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On the zeros of Dirichlet L-functions (V)

by

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§ 1. Introduction. Here we will see a comparative study of the zeros of Dirichlet L-functions.

Let $L(s,\chi)$ be a Dirichlet L-function with a character χ to modulus q. We denote a zero of $L(s,\chi)$ by $\varrho(\chi)=\beta(\chi)+i\gamma(\chi)$. And we assume that the order is given in the set of ordinates of zeros by $0\leqslant\gamma_n(\chi)\leqslant\gamma_{n+1}(\chi)$. In the following we always assume that χ_1 and χ_2 are different primitive characters to the same modulus q. Now we are asking the following questions.

(i) *Have $L(s,\chi_1)$ and $L(s,\chi_2)$ a coincident zero?*

Here we call ϱ a *coincident zero* of $L(s,\chi_1)$ and $L(s,\chi_2)$ if $L(\varrho,\chi_1)=L(\varrho,\chi_2)=0$ with the same multiplicity. Also we call ϱ a *noncoincident* zero if it is not a coincident zero.

(ii) *Does there exist a zero $\varrho(\chi_2)=\beta(\chi_2)+i\gamma(\chi_2)$ of $L(s,\chi_2)$ in*

$$0\leqslant\gamma_n(\chi_1)\leqslant\gamma(\chi_2)\leqslant\gamma_{n+1}(\chi_1)$$

for almost all $\gamma_n(\chi_1)$?

Here we define $\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2)$ by $\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2)$ if $\gamma_n(\chi_1)<\gamma_m(\chi_2)$, and $\gamma_n(\chi_1)\leqslant\gamma_m(\chi_2)\leqslant\gamma_{n+1}(\chi_1)\leqslant\gamma_{m+1}(\chi_2)\leqslant\ldots$ if $\gamma_n(\chi_1)=\gamma_{n+1}(\chi_1)=\ldots$ $\ldots=\gamma_m(\chi_2)=\gamma_{m+1}(\chi_2)=\ldots$

(iii) *Does it happen*

$$\gamma_n(\chi_1)\leqslant\gamma_n(\chi_2)\leqslant\gamma_{n+1}(\chi_1)$$

for almost all n?

For later purposes we define $\Delta_n(\chi_1,\chi_2)$ by $n-m$ if $\gamma_m(\chi_1)\leqslant\gamma_n(\chi_2)\leqslant\gamma_{m+1}(\chi_1)$. Then (iii) asks if $\Delta_n(\chi_1,\chi_2)=0$ for almost all n.

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To these questions, we can show

THEOREM 1. *Assume that χ_1 and χ_2 are different primitive characters to the same modulus q. Then positive proportion of $\gamma_n(\chi_1)$ does not have $\gamma_n(\chi_2)$ in $\gamma_n(\chi_1) \leqslant t \leqslant \gamma_{n+1}(\chi_1)$.*

In particular,

THEOREM 1′. *Under the same assumption as above positive proposition of zeros of $L(s, \chi_1)L(s, \chi_2)$ are non-coincident.*

Further,

THEOREM 2. *Under the same hypothesis as above:*

(i) *For positive proportion of n*

$$\Delta_n(\chi_1, \chi_2) > c_1(\log\log n)^{1/2}$$

and also for positive proportion of n

$$\Delta_n(\chi_1, \chi_2) < -c_1(\log\log n)^{1/2},$$

where c_1 is some positive absolute constant.

(ii) *For any positive increasing function $\Phi(n)$ which tends to ∞ as n tends to ∞,*

$$|\Delta_n(\chi_1, \chi_2)| > \frac{2\pi\sqrt{\log\log n}}{\Phi(n)}$$

for almost all n.

Hence we see $\gamma_n(\chi_2)$ almost never lies in $\gamma_n(\chi_1) \leqslant t \leqslant \gamma_{n+1}(\chi_1)$. Theorem 1 comes from mean value theorem of

$$\big(S(t+h, \chi_1) - S(t, \chi_1)\big) - \big(S(t+h, \chi_2) - S(t, \chi_2)\big)$$

(cf. Lemma 1 in § 2), where

$$S(t, \chi) = \frac{1}{\pi}\arg L(\tfrac12 + it, \chi)$$

as usual. Theorem 2 comes from mean value theorem of $S(t, \chi_1) - S(t, \chi_2)$ (cf. Lemma 2 in § 2).

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§ 2. Lemmas.

2.1. For simplicity we write

$$\Delta_h\big(S(t, \chi_1) - S(t, \chi_2)\big) = \big(S(t+h, \chi_1) - S(t, \chi_1)\big) - \big(S(t+h, \chi_2) - S(t, \chi_2)\big).$$

We will prove the following two lemmas.

