A conjecture of Erdös on continued fractions

by

WAUTER PHILIP (Urbana, Ill.)

1. For \(0 < x \leq 1\) let \([a_1(x), a_2(x), \ldots]\) be the continued fraction expansion of \(x\). Write

\[ L_N(x) = \max_{1 \leq n \leq N} a_n(x). \]

About ten years ago Professor Erdös [4] conjectured that for almost all \(x\)

\[ \liminf_{N \to \infty} N^{-1} L_N(x) \log \log N = 1. \]

Apart from the value of the constant I shall give a proof of Erdös\' conjecture.

**Theorem 1.** For almost all \(x\)

\[ \liminf_{N \to \infty} N^{-1} L_N(x) \log \log N = \frac{1}{\log 2}. \]

By modifying methods developed by Barndoff-Nielsen [1] for similar problems concerning independent identically distributed random variables we get the following refinement of Theorem 1.

**Theorem 2.** Let \(\psi_N\) be nonincreasing such that \(\psi_N N\) is nondecreasing. Then

\[ L_N(x) \leq \psi_N N/\log 2 \]

finitely often or infinitely often for almost all \(x\) according as

\[ \sum \exp(-1/\psi_n) n^{-1} \log n \]

converges or diverges.

**Corollary.** Let \(k \geq 2\) be integer. Then the inequality

\[ N^{-1} L_N(x) (\log^1 N + 2 \log^2 N + \ldots + (1 + \delta) \log^k N) \leq 1/\log 2 \]

has finitely many or infinitely many solutions for almost all \(x\) according as \(\delta > 0\) or \(\delta \leq 0\). If \(k = 3\) then \(1 + \delta\) is to be replaced by \(2 + \delta\). Here \(\log^k\) denotes the \(k\)-fold iterated logarithm.

Theorem 1 follows from the corollary if we put \(k = 2\).
A conjecture of Erdős on continued fractions

There is no analogous result to Theorem 1 with a finite nonzero limit superior. This follows from the well-known

**THEOREM 3.** Let \( \varphi(n) \) be a positive nondecreasing sequence. Then for almost all \( x \) the inequality

\[
L_N(x) > \varphi(N).
\]

has finitely many or infinitely many solutions in integers \( N \) according as the series

\[
\sum \frac{1}{\varphi(n)}
\]

converges or diverges.

**COROLLARY.** Let \( \varphi(n) \) be as in Theorem 3. Then for almost all \( x \)

\[
\limsup_{N \to \infty} \frac{L_N(x)}{\varphi(N)}
\]

is either 0 or \( \infty \).

Theorem 3 is an easy consequence of Bernstein's theorem on continued fractions. Indeed, if \( \sup \varphi(n) < \infty \), then \( \sum \frac{1}{\varphi(n)} \) converges and hence by Bernstein's theorem \( a_n(x) \) holds infinitely often for almost all \( x \). If, however, \( \varphi(n) \to \infty \) then, as is easy to see, \( (1) \) holds finitely often or infinitely often according as the inequality \( a_n(x) > \varphi(n) \) holds finitely often or infinitely often which, in turn, by Bernstein's theorem holds almost everywhere according as the series \( \sum \frac{1}{\varphi(n)} \) converges or diverges.

For the proof of the corollary we distinguish the cases where the series \( \sum \frac{1}{\varphi(n)} \) converges or diverges. If \( \sum \frac{1}{\varphi(n)} \) converges we choose a monotone sequence \( \tau(n) \) tending to \( 0 \) but so slowly that still \( \sum \frac{\tau(n)}{\varphi(n)} < \infty \). Then according to Theorem 3 the inequality \( L_N(x) > \varphi(N) \tau(N) \) holds only a finite number of times for almost all \( x \). Hence \( (3) \) vanishes almost everywhere. If, on the other hand, \( \sum \frac{1}{\varphi(n)} \) diverges we pick a monotone sequence \( \tau(n) \) tending to \( 0 \) such that \( \sum \frac{\tau(n)}{\varphi(n)} = \infty \); then \( L_N(x) > \varphi(N) \tau(N) \) has infinitely many solutions for almost all \( x \) and thus \( (3) \) is infinite almost everywhere.

Theorems 1, 2, 3 and their corollaries improve upon results of Galambos [6], [7]. For the proof of Theorem 1 we use Theorem 4 below which also strengthens a result of Galambos [5]. Except for the application of Theorem 4 the proof of Theorem 1 is different from the one given in [7].

