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Corrigendum to my paper "On twin almost primes" and an addendum on Selberg's sieve

by

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Professor H. Montgomery has brought to my attention the fact that the proof of Lemma 3 in my paper [1] is not satisfactory. Indeed, the argument following (4.12) is incorrect in several places and the quadratic form $\sum \frac{a_\delta}{\delta}$ is not positive definite in general.

On the basis of general principles, one expects Lemma 3 to be true as it stands and that the treatment of [1] deals in fact with the main part of the sum. This is indeed the case.

In order to deal correctly with the right-hand side of (4.10) we note first that

$$(1)\qquad \sum_{\substack{r<\zeta\\ (r,2\delta)=1}} \frac{\mu^2(r)}{\varphi(r)}\Big(\sum_{\substack{n<\zeta/r\\ n|2\delta}} \frac{\mu^2(n)}{\varphi(n)}\Big)^2$$

$$= \frac{2\delta}{\varphi(2\delta)}\Big(\log\zeta + A_0 + A_1\sum_{p|2\delta}\frac{\log p}{p-1}\Big) -$$

$$-\frac{\varphi(2\delta)}{2\delta}\sum_{r|2\delta}\sum_{s|2\delta}\frac{\mu^2(r)}{\varphi(r)}\frac{\mu^2(s)}{\varphi(s)}\max(\log r, \log s) + O(\delta^\varepsilon\zeta^{-1/4})$$

as one can see developing the square in (1) and summing over r using (4.14) of [1]. Now if we use (1) in place of (4.14), in the resulting formula (4.15) there appears an extra term of the type

$$(2)\qquad -(1+o(1))\frac{x}{(\log\zeta)^2}\sum_\delta \frac{a_\delta}{\varphi(\delta)}\Big(\sum_{m|\delta}\theta_m\Big)$$

where θ_m is defined by

$$(3)\qquad \sum_{m|\delta}\theta_m = \Big(\frac{\varphi(2\delta)}{2\delta}\Big)^2\sum_{r|2\delta}\sum_{s|2\delta}\frac{\mu^2(r)}{\varphi(r)}\frac{\mu^2(s)}{\varphi(s)}\max(\log r, \log s)$$

(here δ is odd). We have to show that the double sum appearing in (2) is smaller that the main term of (4.15). In fact we will prove

$$(4)\qquad \Big|\sum_{\delta}\frac{a_\delta}{\varphi(\delta)}\sum_{m|\delta}\theta_m\Big| \ll (\log\log x)^{c_1}\sum_{\delta}\frac{a_\delta}{\varphi(\delta)},$$

which clearly is amply sufficient for the proof of Lemma 3.

Rather than dealing directly with the sum in our case, we have thought that it would be worthwhile to give a general treatment of sums of the type (2), since they constantly appear in Selberg's sieve. In our monograph [2], pp. 65–70, we have given a transformation formula for a sum $\sum_{\delta}\frac{a_\delta}{f(\delta)}$ where f is multiplicative. We begin by generalizing it to a sum $\sum_{\delta}\frac{a_\delta}{f(\delta)}\Big(\sum_{m|\delta}\theta_m\Big)$ where $\theta_m = 0$ if m is not square-free. Let

$$a_\delta = \sum_{[d,\nu_1,\nu_2]=\delta}\alpha_d\lambda_{\nu_1}\lambda_{\nu_2}$$

where $\lambda_\nu = 0$ if ν is not square-free, let $f_1 = f*\mu$, $f_2 = f_1*\mu$ and define

$$\zeta_r = \mu(r)f_1(r)\sum_{\nu}\frac{\lambda_{\nu r}}{f(\nu r)},$$

$$Z(d, m) = \sum_{r|d}\mu(r)\zeta_{mr}.$$

THEOREM. *We have the identity*

$$(5)\qquad \sum_{\delta}\frac{a_\delta}{f(\delta)}\Big(\sum_{m|\delta}\theta_m\Big) = \sum_{d}\sum_{\substack{m\\(d,m)=(d,n)=1}}\sum_{n}\frac{a_d}{f(d)}\frac{\mu(m)}{f_1(m)}\frac{\mu(n)}{f_1(n)}f_1\big((m,n)\big)T(m,n)Z(d,m)Z(d,n)$$

where

$$T(m,n) = \sum_{\substack{t|(m,n)\\r|d}}\frac{f_2(t)}{f_1(t)}\theta_{\frac{[m,n]}{(m,n)}rt}.$$

