Consider $\sum_{n=1}^{\infty} \frac{a_n}{n^w}$; if this series converges absolutely and uniformly for $\operatorname{Re} w \geqslant 1+\varepsilon$, $\varepsilon > 0$, to $\mathscr{A}(w)$ say, then as is easy to see,

$$\sum_{n=1}^{\infty} n^{-w} \sum_{d|n} a_d$$

converges absolutely and uniformly to $\zeta(w)\mathscr{A}(w)$ in the same region.

Hence, since the argument is clearly reversible, it is plain that the above argument will prove the

THEOREM. *If $\sum_{n=1}^{\infty} \frac{a_n}{n^w}$ converges absolutely and uniformly for* $\operatorname{Re} w \geqslant 1+\varepsilon$, $\varepsilon > 0$, *then identity* (2) *holds.* (*In the sense that for a given x either both sides converge to the same value or both diverge.*)

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Normal recurring decimals, normal periodic systems, (j, ε)-normality, and normal numbers

by

R. G. Stoneham (New York, N.Y.)

1. Introduction. In 1946, I. J. Good [1] gave a topological argument (the traversing of a particular planar network) in order to construct what he called "normal recurring decimals possessing normality of order r" [1, p. 167], i.e. all sequences of r digits have normal frequency 10^{-r} in the decimal. Then he says "If r is a given integer, the question arises whether there are recurring decimals possessing normality of order r. Any such recurring decimal (in the base 10) must clearly have a period of *at least* 10^r. Our purpose here is to show that there are such decimals with period 10^r for any given value of r". He also points out that the construction he gives can be done in any base g.

In 1950, Korobov [2, 3 and 4, pp. 64–65] considered the normal recurring decimals of Good from a different point of view and constructed by a different method what he called a "normal periodic system" (still essentially a normal recurring decimal of Good) which is a positive integer $\varrho_n(g)$ that contains sequentially in its representation in a base g all possible n-tuples chosen from $0, 1, \ldots, g-1$. The integer $\varrho_n(g)$ consisting of g^n+n-1 single digits is constructed in such a way that every n-tuple from $00\ldots0, 00\ldots1, \ldots, g-1g-1\ldots g-1$ appears exactly once somewhere in $\varrho_n(g)$. For example, Korobov [3, p. 31] gives the normal periodic system of 2^3+3-1 digits in the base 2, $\varrho_3(2) = 1000101110$ which has each 3-tuple, $000, 001, \ldots, 111$ appearing exactly once in the sequence. Also Korobov [2] proved in 1950, by a method different from Good's, the general existence of normal periodic systems $\varrho_n(g)$. In essence, he gave an algorithm for the construction of a normal periodic system. In [3, § 4, p. 36] Korobov develops a completely general algorithm which will produce every such $\varrho_n(g)$ for a given n and g. Other papers of Korobov referenced in [4, pp. 64–65] studied the use of the $\varrho_n(g)$ in constructing a particular irrational whose distribution of fractional parts approached a uniform distribution.

The purpose of this paper is to show that the normal recurring decimals of Good and the related normal periodic systems of Korobov are very special cases of the quite extensive and fundamental (j, ε)-normal phenomenon [5] which exists in broad classes of rational fractions. In [6], we constructed the first known class of transcendental non-Liouville normal numbers from any rational fraction based on the (j, ε)-normal properties of rational fractions. We will show here that the construction of the irrational α given by Korobov in [3, (18), p. 49] is quite similar to what we presented in [6, p. 242], i.e. the juxtaposition of (j, ε)-normal sets which leads to a transcendental non-Liouville normal number. In essence, the juxtaposition of normal periodic systems which we will show here are (j, ε)-normal sets will produce a normal number. That Korobov's construction leads to a normal number, follows from Wall's theorem [12, p. 110], i.e. α is normal in the base g iff $\{\alpha g^x\}$ for $x = 0, 1, \ldots$ is uniformly distributed. However, Korobov in [2, 3] and also in his other work did not state that the construction with $\varrho_n(g)$ produced a normal number in the sense of Borel [see, for example, 2, Th. 5, p. 237].

Recently, we have shown that $x(g, p) = \sum_{n=1}^{\infty} 1/p^n g^{p^n}$ is the simplest transcendental non-Liouville normal number [7, p. 422] where p is any odd prime and g one of its primitive roots that can be produced using (j, ε)-normality in the rational fractions. This result required some new theorems [7] concerning the distribution of residues *within the periods* of (j, ε)-normal rational fractions.