LEMMA 1. *Let a_1, a_2 be fixed, $0 < a_i \leqslant \frac12$ for $i = 1, 2$. Let χ_i be a primitive character to modulus q_i for $i = 1, 2$, and suppose that $\chi_1 \neq \chi_2$, $q_1, q_2 \leqslant T^{1/4-a_1}$, $T^{1/2+a_2} \leqslant H \leqslant T$, and that $0 < h \ll \dfrac{1}{\log\log T}$. Then*

$$\int_T^{T+H} \big(\Delta_h\big(S(t, \chi_1) - S(t, \chi_2)\big)\big)^l dt$$

$$= \begin{cases} \dfrac{2k!}{(2\pi)^{2k}k!} H\big(4(\log(3 + h\log T))^k\big) + \\ \qquad + O\big((Ak)^{4k} H(\log(3 + h\log T))^{k-1/2}\big) & \text{if} \quad l = 2k, \\ O\big(H(Ak)^{3k}(\log(3 + h\log T))^{k-1}\big) & \text{if} \quad l = 2k-1. \end{cases}$$

LEMMA 2. *Under the same hypothesis as above excluding the hypothesis to h,*

$$\int_T^{T+H} \big(S(t, \chi_1) - S(t, \chi_2)\big)^l dt$$

$$= \begin{cases} \dfrac{2k!}{(2\pi)^{2k}k!} H(2\log\log T)^k + O\big((Ak)^{4k} H(\log\log T)^{k-1/2}\big) & \text{if} \quad l = 2k, \\ O\big((Ak)^{3k} H(\log\log T)^{k-1}\big) & \text{if} \quad l = 2k-1. \end{cases}$$

2.2. We will prove these only for $l = 2k$. Odd cases come similarly. For simplicity we write

$$\|f\| = \Big(\int_T^{T+H} f(t)^{2k} dt\Big)^{1/2k}.$$

We saw in [1] for x in $T^{a_2/20k} \leqslant x \leqslant H^{1/k}$

$$\Big\| S(t, \chi_i) - \frac{1}{\pi}\,\mathrm{Im} \sum_{p<x^3} \frac{\chi_i(p)}{p^{1/2+it}} \Big\| \ll k^2 H^{1/2k} \tag{1}$$

and also

$$\Big\| S(t+h, \chi_i) - \frac{1}{\pi}\,\mathrm{Im} \sum_{p<x^3} \frac{\chi_i(p)}{p^{1/2+i(t+h)}} \Big\| \ll k^2 H^{1/2k} \tag{2}$$

for each $i = 1, 2$ and for h in $0 < h \leqslant H - (H/\sqrt{T})^{1/8}$. Also we saw in [1] that for complex numbers $a(p)$ if $F_a(x) = \sum_{p<x} \dfrac{|a(p)|^{2a}}{p^a} \ll 1$ for $a \geqslant 2$ and $F_{1/2}(x) \ll x^c$ with some $c > 0$, then for $x = T^{(\frac14+a_2)/2k(c+1)}$

$$\Big\| \mathrm{Im} \sum_{p<x} \frac{a(p)}{p^{1/2+it}} \Big\|^{2k} = \frac{2k!}{2^{2k}k!} HF_1(x)^k + O\Big(\frac{2k!}{2^{2k}k!} HF_1(x)^{0\vee(k-2)}\Big) \tag{3}$$

if $F_1(x) \to \infty$ as $x \to \infty$, where $0 \vee (k-2) = \max\{0, k-2\}$.

2.3. Proof of Lemma 1. We take $x = T^{(\frac{1}{2}+a_2)/5k}$. Now

$$\Delta_h\big(S(t,\chi_1)-S(t,\chi_2)\big)$$
$$= \Delta_h\big(S(t,\chi_1)-S(t,\chi_2)\big) - \frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{a(p)}{p^{1/2+it}} + \frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{a(p)}{p^{1/2+it}},$$

where

$$a(p) = \big(\chi_1(p)-\chi_2(p)\big)\big(e^{-ih\log p}-1\big).$$

Hence by (1) and (2)

$$\big\|\Delta_h\big(S(t,\chi_1)-S(t,\chi_2)\big)\big\| = \Big\|\frac{1}{\pi}\operatorname{Im}\sum_{p<x}\frac{a(p)}{p^{\frac12+it}}\Big\| + O(k^2H^{1/2k}).$$

