dt}{t} \right)^{1/2} \geq \frac{1}{2} \int_{T}^{T+H} f^{(2k+1)(2k-1)}(t) \frac{dt}{t}.
\]
Hence we get
\[
\sqrt{|E_M|} \left( \int_{T}^{T+H} f^{2k+1}(t) \frac{dt}{t} \right)^{1/2} \geq \frac{1}{2} \int_{T}^{T+H} f^{(2k+1)(2k-1)}(t) \frac{dt}{t} - M^{2k+1} H + O\left( (2k+1)^{2k} H \log(3 + h \log T) \right)^{1/2}.
\]

Taking \( \log T = 2\pi C \) sufficiently large and \( k = [\log(C)^{1 - \epsilon}] \) with some arbitrary small positive \( \epsilon_1 \), we get
\[
|E_M| \geq H \left( A_1 \frac{h^{2k+1}(2k+1)}{2(2k+1)^{2k+1}} - A_2 \frac{h^{2k+1}(2k+1)}{2(2k+1)^{2k+1}} \right)^2 \geq M.
\]

The last condition is \( A_1 \frac{h^{2k+1}(2k+1)}{2(2k+1)^{2k+1}} > M \) with a suitable positive number \( \epsilon_2 \). Hence we get our conclusion (i). (ii) comes similarly if we consider \( f(t) \varphi_0(t) \) instead of \( f^{2k+1}(t) \varphi_0(t) \). Q.E.D.

3.3. In particular for \( h = 2\pi C / \log T \), and for positive proportion of \( t \) in \((T, T+H)\),
\[
\mathcal{A}_n\mathcal{N}(x, \chi) - \mathcal{N}(t, \chi_0) \geq 2.
\]

For such \( t \) in \((t, t + \frac{2\pi C}{\log T})\) there exists a \( \gamma_n(x) \) such that there is no zero of \( \mathcal{L}(s, \chi_2) \) in \( \gamma_n(x) \leq t \leq \gamma_{n+1}(x) \). Hence we get our Theorem 1 as usual.

\[
\sum_{t \leq x} \mathcal{A}_n(x, \chi) = \mathcal{N}(\gamma_n(x), \chi_2) - \mathcal{N}(\gamma_n(x), \chi_1) - \mathcal{S}(\gamma_n(x), \chi_2) - \mathcal{S}(\gamma_n(x), \chi_1) + O\left( \frac{1}{1 + \gamma_n(x)} \right).
\]

Now
\[
\sum_{t \leq x, \chi_2 \leq T+H} \left( \mathcal{S}(\gamma_n(x), \chi_2) - \mathcal{S}(\gamma_n(x), \chi_1) \right)_{2k}^2 = \int_{(T, T+H)} \left( \mathcal{S}(t, \chi_2) - \mathcal{S}(t, \chi_1) \right)_{2k}^2 dN(t, \chi_2)
\leq \int_{(T, T+H)} \left( \mathcal{S}(t, \chi_2) - \mathcal{S}(t, \chi_1) \right)_{2k}^2 dN(t, \chi_2)
\leq \frac{1}{2\pi} \log \frac{qT}{2\pi} \int_{T}^{T+H} \left( \mathcal{S}(t, \chi_2) - \mathcal{S}(t, \chi_1) \right)_{2k}^2 dt + O\left( \left( \log T \right)^{2k+1}/(2k+1) \right)
\leq \frac{1}{2\pi} \log \frac{qT}{2\pi} \int_{T}^{T+H} \left( \mathcal{S}(t, \chi_2) - \mathcal{S}(t, \chi_1) \right)_{2k}^2 dt + O\left( \left( \log T \right)^{2k+1}/(2k+1) \right)
\leq \frac{2k!}{(2\pi)^{2k+1}} \left( \frac{2\pi}{2\pi} \log \frac{qT}{2\pi} \left( \log \log T \right)^k + O\left( \left( \log T \right)^{2k+1}/(2k+1) \right) \right)
\]