In dealing with continued fractions it is more convenient to use the Gaussian measure \( P \) instead of the Lebesgue measure \( \lambda \). \( P \) is defined on the Lebesgue measurable sets \( E \) by

\[
P(E) = \frac{1}{\log 2} \int_{E} \frac{dx}{1+x}.
\]

**THEOREM 4.** For any \( \delta < 1 \) and \( \gamma > 0 \)

\[
P_{\varphi}: \quad (1) \quad L_N(x) < N \gamma / \log 2 = \exp(-1/\gamma) + O(e^{-(\log N)^{\gamma}})
\]

where the constant implied by \( O \) depends only, perhaps, on \( \delta \).

With an error term of the form \( o(1) \), Theorem 4 is due to Galambos [5] who subsequently [6] showed that his result remains valid if \( P \) is replaced by any probability measure on \([0, 1]\) absolutely continuous with respect to \( \lambda \). (In particular for \( \lambda \) itself.) If we replace \( P \) by probability measures having a "smooth" density then Theorem 4 itself remains valid; in particular, Theorem 4 continues to hold if we replace \( P \) by \( \lambda \). Since we shall not need these facts we omit their proof.

As already indicated above I shall give a direct proof of Theorem 1. The proof of Theorem 2 will only be sketched in order to avoid lengthy repetitions of parts of Barndorff-Nielsen's paper [1]. The proof of Theorem 4 also will be only sketched since it consists only of a modification of Galambos' paper [5].

We remark in passing that Barndorff-Nielsen's Theorems 1 and 2 of [1] remain valid for sequences of random variables satisfying a uniform mixing condition

\[
P(AB) - P(A)P(B) | \leq q | P(A)P(B)
\]

for all \( A \in M_1 \) and \( B \in M_1 \). Here \( M_1 \) denotes the \( \sigma \)-field generated by the random variables \( (X_n, a \leq n \leq b) \) and \( \varphi(\omega) \).

2. **Lemmas on continued fractions.** The shift transformation \( T \) associated with the continued fraction expansion is defined by \( Tx = x + [x] \mod 1 \). \( T \) is called a shift since \( a_{n+1}(x) = a_n(Tx) = a_n(x) \) for all positive integers \( n \). \( T \) maps the unit interval onto itself and preserves the Gaussian measure \( P \), i.e., \( T^{-1}(E) = T(E) \) for any \( \mathcal{L} \)-measurable set \( E \). This explains the importance of \( T \) in investigations dealing with continued fractions. Lemma 1 just proves this point.

**Lemma 1.** For all positive integers \( n \) and \( w \)

\[
P(\{x: a_n(x) > w\}) = P(\{x: a_n(x) > w\}) = \log (1+1/w)/\log 2 = p(w) \quad (\text{say}).
\]

The functions \( a_1(x), a_2(x), \ldots \) considered as random variables on \([0, 1]\) are not independent. However, they satisfy the following mixing condition.

**Lemma 2.** Let \( M_{11} \) be the smallest \( \sigma \)-algebra with respect to which \( a_n(x), a_{n+1}(x) \) are measurable. Then for any sets \( A \in M_{11} \) and \( B \in M_{11} \) we have

\[
|P(AB) - P(A)P(B)| \leq q(P(A)P(B)
\]

where \( 0 < q < 1 \) and \( c \geq 1 \) are numerical constants.
The inequalities (7), (8) and (9) replace (5), (10), (13) and (14). The remaining changes are only minor.

4. Proof of Theorem 1. Since \( \lambda(E)/\log 4 \leq P(E) \leq \lambda(E)/\log 2 \) the measures \( P \) and \( \lambda \) are equivalent, i.e., they have the same sets of measure 0. Hence we are to show Theorem 1 for all \( \alpha \) except a set of \( P \)-measure 0. For integers \( M, N \geq 0 \) we put

\[
L(M, N, \alpha) = \max_{M < n < M + N} a_n(\alpha)
\]

and

\[
\psi(n) = n/(\log n \log \log 2).
\]

Since the transformation \( T \) preserves \( P \) we have by Theorem 4 for any integer \( k \geq 4 \)

\[
P(E_k) = P(x; L(0, k^{2^{k+1}}, k^{2(k+1)})) \leq \psi(k^{2(k+1)}))
\]

\[
= P(x; L(0, k^{2(k+1)}, \alpha) \leq \psi(k^{2(k+1)}))
\]

\[
\geq \frac{1}{2} \exp(- \log k^{2(k+1)}) \geq \frac{1}{2} (k \log k)^{-1}.
\]