Furthermore, we will show that the results concerning the (j, ε)-normality in the rational fractions can produce a great variety of periodic sequences which closely resemble the normal recurring decimals of Good and the normal periodic systems of Korobov. Therefore, the result of Korobov where he constructs an irrational from normal periodic systems is a special case of the construction in [6] which produces a normal number from (j, ε)-normal sets. Consequently, either aspect of the properties of the irrational constructed therefrom, i.e. the normality or the uniform distribution of fractional parts follows from Wall's theorem of 1949. In Postnikov's [4, p. 62, Th.] work of 1966, we find Wall's theorem stated and proved without reference, no doubt due to the fact that Wall's theorem was in a Ph. D. unpublished thesis [12, p. 110].

2. On (j, ε)-normality. In [5, p. 222], we gave a definition of (j, ε)-normality so as to apply to the distribution of the digits in the infinite periodic representation of a rational fraction Z/m in some base g. Let $N(B_j, g)$ denote the number of occurrences of the block B_j consisting of any combination of j digits chosen from $0, 1, \ldots, g-1$ commencing in any period of the representation of Z/m in the base g and terminating in at most $j-1$ digits of the next period. Let $x = .x_1 x_2 \ldots$ be the representation in the scale g and let X_λ denote the block of the first λ digits

in x where $N(B_j, X_\lambda)$ denotes the number of occurrences of the block B_j in X_λ. Therefore, we have

Definition. (j, ε)-normal rational fractions. Let $Z/m < 1$ in lowest terms have a periodic representation that may or may not have a non-periodic part in a scale g such that $2 \leqslant g < m$. If for a given j and $\varepsilon \geqslant 0$, every j digit sequence B_j which occurs in the expansion is such that

$$\lim_{\lambda\to\infty} |N(B_j, X_\lambda)/\lambda - 1/g^j| = |N(B_j, g)/\omega(m) - 1/g^j| \leqslant \varepsilon \tag{2.0}$$

then Z/m is (j, ε)-*normal* in the scale g where $\omega(m) = \mathrm{ord}_m g$.

Then we defined a uniform ε-distribution for the associated discrete approximately uniform distribution of fractional parts $\{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ on $[0, 1]$ which leads to a necessary and sufficient condition for the (j, ε)-normality of Z/m [5, Th. 2, p. 224] which is completely analogous to Wall's theorem for normal numbers.

In the application of this definition here, we naturally prefer the original view of Good [1, pp. 167–168] in that we shall consider the "normal recurring decimals" as the periodic expansion of some rational fraction with period g^n, requiring at most $n-1$ digits into the next period to complete the set of all possible n-tuples. For example Good [1, p. 168], gives for $g = 2$ with period length 2^6, the normal recurring decimal $Z/m = .\dot{0}00000\ldots(64\text{ digits})\ldots00011\dot{1}\ 000000$ (which we will show is (j, ε)-normal in our studies) where we see that all possible 6-tuples B_j for $j = 6$ from 0 to 2^6-1 appear exactly once in the periodic sequence. Also, note that we must take counts extending at most $j-1 = 5$ places into the next period to complete all 6-tuples, i.e. $0011\dot{1}|0$, $011\dot{1}|00$, $11\dot{1}|000$, $1\dot{1}|0000$, $\dot{1}|00000$. In Korobov's normal periodic systems, he defines a positive integer $\varrho_n(g)$ of length g^n+n-1, which in the above case, would be designated by $\varrho_6(2)$ and consists of a total of 2^6+6-1 digits. As we said above, it is possible to identify more precisely the connection between these special sequences and (j, ε)-normality by using the original "recurring decimal" concept of Good.

Every $\varrho_n(g)$ of a normal periodic system of Korobov can be written as a periodic representation in the base g of some rational fraction Z/m whose period is g^n where the first $n-1$ digits of the next period are necessary to complete the counts of exactly one for every n-tuple.

As a matter of fact, in 1964, we [9, pp. 204–205] called these particular blocks which extend over the end of the period "anomalous blocks" for at most $j-1$ places. In our later work, we defined (j, ε)-normality [5, p. 222] over the full infinite periodic sequence rather than over one period which was our emphasis in [8].