Now

$$\sum_{p<x}\frac{|a(p)|^2}{p} = \sum_{p<x}\frac{|\chi_1(p)-\chi_2(p)|^2}{p}\cdot|e^{-ih\log p}-1|^2$$
$$= 4\sum_{\substack{p<x\\ p\nmid q}}\frac{1-\cos(h\log p)}{p} - 2\sum_{p<x}\frac{\overline{\chi_1(p)}\chi_2(p)}{p}\big(1-\cos(h\log p)\big) -$$
$$-2\sum_{p<x}\frac{\chi_1(p)\bar{\chi}_2(p)}{p}\big(1-\cos(h\log p)\big)$$
$$= 4\sum_{p<x}\frac{1-\cos(h\log p)}{p} - 4\sum_{\substack{p\mid q\\ p<x}}\frac{1-\cos(h\log p)}{p} + O(1)$$

by Mertens's Theorem

$$= 4\log(h\log x)+O(1) \quad \text{under our assumption on } h \text{ and } q.$$

Hence by (3) we get

$$\big\|\Delta_h\big(S(t,\chi_1)-S(t,\chi_2)\big)\big\|^{2k} = \frac{2k!}{(2\pi)^{2k}k!}H\big(4\log(3+h\log T)\big)^k +$$
$$+O\Big((Ak)^{4k}H\big(\log(3+h\log T)\big)^{k-1/2}\Big).$$

2.4. Proof of Lemma 2. In this case $a(p) = \chi_1(p)-\chi_2(p)$ and

$$\sum_{p<x}\frac{|a(p)|^2}{p} = 2\sum_{p<x}\frac{1}{p} - 2\sum_{p\mid q}\frac{1}{p} + O(1)$$

by Mertens's Theorem. This is equal to

$$2\log\log x + O(\log\log\log q).$$

Hence under our assumption on q, taking $x = T^{(1/2+a_2)/5k}$, we get our conclusion in the same way as above.

§ 3. Proof of theorems.

3.1. Let $N(t,\chi)$ be the number of zeros $\varrho = \beta+i\gamma$ of $L(s,\chi)$ in $0<\beta<1$, $0\leqslant\gamma\leqslant t$, possible zeros on $\operatorname{Im}s = 0$ or t counted one-half only. As is well known

$$N(t,\chi) = \frac{t}{2\pi}\log t - \frac{1+\log\frac{2\pi}{q}}{2\pi}t - \frac{\chi(-1)}{8} + S(t,\chi) - S(0,\chi) + O\Big(\frac{1}{1+t}\Big) \quad \text{for} \quad t>0.$$

Hence

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big)$$
$$= \big(N(t+h,\chi_1)-N(t,\chi_1)\big) - \big(N(t+h,\chi_2)-N(t,\chi_2)\big)$$

is essentially $\Delta_h\big(S(t,\chi_1)-S(t,\chi_2)\big)$. From Lemma 1 we see the following

COROLLARY. (i) *Under the same hypothesis as Lemma 2, for $h = 2\pi C/\log T$,*

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big) > c_2(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$$

for positive proportion of t in $(T, T+H)$ if $C>C_0$. Also for $h = 2\pi C/\log T$,

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big) < -c_2(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$$

for positive proportion of t in $(T, T+H)$ if $C>C_0$, where the Lebesgue measure of such t's is at least $c_3H\exp\big(-(\log\log C)^{1-c_4}\big)$, c_2 and c_3 are some positive absolute constants, and c_4 is a suitable small positive number.

(ii) *Under the same situation as* (i) *the Lebesgue measure of t for which $t\in(T, T+H)$ and*

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big) > c_5(\log C)^{1/2}$$

is at least c_6H, where c_6 is some positive absolute constant which does not depend on C. Same is true for

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big) < -c_5(\log C)^{1/2}.$$

Proof. We write $f(t) = \Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big)$. From Lemma 1

$$\int_T^{T+H} f^{2k}(t)\,dt = \frac{2k!}{(2\pi)^{2k}k!}H\big(4\log(3+h\log T)\big)^k +$$
$$+O\Big((Ak)^{4k}H\big(\log(3+h\log T)\big)^{k-1/2}\Big)$$

and

$$\int_T^{T+H} f^{2k-1}(t)\,dt = O\Big((Ak)^{3k}H\big(\log(3+h\log T)\big)^{k-1}\Big).$$

