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On Eisenstein series with characters and the values of Dirichlet L-functions

by

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1. Introduction. Throughout the sequel, let χ denote a primitive character of modulus k. As usual, the Dirichlet L-function $L(s, \chi)$ is defined for $\sigma = \mathrm{Re}(s) > 0$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

If n is a positive integer, the values $L(2n, \chi)$, when χ is even, and $L(2n-1, \chi)$, when χ is odd, are easily calculated [3], [4], [5], [6], [9], [14]. In fact, at the end of Section 5 below, we derive an infinite number of closed form expressions for $L(2n, \chi)$, when χ is even, and $L(2n-1, \chi)$, when χ is odd. One of these is the familiar closed form expression found in [6] and [14]. However, nothing arithmetically is known about the values $L(2n, \chi)$, when χ is odd, and $L(2n+1, \chi)$, when χ is even. The situation is analogous to that of the Riemann zeta-function $\zeta(s)$; the arithmetical nature of $\zeta(2n+1)$ is completely unknown.

E. Grosswald [11] recently discussed the arithmetical nature of a certain series $G_n(i, \chi)$ defined below in Section 3. This series is a character analogue of a series which occurs in a formula for $\zeta(2n+1)$ that is found in Ramanujan's notebooks ([16]; vol. I, p. 259, no. 15; vol. II, p. 177, no. 21): if $\alpha > 0$, then

$$(1.1)\quad (-1)^n \alpha^{2n} \sum_{r=1}^{\infty} \sigma_{-2n-1}(r) e^{-2\pi r/\alpha} = \sum_{r=1}^{\infty} \sigma_{-2n-1}(r) e^{-2\pi r\alpha} + \frac{1}{2}\big(1-(-1)^n \alpha^{2n}\big)\zeta(2n+1) + \frac{1}{\pi} \sum_{k=0}^{n+1} (-1)^{k+1} \zeta(2k)\zeta(2n+2-2k)\alpha^{2k-1},$$

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where

$$\sigma_\nu(r) = \sum_{d|r} d^\nu.$$

Grosswald [10], [12] and J. R. Smart [17] have recently given proofs of Ramanujan's formula. For references to several other proofs of Ramanujan's formula, see [8]. The expressions $G_n(i, \chi)$ do not actually occur in formulas for $L(n, \chi)$. However, K. Katayama [13] has recently proven character analogues of Ramanujan's formula.

In this paper, we develop transformation formulae for analytic Eisenstein series with characters. By the use of the Lipschitz summation formula or a character analogue of the Lipschitz summation formula, the theorems may be converted into theorems giving transformation formulae for certain Lambert series with characters or certain character generalizations of the classical Dedekind eta-function. The Eisenstein series considered here are very similar to those considered by the author in [7]. Throughout the sequel, the transformations under consideration are the modular transformations $Vz = V(z) = (az+b)/(cz+d)$, where a, b, c, and d are rational integers such that $c > 0$ and $ad-bc = 1$.

The transformation formulae yield immediately formulae for L-functions or certain generalizations thereof. Grosswald's result on $G_n(i, \chi)$ is an immediate consequence of one of our theorems and a very special case of a large class of such results. Katayama's analogues of Ramanujan's formula are seen to be special cases of an infinite class of similar formulae.

Appearing in the transformation formulae are certain generalizations of the classical Dedekind sums. These new sums involve characters and generalized Bernoulli functions. In the simplest cases involving the first Bernoulli function and/or the first generalized Bernoulli function, we prove reciprocity theorems for these sums. It will be clear, however, that one can prove reciprocity theorems for the character analogues of Dedekind sums involving higher order Bernoulli functions.

We emphasize that there are essentially no new ideas in this paper. The method used to derive the transformation formulae is precisely the method developed by the author in [7]. For this reason, proofs in Sections 3 and 4 will not be given in detail; only the necessary changes from the proofs in [7] will be indicated. The reader familiar with [7], pp. 12–17, will be able to fill in the details with no difficulty whatsoever.

2. Notation and preliminary results. Let $\mathscr{H} = \{z\colon \operatorname{Im}(z) > 0\}$ denote the upper half-plane. We write $e(z)$ for $e^{2\pi iz}$. Unless otherwise stated, we choose that branch of $\log z$ with $-\pi \leqslant \arg z < \pi$. As customary, the fractional part of x is denoted by $\{x\}$, and the greatest integer less than or equal to x is denoted by $[x]$.