In [5, p. 229], we found it possible to classify all rational fractions into three Types A, B, and C according to the prime decomposition of m in Z/m

as related to their (j, ε)-normal properties. These classifications came about as a logical consequence of the existence or not of residue progressions [5, Th. 4, p. 227] which was a general property of the associated sequences of reduced power residues necessary to prove the (j, ε)-normality [5, Ths. 5, 6, and 7, pp. 231–233] of the various types of rational fractions. In [5, p. 230], we discussed a particular kind of Type B, i.e.

$$Z/m = Z/g^{\lambda}+Z/g^{2\lambda}+\ldots = Z/(g^{\lambda}-1) \tag{2.1}$$

where the positive integer Z can be any prescribed sequence of λ digits in the base g so chosen to be (j, ε)-normal or not. It is precisely this Type B rational fraction which will produce the normal recurring decimals of Good or the normal periodic systems of Korobov. Let us begin with an example. Consider the rational fraction

$$23/(2^8-1) = 10111/(2^8-1) = .\dot{0}0010111\dot{1}|00\ldots \tag{2.2}$$

which produces a normal recurring decimal of Good where we have a count of exactly one for each of $000, 001, \ldots, 111$ such that $\lim_{\lambda\to\infty} N(B_3, X_\lambda)/\lambda = 1/2^3$. The complete set of reduced power residues for $23\cdot 2^i \equiv r_i \bmod (2^8-1)$ with period 2^3 are $r_i = 23, 46, 92, 184, 113, 226, 197$, and 139 for $i = 0, 1, \ldots, 7$, resp. where the digits b_i in $23/255$ are given by the greatest integer $b_i = [2r_i/(2^8-1)]$. Each of the fractions $r_i/(2^8-1)$ for each i produces a periodic set of digits which will give a $\varrho_3(2)$ of Korobov (or a normal recurring decimal of Good) that is a cyclic permutation of the digits in (2.2). Now there is also a completely distinct set of cyclic power residues whose least positive residue can be obtained from the 2-adic form of $23 = 10111$ by a "mirror reflection", i.e. $23 = 10111|11101 = 29$. Using 29, we find from $29\cdot 2^i \equiv r_i \bmod (2^8-1)$ that $r_i = 29, 58, 116, 232, 209, 163, 71, 142$ for $i = 0, 1, \ldots, 7$, resp. Again these produce a set of $\varrho_3(2)$, each of which are a cyclic permutation, but are distinct from any of those produced by 23.

Without loss of generality, one may begin an orderly search for all such forms above for $n = 3$ by initially requiring only those Z contained in $255/2^4 < Z < 255/2^3$ so that $Z\cdot 2^i$ would surely commence with exactly 3 zeros. From the set $Z = 16, 17, \ldots, \underline{23}, \ldots, \underline{29}, 30, 31$, one finds that only 23 and 29 produce normal recurring sequences. Some of the Z in the range can be quickly rejected since Z is relatively prime to $2^8-1 = 3\cdot 5\cdot 17$. Others must produce the appropriate number of zeros and ones; in particular, no more that 3 consecutive ones. The 2 distinct sequences produced by 23 and 29 also agrees with a result of De Bruijn [11] which states that for the base $g = 2$, the number of what Korobov calls [3, p. 36] "essentially distinct" systems in $\varrho_n(2)$ is 2^r where $r = 2^{n-1}-n$ or $r = 2^{3-1}-3 = 1$ which implies 2 distinct systems which correspond to 23 and 29. Thus, it appears that the statements made by Korobov in 3, p. 36, top] with reference to systems being "distinct" or "essentially distinct" actually reflect the arithmetic properties of the reduced power residue systems.

However, our purpose here is not to elucidate how to generate systems $\varrho_n(g)$ by this congruence method for the Type B fraction $Z/(g^{\lambda}-1)$ where $\lambda = g^n$ for n-tuple systems but to assume we are given the construction of Good or normal periodic systems by Korobov and study the connection with (j, ε)-normality for Type B. In the following theorem, we prove that $Z/(g^{\lambda}-1)$ where $\lambda = g^n$ is (j, ε)-normal for $j = 1, 2, \ldots, n$ and $\varepsilon = 0$. First, a few remarks. The count of the number of $N(B_j, g)$ contained in the infinite periodic representation of $Z/(g^{\lambda}-1)$ in the base g where B_j is any j-tuple of digits from 0 to g^j-1, $\lambda = g^n$, and $j = 1, 2, \ldots, n$; ultimately depends upon the distribution of the fractional parts $\{Z\cdot g^i/(g^{\lambda}-1)\} = r_i/(g^{\lambda}-1)$ on $[0, 1]$ where $i = 0, 1, \ldots, g^n-1$ and $Z\cdot g^i \equiv r_i \bmod (g^{\lambda}-1)$. From the example above, if $Z = 23$, then

$$23\cdot 2^i \equiv r_i \bmod (2^8-1) \Rightarrow r_i/(2^8-1) = 23/255,\ 46/255, \ldots, 139/255;$$

and thus due to the special construction of the integer Z based on the procedure of Good or Korobov, we must find *exactly one* value of $r_i/(2^8-1)$ contained in every half open interval $[a/2^3, (a+1)/2^3)$ for $a = 0, 1, \ldots \ldots, 2^3-1$. Therefore, for $j = 3$, we have $N(B_3, 2) = 1$ which indicates a count of one for each $000, 001, \ldots, 111$ appearing in one period of $23/(2^3-1)$ and at most, 2 digits of the next period.