We write $E_M = \{t \in (T, T+H) \colon f(t) > M\}$ for $M \geqslant 0$. And let $\varphi_M(t)$ be the characteristic function of E_M. Now

$$\int_T^{T+H} f^{2k+1}(t)\varphi_0(t)\,dt = \int_T^{T+H} f^{2k+1}(t)\varphi_M(t)\,dt + \int_T^{T+H} f^{2k+1}(t)\big(1-\varphi_M(t)\big)\varphi_0(t)\,dt$$
$$\leqslant \sqrt{|E_M|}\cdot\Big(\int_T^{T+H} f^{2(2k+1)}(t)\,dt\Big)^{1/2} + M^{2k+1}H,$$

where $|E_M|$ is the Lebesgue measure of E_M. On the other hand, by Hölder inequality

$$\int_T^{T+H} f^{2k+1}(t)\varphi_0(t)\,dt = \tfrac{1}{2}\int_T^{T+H} |f^{2k+1}(t)|\,dt + \tfrac{1}{2}\int_T^{T+H} f^{2k+1}(t)\,dt$$
$$\geqslant \frac{1}{2}\,\frac{\Big(\int_T^{T+H} |f(t)|^{2k}\,dt\Big)^{(2k-1)/2(k-1)}}{\Big(\int_T^{T+H} |f(t)|^{2}\,dt\Big)^{1/2(k-1)}} + \tfrac{1}{2}\int_T^{T+H} f^{2k+1}(t)\,dt.$$

Hence we get

$$\sqrt{|E_M|}\Big(\int_T^{T+H} f^{2(2k+1)}(t)\,dt\Big)^{1/2}$$
$$\geqslant \frac{1}{2}\,\frac{\Big(\int_T^{T+H} |f(t)|^{2k}\,dt\Big)^{(2k-1)/2(k-1)}}{\Big(\int_T^{T+H} |f|^{2}\,dt\Big)^{1/2(k-1)}} - M^{2k+1}H + O\Big((Ak)^{3k}H\big(\log(3+h\log T)\big)^{k-1}\Big).$$

Taking $h\log T = 2\pi C$ sufficiently large and $k = [(\log\log C)^{1-\varepsilon_1}]$ with some arbitrary small positive ε_1, we get

$$|E_M| \geqslant H\left(A_1\frac{k^{k(2k-1)/2(k-1)}}{(2k+1)^{k+1/2}} - A_2\frac{M^{2k+1}}{(2k+1)^{k+1/2}(\log C)^{k+1/2}}\right)^2$$

if $\big(A_3 k^{k(2k-1)/2(k-1)}(\log C)^{k+1/2}\big)^{1/(2k+1)} > M$.

The last condition is $A_4(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon_2} > M$ with a suitable positive number ε_2. Hence we get our conclusion (i). (ii) comes similarly if we consider $f(t)\varphi_0(t)$ instead of $f^{2k+1}(t)\varphi_0(t)$. Q.E.D.

3.2. In particular for $h = 2\pi C/\log T$, and for positive proportion of t in $(T, T+H)$,

$$\Delta_h\big(N(t,\chi_1) - N(t,\chi_2)\big) \geqslant 2.$$

For such t, in $\left(t, t+\dfrac{2\pi C}{\log T}\right)$, there exists a $\gamma_n(\chi_1)$ such that there is no zero of $L(s,\chi_2)$ in $\gamma_n(\chi_1) \leqslant t \leqslant \gamma_{n+1}(\chi_1)$. Hence we get our Theorem 1 as usual.

3.3. We assume the same hypothesis as Lemma 2. By the definition of $\Delta_n(\chi_1, \chi_2)$

$$\Delta_n(\chi_1,\chi_2) = N\big(\gamma_n(\chi_2),\chi_2\big) - N\big(\gamma_n(\chi_2),\chi_1\big)$$
$$= S\big(\gamma_n(\chi_2),\chi_2\big) - S\big(\gamma_n(\chi_2),\chi_1\big) + O\left(\frac{1}{1+\gamma_n(\chi_2)}\right).$$

Now

$$\sum_{T<\gamma_n(\chi_2)\leqslant T+H}\Big(S\big(\gamma_n(\chi_2),\chi_2\big) - S\big(\gamma_n(\chi_2),\chi_1\big)\Big)^{2k}$$
$$= \int_T^{T+H}\big(S(t,\chi_2)-S(t,\chi_1)\big)^{2k}\,dN(t,\chi_2)$$
$$= \int_T^{T+H}\big(S(t,\chi_2)-S(t,\chi_1)\big)^{2k}\,d\left(\frac{t}{2\pi}\log t - \frac{1+\log\frac{2\pi}{q}}{2\pi}t + \right.$$
$$\left. + S(t,\chi_2) - S(0,\chi_2) + O\left(\frac{1}{1+t}\right) - \frac{\chi(-1)}{8}\right)$$
$$= \frac{1}{2\pi}\log\frac{qT}{2\pi}\int_T^{T+H}\big(\mathring{S}(t,\chi_1) - S(t,\chi_2)\big)^{2k}\,dt +$$
$$+ O\left(\frac{H}{T}\int_T^{T+H}\big(S(t,\chi_1)-S(t,\chi_2)\big)^{2k}\,dt\right) + O\big((\log T)^{2k+1}/(2k+1)\big)$$
$$= \frac{2k!}{(2\pi)^{2k}k!}\cdot\frac{H}{2\pi}\log\frac{qT}{2\pi}(\log\log T)^k +$$
$$+ O\big(H\log T(\log\log T)^{k-1/2}(Ak)^{4k}\big) + O\big((\log T)^{2k+1}/(2k+1)\big).$$