Now \( E_k \) depends only on \( a_n(\alpha) \) with \( k^{2k} < n \leq k^{2(k+1)} + k^{2k} \). Hence by Lemma 2 we have for any pair \( k < l \) of integers.

\[
P(E_l \cap E_k) = P(E_k) P(E_l) \leq e^{-2 \log 2} P(E_k) P(E_l)
\]

since \( (k+1)^{2(k+1)} - k^{2(k+1)} - k^{2k} \geq 1 \). Consequently, Lemma 3 implies that for almost all \( \alpha \) the events \( E_k \) occur infinitely often since \( \psi(N) \geq \log N \) by (10). On the other hand, by Lemma 1

\[
P(F_k) = P(x; L(0, k^{2k}, \alpha) \geq \psi(k^{2(k+1)})) \leq \sum_{n \geq k} \psi(k^{2(k+1)}))
\]

\[
= k^{2k} \psi(k^{2(k+1)})) \geq k^{2k} \log k^{2(k+1)}, k^{2k} \geq k^{-3/2}.
\]

Thus by the convergence part of Lemma 3 (which in fact is the convergence part of the Borel–Cantelli lemma) for almost all \( \alpha \) the events \( F_k \) occur only a finite numbers of times. Hence the events

\[
E_k \cap F_k = \{x; L(0, k^{2k} + k^{2(k+1)}, \alpha) \leq \psi(k^{2(k+1)}))
\]

occur infinitely often for almost all \( x \). But this implies that the events

\[
L(0, k^{2(k+1)}, \alpha) \leq \psi(k^{2(k+1)}))
\]

occur infinitely often for almost all \( \alpha \). Consequently,

\[
\liminf_{N \to \infty} N^{-1} L_N(\alpha) \log n \log 2 \leq 1 \quad \text{a.e.}
\]

This proves half of Theorem 1.

We now prove the opposite inequality. Let \( \tau \geq 2 \). Again by Theorem 4

\[
P(G_k) = P(x; L(0, k^{2k}, \alpha) \leq \tau^{-2} \psi(k^{2(k+1)})) \leq \exp(- \log \log \tau) \leq k^{-\tau}.
\]
Since $\sum k^{-r} < \infty$ the convergence part of Lemma 3 implies that the events $\Theta_k$ occur only a finite number of times for almost all $\omega$, or

$$L(0, [r^k], \omega) > r^{-\psi([r^{k+1}])}$$

for all $k \geq k_0(\omega, r)$. Now let $N = N_k(\omega, r)$ be given. Define $k$ by $[r^k] \leq N < [r^{k+1}]$. Since $L(0, [r^k], \omega) \leq L_N(\omega)$ and $\psi(N) \leq \psi([r^{k+1}])$ we conclude that for almost all $\omega$ and all $N \geq N_k$

$$L_N(\omega) > r^{-\psi(N)}.$$

Since $r > 1$ was arbitrary we obtain

$$\liminf_{N \to \infty} N^{-1} L_N(\omega) \log \log N \log 2 \geq 1 \quad \text{a.e.}$$

This together with (11) proves Theorem 1.

5. Proof of Theorem 2. The following lemma corresponds to [1], Lemma 4.

**Lemma 4.** Without loss of generality we may assume that

$$2(2 \log \log n)^{-1} \leq \psi_n \leq 2(2 \log \log n).$$

**Proof.** Suppose that Theorem 2 has been proved for sequences $\psi_n$ satisfying (12). To any nonincreasing sequence $\psi_n$ such that $\omega \psi_n$ is non-decreasing we define a sequence

$$\psi_n' = \begin{cases} 
2 \log \log n & \text{if } \psi_n > 2 \log \log n, \\
\psi_n & \text{if } (2 \log \log n)^{-1} \leq \psi_n \leq 2 \log \log n, \\
(2 \log \log n)^{-1} & \text{if } \psi_n < (2 \log \log n)^{-1}.
\end{cases}$$

Then the series

$$\sum \exp(-1/\psi_n') n^{-1} \log n$$

and

$$\sum \exp(-1/\psi_n) n^{-1} \log n$$

converge or diverge simultaneously. Indeed if $\psi_n > \psi_n'$ for infinitely many $n$, say $n_1, n_2, \ldots, n_k$, then the $n_k$-th partial sums of (13) and (14) are not less than

$$\exp(-\frac{1}{2} \log \log n_k) \sum_{n=1}^{n_k} n^{-1} \to \infty.$$

On the other hand, since

$$\sum_{n=3}^{\infty} \exp(-2 \log \log n) n^{-1} \log n < \infty$$

the terms with $\psi_n < \psi_n'$ cannot influence the simultaneous convergence of (13) and (14).