Therefore, in general, consider $\{Z\cdot g^i/(g^{\lambda}-1)\} = r_i/(g^{\lambda}-1)$ for $i = 0, 1, \ldots, \lambda-1 = g^n-1$ on $[0, 1]$. The requirement that there be exactly one of each n-tuple B_n from 0 to g^n-1 whose 1st digit commences somewhere in one period and may terminate in at most $n-1$ places of the next in the infinite periodic sequence of $Z/(g^{\lambda}-1)$ with period $\lambda = g^n$, implies that every half open sub-interval $[a/g^n, (a+1)/g^n)$ on $[0, 1]$ for $a = 0, 1, \ldots, g^n-1$ contains exactly one point. Thus, $N(B_n, g) = 1$, for every n-tuple B_n, and consequently, the uniformity in the n sub-intervals implies $N(B_{n-1}, g) = g$ in $[a/g^{n-1}, (a+1)/g^{n-1})$ for $a = 0, 1, \ldots, g^{n-1}-1, \ldots, N(B_j, g) = g^{n-j}$ in $[a/g^j, (a+1)/g^j)$ for $j = 1, 2, \ldots, n$. If $N(B_j, X_t)$ denotes the number of occurrences of the j-tuple B_j in the 1st t digits of the representation of $Z/(g^{\lambda}-1)$ in the base g, we have proved

THEOREM 1. *If the rational fraction $Z/(g^{\lambda}-1)$ of Type B with period $\lambda = g^n$ has a representation in the base g such that the infinite periodic sequence is a normal recurring decimal of Good or contains a normal periodic system of Korobov, then $Z/(g^{\lambda}-1)$ is (j, ε)-normal with*

$$\lim_{t\to\infty} N(B_j, X_t)/t = N(B_j, g)/g^n = 1/g^j \tag{2.3}$$

for $j = 1, 2, \ldots, n$ and $\varepsilon = 0$.

Let us be more precise about the relation between the normal periodic systems of Korobov, normal recurring decimals of Good, and our Type B rational fraction. In Korobov [3, p. 35, eq. (15)'], he defines the 1st g^n digits $d_1 d_2 \ldots d_\lambda = \varrho_n'(g)$ where $\lambda = g^n$ to be a "system", then if we repeat the first $n-1$ digits of $\varrho_n'(g)$, we have his so-called "normal periodic system". Therefore, if we write

$$(2.4)\qquad \varrho_n'(g)/(g^\lambda-1) = .d_1 d_2 \ldots d_\lambda d_1 d_2 \ldots d_{n-1} \ldots d_\lambda \ldots = .\varrho_n'(g)\,\varrho_n'(g)\ldots$$

we have all 3 forms in one statement. The rational fraction $\varrho_n'(g)/(g^\lambda-1)$ with period $\omega(g^\lambda-1) = \lambda = g^n$ is our Type B, the set of digits $.d_1 d_2 \ldots d_\lambda \ldots$ is Good's normal recurring decimal, and $\varrho_n'(g)$ is the first g^n digits in Korobov's normal periodic system where he will add on $n-1$ digits of a juxtaposition repetition of $\varrho_n'(g)$ [see 3, p. 35, eq. (15) or (17)].

3. A transcendental non-Liouville normal number. In this section, we prove that one may construct a transcendental non-Liouville normal number from the normal recurring decimals of Good, the normal periodic systems of Korobov, or our particular Type B rational fraction.

The construction is similar to that of Korobov in [2, or 3, p. 49, eq. (18)], however there are a number of new features in the proof we give here in that the proof depends upon the (j, ε)-normality of the Type B rational fraction $\varrho_n'(g)/(g^\lambda-1)$ and also we will show that the irrational produced by the construction is a transcendental of the non-Liouville type similar to our result in [6, p. 247, Th. 2].