Proof. We follow closely the arguments of [2], pp. 65–70. A divisor of $[d,\nu_1,\nu_2]$ is written uniquely in the form $m[u,v]$ where

$$m|d,\quad u|\nu_1,\quad v|\nu_2,\quad ([u,v],d)=1,$$

$$\Big(\frac{[u,v]}{u},\nu_1\Big) = \Big(\frac{[u,v]}{v},\nu_2\Big) = 1$$

because $[\nu_1,\nu_2]$ is square-free. Performing the same transformations as in [2], pp. 65–68, and defining for $(a,b)=1$

$$y(d,a,b) = \sum_{(\nu,a)=1}\frac{\lambda_{b\nu}f\big((d,b\nu)\big)}{f(b\nu)}$$

we find, denoting by S the left-hand side of (5):

$$S = \sum\frac{a_d}{f(d)}\theta_{[u,v]m}f_1(n)\,y\Big(d,\frac{[u,v]}{u},[u,n]\Big)y\Big(d,\frac{[u,v]}{v},[v,n]\Big)$$

where the summation runs over m,n,u,v,d with $m|d$, $(n,d)=1$,

$$([u,v],d)=1,\qquad \Big(\frac{[u,v]}{u},[u,n]\Big) = \Big(\frac{[u,v]}{v},[v,n]\Big) = 1.$$

We can express $y(d,a,b)$ by means of $Z(d,t)$ as follows:

$$y(d,a,b) = \frac{\mu(b)}{f_1(b)}\sum_{r|a}\frac{\mu^2(r)}{f_1(r)}Z(d,br)$$

as one can prove along the lines of [2], pp. 69–70. We deduce that

$$S = \sum_{d}\sum_{\substack{M\\(d,M)=(d,N)=1}}\sum_{N}\frac{a_d}{f(d)}(\Sigma)\,Z(d,M)Z(d,N)$$

where we have written (Σ) for

$$\sum\theta_{[u,v]m}\frac{\mu^2(s)\mu^2(t)f_1(n)}{f_1(s)f_1(t)f_1([u,n])f_1([v,n])}\mu([u,n])\mu([v,n])$$

summed over s,t,u,v,n,m satisfying

$$[u,n]s = M,\qquad [v,n]t = N,\qquad s\,\Big|\,\frac{[u,v]}{u},\qquad t\,\Big|\,\frac{[u,v]}{v},$$

$$m|d,\quad ([u,v],d)=(n,d)=1,\quad \Big(\frac{[u,v]}{u},[u,n]\Big) = \Big(\frac{[u,v]}{v},[v,n]\Big) = 1.$$

Since M,N are square-free the last sum is

$$(6)\qquad \frac{\mu(M)}{f_1(M)}\frac{\mu(N)}{f_1(N)}\sum\theta_{[u,v]m}f_1(n)\mu(s)\mu(t)$$

where the summation is as before. Thus we have to show that the sum in (6) is equal to $f_1\big((M,N)\big)T(M,N)$, if M,N are square-free. We write

$$u = txw,\qquad v = syw,\qquad w = (u,v).$$

The sum in (6) becomes

$$\sum \theta_{stxywm} f_1(n)\mu(s)\mu(t)$$

summed over s, t, x, y, w, m, n such that

$$xst[w,n] = M, \quad yst[w,n] = N, \quad st[w,n] \mid (M,N), \quad m \mid d.$$

We have

$$stxywm = \frac{[M,N]}{(M,N)} st \left(\frac{(M,N)}{st[w,n]}\right)^2 wm$$

and this must be square-free otherwise θ will be 0. It follows that

$$st[w,n] = (M,N), \quad stxywm = \frac{[M,N]}{(M,N)} stwm.$$

Now we write $w = rh$, $n = rk$ where $r = (w,n)$. The sum becomes

$$\sum_{\substack{rsthk=(M,N)\\ m\mid d}} \theta_{\frac{[M,N]}{(M,N)}strhm} f_1(r) f_1(k)\mu(s)\mu(t)$$