The remainder of the proof is the same as in [1].

As became evident from the proof of Lemma 4 our situation is somewhat simpler than the one considered in [1], at least in one respect, because the distribution function $F(t)$ of the partial quotients $a_n$ is explicitly known $F(t) = 1 - p(t)$). On the other hand, since in [1] the random variables are assumed to be independent, the distribution function of the maximum of the first $N$ random variables is simply $F(t)^N$. But we already have proved Theorem 4 which gives an estimate of the distribution function of $L_N$. By Theorem 4

$$P(L_N(\omega) \leq N \psi_N/\log 2) = \exp(-1/\psi_N + O(\exp(-\log N)^2))$$

$$= \exp(-1/\psi_N)(1 + O(1))$$

in view of Lemma 4. Consequently, we can replace the factors $(P(L_n)^n$ in (1), pp. 388–392 simply by $\exp(-1/\psi_n)$ without affecting the convergence properties of the series under consideration. The proof of Theorem 2 is entirely parallel to the proof of [1], Theorem 1, pp. 388–392. We only have to take precautions at these steps where the independence of the random variables is used. In most cases Lemma 2 will take care of that. For example Kolmogorov’s zero-one law continues to hold for mixing sequences of random variables (see [3]). The only place which requires a slight modification is the estimate of $S_1$ defined in [1], p. 388. Choose $k$ so that $1 + q_k < e^{1/8}$ where $c$ and $q$ are the constants occurring in Lemma 2. Using the notation of [1] we have for $n \geq n_0$

$$S_1 = P(E_{m_{n+1}} \cap E_{m_{n+1}+1})$$

$$\leq P(E_{m_{n+1}} \cap [a_n(a_n] \leq \psi_{m_{n+1}} m_{n+1}/\log 2, m_n + k \leq r \leq m_{n+1}])$$

$$\leq P(E_{m_{n+1}})P([a_n] L(0, m_{n+1} - m_n - k, a) \leq \psi_{m_{n+1}} m_{n+1}/\log 2)(1 + q_k^2)$$

$$\leq P(E_{m_{n+1}}) \exp(-1/4)$$

by Lemma 2 and Theorem 4. The other modifications are only of a routine nature.

References


On composite $n$ for which $\varphi(n) \mid n - 1$

by

C. Pomerance (Athens, Ga.)

§ 1. Introduction. In [4], D. H. Lehmer asked if there are any composite natural numbers $n$ for which $\varphi(n) \mid n - 1$, where $\varphi$ is Euler's function. This is still an unanswered question, many people feeling it is as difficult as the odd perfect number problem. There have been partial results however, such as if such an $n$ exists then $n$ is divisible by at least 11 distinct primes, and if $3 \mid n$, then $n > 5.5 \cdot 10^{28}$ and $n$ is divisible by at least 213 distinct primes (Lieuwens [5]).

If $A$ is an arbitrary set of positive integers, then we denote by $N(A, x)$ the number of members of $A$ which do not exceed $x$. Let $F$ denote the set of composite $n$ for which $\varphi(n) \mid n - 1$. In [6] we proved

(1)

$$N(F, x) = O(x \exp (-c_1 \log \log \log x)^{1/2})$$

for some $c_1 > 0$. If $n \in F$, then $a^{n-1} \equiv 1 \pmod{n}$ for every $a$ with $(a, n) = 1$, that is, $n$ is a Carmichael number (also called an absolute pseudoprime). Hence a result of Knödel [3] dealing with Carmichael numbers also implies (1). However, a result of Erdős [1], also dealing with Carmichael numbers, gives the better estimate

$$N(F, x) = O(x \exp (-c_2 \log \log \log \log x))$$

for some $c_2 > 0$. In the present note, borrowing somewhat the methods of Knödel and Erdős, we prove

(2)

$$N(F, x) = O(x^{1/3} (\log \log x)^{13/2}).$$

In fact we prove a more general theorem for which (2) is a special case. Indeed, in [6] we considered the sets

$$F(a) = \{ n: n = a \mod \varphi(n) \},$$

$$F'(a) = \{ n \in F(a): n \neq pa \text{ for each prime } p \mid a \},$$

where $a$ is an arbitrary integer. We prove that for any $a$,

(3)

$$N(F'(a), x) = O(a^{1/3} (\log \log x)^{13/2}).$$