Let $\varrho_i'(g)\,a_i\,\varrho_i'(g)$ denote a_i repetitions of the set of g^i digits in $\varrho_i'(g)$, then we write in juxtaposition

$$(3.0)\qquad \alpha(\varrho_n'(g), n) = .\varrho_1'(g)\,a_1\,\varrho_1'(g)\,\varrho_2'(g)\,a_2\,\varrho_2'(g)\ldots\varrho_{n-1}'(g)\,a_{n-1}\,\varrho_{n-1}'(g)\,\varrho_n'(g)\,k\,\varrho_n'(g)\,B_r$$

where B_r is the 1st r digits into the kth repetition of $\varrho_n'(g)$ with $0 \leqslant r \leqslant g^n$. We distinguish 2 cases for k, i.e. Case 1 where $1 \leqslant k < a_n$, Case 2 where $k = a_n$, and the a_i is any sequence of increasing positive integers such that $\lim_{i\to\infty} a_i \to \infty$. Now Case 1 has been written in (3.0), and if we have Case 2, then the block B_r consists of the 1st r digits into the 1st writing of $\varrho_{n+1}'(g)$ with $0 \leqslant r \leqslant g^{n+1}$, i.e. we have joined complete sets of $\varrho_1'(g)$, $\varrho_2'(g), \ldots, \varrho_n'(g)$ repeated $a_1, a_2, \ldots, a_n$ times, resp.

Let $N(t, \alpha, B_j)$ denote the number of occurrences of the block B_j in the 1st t digits of $\alpha(\varrho_n'(g), n)$, and $N(B_j, \varrho_i'(g))$, the number of occurrences of the block B_j commencing in one period of $\varrho_i'(g)$ and terminating in at most $j-1$ places of the next $\varrho_i'(g)$. We define for Case 1 (it will serve for Case 2 as well with a slight adjustment),

$$(3.1)\qquad I = \Big(\sum_{i=1}^{n-1} a_i N(B_j, \varrho_i'(g)) + kN(B_j, \varrho_n'(g)) + N(B_j, r)\Big)/t$$

where

$$(3.2)\qquad t = \sum_{i=1}^{n-1} a_i g^i + kg^n + r$$

and $N(B_j, r)$ denotes the number of occurrences of B_j in the set of r digits into the $(k+1)$st repetition of $\varrho_n'(g)$. Accounting for anomalous blocks across the end of $\varrho_i'(g)$ and $\varrho_{i+1}'(g), \ldots$, etc., we have [6, p. 243, (2.5)]

$$(3.3)\qquad |N(t, \alpha, B_j)/t - I| \leqslant n(j-1)/t < n(j-1)/a_{n-1}g^{n-1}$$

since it is clear from (3.2) that $t > a_{n-1}g^{n-1}$. Thus as $n\to\infty$ (even for a constant a_{n-1}), we have

$$(3.4)\qquad \lim_{n\to\infty} |N(t, \alpha, B_j)/t - I| = 0$$

which implies that

$$(3.5)\qquad \lim_{t\to\infty} N(t, \alpha, B_j)/t = \lim_{n\to\infty} I.$$

From this point on, we may follow the arguments in [6, pp. 244–246, Lemma 2, through (2.22)] where now we have Z_i/m^i replaced by $\varrho_i'(g)/(g^{\lambda_i}-1)$ with $\lambda_i = g^i$ and $\omega(m^i)$ replaced by $\omega(g^{\lambda_i}-1) = \lambda_i = g^i$. In this way, we will arrive at (similar to 6, p. 246, (2.21), (2.22))

$$(3.6)\qquad \lim_{n\to\infty} I = \lim_{n\to\infty} N(B_j, \varrho_n'(g))/g^n = 1/g^j$$

where we have made use of the fact that $\varrho_n'(g)/(g^\lambda-1)$ is (j, ε)-normal as shown in Theorem 1, (2.3). Also, (3.6) holds for all $j = 1, 2, \ldots$ and therefore, the construction of α in (3.0) is a normal number [6, p. 239, (1.0)] for $\alpha = \lim_{n\to\infty} \alpha(\varrho_n'(g), n)$.

For the transcendental non-Liouville nature of α, due to the similarity of the arguments, α will be a non-Liouville [6, Th. 2, p. 247] transcendental if there exist 2 positive constants, such that

$$(3.7)\qquad \delta < a_{n+1}g^{n+1}/S(n, g) < \beta$$

where δ and β are independent of n. We define

$$(3.8)\qquad S(n, g) = \sum_{i=1}^{n} a_i g^i.$$

We have proved the following theorem:

THEOREM 2. *The irrational α constructed in* (3.0) *is a transcendental non-Liouville normal number if there exists 2 positive constants independent of n such that*

$$\delta < a_{n+1}g^{n+1}/S(n, g) < \beta \quad \textit{where} \quad S(n, g) = \sum_{i=1}^{n} a_i g^i.$$