Hence we get
\[
\sum_{t \leq x, \chi_2 \leq T+H} \left( \mathcal{A}_n(x, \chi_2) - \mathcal{A}_n(x, \chi_1) \right)_{2k}^2 = \frac{2k!}{(2\pi)^{2k+1}} \left( \log \log T \right)^k + O\left( \left( \log T \right)^{2k+1}/(2k+1) \right).
\]

In particular, we get
\[
\sum_{\chi_2 \leq T+H} \left( \mathcal{A}_n(x, \chi_2) - \mathcal{A}_n(x, \chi_1) \right)_{2k}^2 = \frac{2k!}{(2\pi)^{2k+1}} \left( \log \log T \right)^k + O\left( \left( \log T \right)^{2k+1}/(2k+1) \right).
\]
Similarly, we get
\[
\sum_{n=1}^{N} \left( \mathcal{A}_n(z_1, z_2) \right)^{2k-1} = O(N(\log \log N)^{2k-\frac{1}{2}} (Ab)^{2k}) + O((\log N)^{2k}/2k).
\]

3.4. (i) of Theorem 2 comes similarly as in 3.2 if we use the above mean value estimate \( \sum_{n=1}^{N} \left( \mathcal{A}_n(z_1, z_2) \right)^{l} \) for \( l = 1, 2, 4 \) or comes from 3.5 below.

3.5. We write
\[
F_N(u) = \frac{1}{N} \left| \left\{ N < n \leq 2N; -\infty < \mathcal{A}_n(z_1, z_2) < \frac{w \sqrt{\log \log n}}{2\pi} + O\left( \frac{u}{\sqrt{\log N}} \right) \right\} \right|.
\]
Then from 3.3 we see
\[
\int_{-\infty}^{\infty} u^l dF_N(u) = \mu_l(N)
\]
\[
= \begin{cases} 
\frac{2k!}{k!} + O\left( \frac{(Ab)^{2k}}{\sqrt{\log \log N}} \right) + O\left( \frac{(\log N)^{2k+1}}{N(\log \log N)^{(2k+1)}} \right) & \text{if } l = 2k, \\
O\left( \frac{(Ab)^{2k}}{\sqrt{\log \log N}} \right) + O\left( \frac{(\log N)^{2k}}{(\log \log N)^{2k-\frac{1}{2}} N^{2k}} \right) & \text{if } l = 2k - 1.
\end{cases}
\]

Since
\[
\mu_l(N) \to \mu_l = \begin{cases} 
\frac{2k!}{k!} & \text{if } l = 2k, \\
0 & \text{if } l = 2k - 1
\end{cases}
\]
as \( N \to \infty \) and the distribution function determined by \( \{ \mu_l; l = 0, 1, 2, \ldots \} \) is
\[
\lim_{N \to \infty} F_N(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{u} e^{-\frac{x^2}{4}} \, dx.
\]

(cf. 4.24 and 3.4 of [4]).

Hence for any positive increasing function \( \Phi(u) \) which tends to \( \infty \) as \( n \to \infty \),
\[
|\mathcal{A}_n(z_1, z_2)| > \frac{\sqrt{\log n}}{2\pi \Phi(u)}
\]
for almost all \( n \).

§ 4. Concluding remark. As is seen from 3.1 and 3.2 in § 3, we do not have to assume that \( \chi_1 \) and \( \chi_2 \) have the same modulus in Theorem 1. Namely, the inequality in the Corollary of 3.1 becomes
\[
\mathcal{A}_n(Nt, x_1) - Nt, x_2) \geq \frac{1}{\log T} \frac{O}{\log \frac{q_1}{q_2} + c_2(\log \log T)^{1/2}} (\log \log T)^{1/2+4}
\]
for \( h = 2\pi/\log T \), where \( \chi_2 \) has a modulus \( \Omega \) for \( i = 1, 2 \).

References


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