$$= \sum_{\substack{l\mid(M,N)\\ m\mid d}} \theta_{\frac{[M,N]}{(M,N)}lm} \sum_{\substack{strh=l\\ lk=(M,N)}} f_1(r)f_1(k)\mu(s)\mu(t)$$

$$= \sum_{\substack{l\mid(M,N)\\ m\mid d}} \theta_{\frac{[M,N]}{(M,N)}lm} f_1\left(\frac{(M,N)}{l}\right) f_2(l),$$

and the proof of the Theorem is complete.

Inequality (4) is now a consequence of

COROLLARY. *Assume that there are A, B, C such that $f(p) \geqslant p - C$ and*

$$|\theta_m| \ll \frac{\mu^2(m)}{f_1(m)} d(m)^A (1+\log m)^B. \tag{7}$$

Then if $a_d \geqslant 0$ we have the inequality

$$\left|\sum \frac{a_\delta}{f(\delta)}\left(\sum_{m\mid\delta}\theta_m\right)\right| \ll (\log\log\xi)^{c_1} \sum \frac{a_\delta}{f(\delta)} \tag{8}$$

where $\xi = \max\delta$, for some constant $c_1 = c_1(A, B, C)$.

Proof. Using (7), one finds easily

$$|T(m,n)| \ll \frac{d\left(\frac{[m,n]}{(m,n)}\right)^A}{f_1\left(\frac{[m,n]}{(m,n)}\right)} \left(1+\log\frac{[m,n]}{(m,n)}\right)^B \times$$

$$\times \sum_{t\mid(m,n)} \frac{|f_2(t)|}{f_1(t)^2} d(t)^A (1+\log t)^B \sum_{r\mid d} \frac{d(r)^A(1+\log r)^B}{f_1(r)}$$

and since $f(p) \geqslant p - C$ the two last sums are majorized by some power of $\log\log\xi$.

Now write

$$m = \Delta M, \quad n = \Delta N, \quad (m,n) = \Delta$$

and use the identity (5). Since $a_d \geqslant 0$, we get

$$\left|\sum_\delta \frac{a_\delta}{f(\delta)}\left(\sum_{m\mid\delta}\theta_m\right)\right|$$

$$\ll (\log\log\xi)^{c_1} \sum_{\substack{d\\ (d,\Delta)=1}} \sum_{\Delta} \frac{a_d}{f(d)} \frac{1}{f_1(\Delta)} \times$$

$$\times \left(\sum_{\substack{M\\ (d,\Delta M)=1}} \frac{\mu^2(\Delta M)}{f_1(M)^2} d(M)^A (1+\log M)^B |Z(d,\Delta M)|\right)^2$$

and by Cauchy's inequality the last square is

$$\ll \sum_{(d,\Delta M)=1} \frac{\mu^2(\Delta M)}{f_1(M)^{5/2}} Z(d,\Delta M)^2.$$

It follows that everything is majorized by

$$(\log\log\xi)^{c_1} \sum_{\substack{d\\ (d,\Delta M)=1}} \sum_{\Delta M} \frac{a_d}{f(d)} \frac{\mu^2(\Delta M)}{f_1(\Delta M)} \frac{1}{f_1(M)^{3/2}} Z(d,\Delta M)^2.$$

Collecting together the terms with a same ΔM, and noting that

$$\sum_{r\mid m} \frac{1}{f_1(r)^{3/2}} \ll 1$$

and

$$\sum \frac{a_\delta}{f(\delta)} = \sum_{\substack{d\\ (d,m)=1}} \sum_{m} \frac{a_d}{f(d)} \frac{\mu^2(m)}{f_1(m)} Z(d,m)^2,$$

we get the result of the Corollary.

References

[1] E. Bombieri, *On twin almost primes*, Acta Arith. 28 (1975), pp. 177–193.

[2] — *Le grand crible dans la théorie analytique des nombres*, Astérisque 18 (1974), Soc. Math. de France.