As an illustration of this theorem, suppose $a_i = i$, then we find (we can actually carry out the sums in this case!)

$$(3.9)\quad a_{n+1}g^{n+1}/S(n,g) = a_{n+1}g^{n+1}/(g^{n+1}[n(g-1)-1]+g)/(g-1)^2 = \frac{(g-1)^2+(g-1)^2/n}{(g-1)-1/n+g/ng^{n+1}} \to (g-1) \quad \text{as } n\to\infty$$

since $\sum_{i=1}^{n} ig^i = (g^{n+1}[n(g-1)-1]+g)/(g-1)^2$. Therefore, it is clear that 2 positive constants can be chosen for n sufficiently large such that $\delta < g-1 < \beta$ since g is a positive integer $\geqslant 2$.

In [7, Th. 7, (4.1)], we have discussed in considerable detail the consequences and restrictions of the repetition sequence a_i in relation to the existence of the positive constants δ and β which this theorem requires for the transcendental non-Liouville nature of these irrationals produced by a juxtaposition construction of (j, ε)-normal sets.

4. Normal recurring decimals of Good and (j, ε)-normality. Apparently, it has not been clearly recognized yet in the literature that our results concerning (j, ε)-normality in the rational fractions [5–9] show that Good's original question [1] in 1946, i.e.... "the question arises whether there are recurring decimals possessing normality of order r"... is answered very broadly in the affirmative.

These results [5–9] give the classification and study of *all* rational fractions into three basic Types A, B, and C which may or may not have "normal recurring" expansions or are (j, ε)-normal (in our language) and have an approximately uniform discrete distribution of the fractional parts $\{Zg^i/m\}$ on $[0, 1]$ for $i = 0, 1, \ldots, \omega(m)-1 = \mathrm{ord}_m g - 1$. These, and many other results, in essence, are the study of normal recurring sequences in the sense of Good which we have shown occurs extensively in the periods of broad classes of rational fractions. We have only recently seen the connection of our (j, ε)-normal studies and the normal periodic systems of Korobov.

Since it appears awkward to construct, in a general way, normal recurring sequences either by the topological method of Good, the algorithmic procedure of Korobov, and to determine the Z in the Type B rational fraction $Z/(g^\lambda - 1)$ where $\lambda = g^n$, let us show some examples which are easily set down and whose general distribution properties are easily described that closely resemble the desired normal recurring decimals or normal periodic systems of Korobov.

One difficulty in the algorithmic methods of Korobov presented in [3, Method A (p. 32), Method A_1 (p. 33), Method A_2 (pp. 36–40)] is that all of these require a "human decision", i.e. *looking back* on what one has "written down so far" and produce such and such an n-tuple which "has not been written before" [3, Method A, p. 32]. Also in [3, Method A_1, p. 33], we find a "rule" which requires a "human decision", he says, ... "if there is no such digit δ_{k+n} (i.e. if any value of $\delta_{k+n} \neq \delta_{\mu+1}\ldots$". Clearly, this shows that the construction of a normal periodic system requires the type of decision we just described. Now what we have just pointed out in the procedure of Korobov is not intended to be a criticism of the work, but we wish to emphasize a mathematical point. It is not surprising that this is necessary for this particular Type B since as we said in [5, p. 230]... "An interesting case for Type B shows that we may *construct* rational fractions of Type B which *may or may not* be (j, ε)-normal"..., i.e. be a normal periodic system or not. In the topological procedure of Good, one proceeds around a particular network and writes the normal recurring sequence as determined by the circuit [1, p. 169] and to this extent is "computational". However, in the example we now present of Type B (and earlier with $Z = (23 \text{ or } 29)/(2^8-1)$), the sequence of digits in the expansion are determined by a congruence and also, one can give precise inequalities which hold over the whole period in relation to the observed and expected frequencies of j-tuples. Thus, we can state that $23/(2^8-1)$ is a normal recurring decimal (or it contains a normal periodic system). On the other hand, it can be proved [9, p. 201, (1.5)] that if p is any odd prime, and g, a primitive root mod p^2, then $Z/p < 1$ in lowest terms is (j, ε)-normal for all $j = 1, 2, \ldots, [\log_g p]$ and $\varepsilon = 2/(p-1)+1/(p-1)g^j$, i.e.