Hence we get

$$\sum_{T\leqslant\gamma_n(\chi_2)\leqslant T+H}\big(\Delta_n(\chi_1,\chi_2)\big)^{2k} = \frac{2k!}{(2\pi)^{2k}k!}\,\frac{H}{2\pi}\log\frac{qT}{2\pi}(\log\log T)^k +$$
$$+ O\big(H\log T(\log\log T)^{k-1/2}(Ak)^{4k}\big) + O\big((\log T)^{2k+1}/(2k+1)\big).$$

In particular, we get

$$\sum_{n=1}^{N}\big(\Delta_n(\chi_1,\chi_2)\big)^{2k} = \frac{2k!}{(2\pi)^{2k}k!}N(\log\log N)^k +$$
$$+ O\big(N(\log\log N)^{k-1/2}(Ak)^{4k}\big) + O\big((\log N)^{2k+1}/(2k+1)\big).$$

Similarly, we get

$$\sum_{n=1}^{N}(\Delta_n(\chi_1,\chi_2))^{2k-1} = O\big(N(\log\log N)^{k-1}(Ak)^{3k}\big)+O\big((\log N)^{2k}/2k\big).$$

3.4. (i) of Theorem 2 comes similarly as in 3.2 if we use the above mean value estimate $\sum_{n=1}^{N}(\Delta_n(\chi_1,\chi_2))^l$ for $l = 1, 2, 4$ or comes from 3.5 below.

3.5. We write

$$F_N(u) = \frac{1}{N}\left|\left\{N<n\leqslant 2N;\, -\infty<\Delta_n(\chi_1,\chi_2)\leqslant\frac{u\sqrt{\log\log n}}{2\pi}+O\left(\frac{u}{\sqrt{\log N}}\right)\right\}\right|.$$

Then from 3.3 we see

$$\int_{-\infty}^{\infty} u^l\,dF_N(u) = \mu_l(N)$$

$$= \begin{cases} \dfrac{2k!}{k!}+O\left(\dfrac{(Ak)^{4k}}{\sqrt{\log\log N}}\right)+O\left(\dfrac{(\log N)^{2k+1}}{N(\log\log N)^k(2k+1)}\right) & \text{if} \quad l = 2k, \\ O\left(\dfrac{(Ak)^{3k}}{\sqrt{\log\log N}}\right)+O\left(\dfrac{(\log N)^{2k}}{(\log\log N)^{k-1/2}N\cdot 2k}\right) & \text{if} \quad l = 2k-1. \end{cases}$$

Since

$$\mu_l(N)\to\mu_l = \begin{cases} \dfrac{2k!}{k!} & \text{if} \quad l = 2k, \\ 0 & \text{if} \quad l = 2k-1 \end{cases}$$

as $N\to\infty$ and the distribution function determined by $\{\mu_l;\ l = 0, 1, 2, \ldots\}$ is

$$\int_{-\infty}^{u}\frac{1}{\sqrt{4\pi}}e^{-x^2/4}dx, \quad \lim_{N\to\infty}F_N(u) = \int_{-\infty}^{u}\frac{1}{\sqrt{4\pi}}e^{-x^2/4}dx$$

(cf. 4.24 and 3.4 of [4]).

Hence for any positive increasing function $\Phi(n)$ which tends to ∞ as $n\to\infty$,

$$|\Delta_n(\chi_1,\chi_2)| > \frac{\sqrt{\log\log n}}{2\pi\Phi(n)}$$

for almost all n.

§ 4. Concluding remark. As is seen from 3.1 and 3.2 in § 3, we do not have to assume that χ_1 and χ_2 have the same modulus in Theorem 1. Namely, the inequality in the Corollary of 3.1 becomes

$$\Delta_h\big(N(t,\chi_1)-N(t,\chi_2)\big)\gtrless\frac{C}{\log T}\log\frac{q_1}{q_2}+c_2(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$$

for $h = 2\pi C/\log T$, where χ_i has a modulus q_i for $i = 1, 2$.

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