$$(4.0)\qquad |N(B_j, g)/(p-1) - 1/g^j| < \varepsilon = 2/(p-1)+1/(p-1)g^j.$$

For example, a case which contains *almost* a normal periodic system of Korobov (or a normal recurring decimal) is $p = 19$ with $g = 2$. Thus

$$(4.1)\qquad 1/19 = .\dot{0}0001101011110010\dot{1}\,|\,00001101\ldots$$

contains all 4-tuples from 0 to 2^4-1 at least once. In fact, every 4-tuple has a count of one except the blocks 1010 and 0101 which appear twice (naturally one completes the counts over the end of the period and at most, 3 digits into the next repetition). For the calculation, one computes the residue distribution $2^i \equiv r_i \bmod 19$ where each digit b_i in $1/19$ is given by $b_i = [2r_i/19]$ where each digit b_i is in a 1-1 correspondance with the r_i which for $i = 0, 1, 2, \ldots, p-2$ appear in a scattered or "random" order. In this, no "human decision" is envolved with respect to what residue or digit in the expansion is to appear next, but nevertheless, when the complete period has been set down, we do know some general inequalities about the relative frequencies of the j-tuples.

The (j, ε)-normality illustrated in the above example of Z/p^n for $n \geqslant 1$ where p is an odd prime and g, a primitive root mod p^2, also holds when g is not a primitive root, i.e. only $(g, p) = 1$ and n is sufficiently large [see 5, p. 233, Th. 7]. Finally, we have found [5, p. 233, Th. 6] (j, ε)-

normality in rational fractions $Z/m < 1$ in lowest terms where m is a general composite and $(g, m) = 1$, i.e. if $\omega(m) = \mathrm{ord}_m g$, then

$$|N(B_j, g)/\omega(m) - 1/g^j| < \varepsilon = D/m \tag{4.2}$$

where D/m can be arbitrarily small (i.e. D is fixed for a given set of primes in m and the exponents of those primes in m can increase indefinitely) for $j = 1, 2, \ldots, [\log_g m/D]$. Therefore, many recurring decimals exist in the rationals which meet the "question" posed by Good in 1946 whose distribution properties can be completely described.

5. Final remarks, normal numbers, etc. In all these results, it is interesting to note, for example, that we can prove that in the full period of, say, 1/10687 which has 10686 places in the period since 10 is a primitive root that any j-tuple for $j = 1, 2, \ldots, [\log_{10} 10687] = 4$ *will appear* in the period of 1/10687 with certainty with a frequency of $[10687/10^j]$ or $[10687/10^j]+1$. But, we cannot say, precisely where some particular choice of digits, say, 7648 will make its first appearance, i.e. if

$$1/10687 = .\dot{0}0009357\ldots(7648)\ldots\dot{b}\ldots \tag{5.0}$$

then, we know 7648 will appear once or twice at most, but we cannot predetermine its location.

Deep underneath what we are discussing here, is the difficult, unsolved problem of proving that some given irrational like π, "e", or $\sqrt{2}$ is normal or not in some base. Borel proved that almost all real numbers are absolutely normal (i.e. normal in every positive integer base $\geqslant 2$) where the non-normal numbers have measure zero [11, p. 103], but this existential result does not help in any way to prove that a *given* irrational is normal in any base.

Apparently, we *can answer* the analogous question in the rationals [5, pp. 234 (bottom)–235], i.e. determine those rationals which are (j, ε)-normal or not and show that the frequencies with respect to particular blocks have properties analogous to what one would expect in a normal number. By this we mean to say that in a normal number we can say that all possible j-tuples will occur with a relative frequency in the limit of $1/g^j$ for all j, but we cannot say where a prescribed block will occur.

In the case of the rationals, recently in [7], we have been able to improve on our (j, ε)-normal results over a whole period, and prove that prescribed j-tuples *will appear* somewhere within a block of digits *within the period* of rational fractions of Type A and B of length slightly greater than the square root of the period length. Also, in [8], we showed that there exists in the rationals, a precise dual of Borel's result, i.e. there also exists what we called "absolute (j, ε)-normality" where a given fraction is (j, ε)-normal in a bounded consecutive set of positive integers $g = 2, 3, \ldots, B$.

In 1925, L. E. J. Brouwer [5, p. 234] stated what he thought was an "undecidable" proposition, i.e. to prove that the block 0123456789 will occur somewhere in the infinite sequence of digits of π when represented in the base 10. Of course, this would be decided if it could be shown that π is a normal number.

In forthcoming paper, [10], we submit the mathematical details of a result that we first mentioned in [5, p. 235] related to the (j, ε)-normality of the sequence of nth partial products of the Wallis infinite product which permits us to make a small advance on the Brouwer conjecture in relation to π. By applying our results in [7], we can narrow down the occurrence of the block 0123456789 to a block of digits slightly greater the square root of the period length of the sequence of rational approximations that result from the nth partial products.

Addendum

After completing the foregoing paper, we came across a considerable number of references in [13, pp. 120–121] published in a book by S.K. Stein in 1963 which show that the construction of normal recurring decimals of Good and the normal periodic systems of Korobov have been studied in a wide variety of forms and conceptual views by a number of authors for over 80 years.

There is an extensive discussion in [13, Chap. 9, in particular, see the summary on p. 117] with a historical background of various techniques that have been used to generate normal periodic systems. The earliest result of a general type which we have examined in this reference material is a paper of M. H. Martin [14] in 1934. We find [14, p. 859], "Let us consider the n^r permutations of n different symbols $e_1, e_2, \ldots, e_n$ taken r at a time with repetitions allowed. Can a sequence of these symbols be constructed such that each of these n^r permutations is found exactly once as a subsequence of r consecutive symbols in this sequence?"

We can translate the notation of Martin's paper into that of Korobov if we identify the n different symbols $e_1, e_2, \ldots, e_n$ with the g symbols $0, 1, 2, \ldots, g-1$ where $e_1 = 0, e_2 = 1, \ldots, e_g = g-1$ (other orders of the identification could be chosen as well). Therefore, the sequence for which Martin gives a general algorithmic construction for any g and n is a normal recurring decimal of Good or $\varrho_n(g)$ in Korobov's notation if we further identify "r" in Martin with "n" in Korobov. Thus, Martin's "problem in arrangements" paraphrased producing a normal periodic system would read "Let us consider the g^n permutations of g different symbols $0, 1, 2, \ldots, g-1$ taken n at a time with repetitions allowed. Can a sequence of these symbols (digits!) be constructed such that each of these g^n permutations is found exactly once as a subsequence of n consecutive symbols in this sequence?"

Furthermore, Martin gives an example [14, p. 859] for $g = 3$, $n = 2$, $g^n = 3^2$ of length $g^n+n-1 = 3^2+2-1 = 10$ digits which is the normal periodic system $\varrho_2(3)$, i.e. Martin gives $e_1e_3e_3e_2e_3e_1e_2e_2e_1|e_1 = 022120110|0$ which contains exactly once each of $00, 01, 02, 10, 11, \ldots, 22$ in sequence. He then proceeds to give a general algorithm for the construction of a class of $\varrho_n(g)$ and a theorem [14, p. 862] which shows that the algorithm produces the desired properties of normal periodic system. All of Martin's techniques for the algorithm and proofs follow closely the style of Korobov, i.e. combinatorial, arrangements, selections, etc. rather than topological in the sense of Good.

We might also mention here that since the original work of Martin, Good, de Bruijn, etc. in 1946 that a fairly large literature [15, pp. 128–141, in particular, p. 129, and p. 131, Th. 15 (de Bruijn's, i.e. the number of possible $\varrho_n(2)$ is $2^{2^{n-1}-n}$)] has grown in relation to the applications of normal periodic systems in base 2. Most significant is in electronic logical design [15, Chap. VI, *Nonlinear shift register sequences*] where the binary configuration in groups of n consecutive digits in the sequence of length 2^n+n-1 can be used to sense information at sources identified with $0, 1, 2, \ldots, 2^n-1$ represented in binary form. The unique feature being that each source is tested (or requested for information!) only once with no duplications. Apparently in combinatorial theory, we frequently see $\varrho_n(2)$ called a "de Bruijn" sequence [16].

We see that the result of Martin in 1934 gives a general answer to the question raised by I. J. Good 12 years later in 1946, and to which Korobov gave in 1951 a completely general algorithm to produce all possible $\varrho_n(g)$.

Recently, we have learned that de Bruijn's result [11] for $g = 2$ was anticipated by Flye-Ste-Marie [17] in 1894 who gave the same result as a solution to a combinatorial problem in arrangements.

In 1956, de Bruijn and van Aardenne-Ehrenfest [18] generalized the result of Flye-Ste-Marie and de Bruijn for base 2 and proved that the number of distinct normal recurring decimals $\varrho_n(g)$ of Type B as stated in Theorem 1 of this paper for any base $g \geqslant 2$ is given by $(g!)^{g^{n-1}}/g^n$. An easily available proof of this results was given by J. H. van Lint in 1973 in [20, p. 84, Th. 9.1.3].

H. Fredricksen [19] has made a detailed study of most of the known literature for de Bruijn sequence algorithms for base 2.

Finally, we emphasize that (2.3) of Theorem 1 in this paper shows that, henceforth, the definition of (j, ε)-normality in the rationals must include the case $\varepsilon = 0$. Accordingly, we have introduced this requirement in the basic definition.

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