Let

$$G(z, \chi) = \sum_{h=1}^{k-1} \chi(h) e(hz/k)$$

denote the classical Gauss sum. Put $G(\chi) = G(1, \chi)$. We shall need the fundamental property of Gaussian sums ([2], p. 313),

$$G(\chi)G(\bar{\chi}) = \chi(-1)k. \tag{2.1}$$

For $z \in \mathscr{H}$ and $\sigma > 1$, the Lipschitz summation formula ([15], p. 77)

$$\sum_{n=-\infty}^{\infty} (n+z)^{-s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} e(nz) n^{s-1} \tag{2.2}$$

and the character analogue of the Lipschitz summation formula ([6], Example 3, [7], equation (2.5))

$$\sum_{n=-\infty}^{\infty} \chi(n)(n+z)^{-s} = \frac{G(\chi)(-2\pi i/k)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \bar{\chi}(n) e(nz/k) n^{s-1} \tag{2.3}$$

are valid.

The Bernoulli polynomials $B_n(x)$, $-\infty < x < \infty$, $n \geqslant 0$, are generated by ([1], p. 804)

$$\frac{ue^{xu}}{e^u - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{u^n}{n!} \quad (|u| < 2\pi). \tag{2.4}$$

The Bernoulli numbers B_n, $n \geqslant 0$, are defined by $B_n = B_n(0)$. The Bernoulli polynomials satisfy the multiplication theorem ([1], p. 804)

$$B_n(mx) = m^{n-1} \sum_{h=0}^{m-1} B_n(x + h/m). \tag{2.5}$$

The Bernoulli functions $\mathscr{B}_n(x)$ are defined for all real x by

$$\mathscr{B}_n(x) = B_n(x - [x]),$$

except when $n = 1$ and x is an integer m in which case we define $\mathscr{B}_1(m) = 0$. The generalized Bernoulli functions $\mathscr{B}_n(x, \chi)$, $0 \leqslant n < \infty$, are functions with period k that may be defined for all real x by ([6], Theorem 3.1, [7], equation (2.4))

$$\mathscr{B}_n(x, \chi) = k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) \mathscr{B}_n\left(\frac{x+h}{k}\right). \tag{2.6}$$

The generalized Bernoulli numbers $B_n(\chi)$, $0 \leqslant n < \infty$, are defined by

$$B_n(\chi) = \mathscr{B}_n(0, \chi).$$

Unfortunately, our notation conflicts with that of Leopoldt [14]. More precisely, $B_n(\chi) = B_{\bar\chi}^n$, where B_χ^n denotes the nth generalized Bernoulli number in Leopoldt's notation.

3. Transformation formulae for the first class of Eisenstein series. Let χ_1 and χ_2 be primitive character, each of modulus k. Let r_1 and r_2 be arbitrary real numbers. For $\sigma > 2$ and $z \in \mathscr{H}$, define

$$G(z, s;\ \chi_1, \chi_2;\ r_1, r_2) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\chi_1(m)\chi_2(n)}{\big((m+r_1)z+n+r_2\big)^s},$$

where the dash $'$ on the summation sign means that if r_1 and r_2 are both integers, then the pair $m = -r_1$, $n = -r_2$ is to be omitted from the summation. Extend the definition of χ_1 to the set of all real numbers by defining $\chi_1(r) = 0$ if r is not an integer. Define for $\sigma > 0$ and a real,

$$L(s, \chi, a) = \sum_{n>-a} \chi(n)(n+a)^{-s},$$

so that $L(s, \chi, 0) = L(s, \chi)$. It is easily shown that $L(s, \chi, a)$ can be analytically continued to an entire function of s ([7], equation (2.8)). Let

$$\mathscr{L}_{\pm}(s, \chi, a) = L(s, \chi, a) + \chi(-1)e(\pm s/2)L(s, \chi, -a).$$

Lastly, for $z \in \mathscr{H}$ and s complex, define

$$A(z, s;\ \chi_1, \chi_2;\ r_1, r_2) = \sum_{m>-r_1} \chi_1(m) \sum_{n=1}^{\infty} \chi_2(n) e\big(n((m+r_1)z+r_2)/k\big) n^{s-1}$$

and

$$H(z, s;\ \chi_1, \chi_2;\ r_1, r_2) = A(z, s;\ \chi_1, \chi_2;\ r_1, r_2) + \chi_1(-1)\chi_2(-1)e(s/2)A(z, s;\ \chi_1, \chi_2;\ -r_1, -r_2).$$

Proceeding as in [7], pp. 12, 13, we find with the use of (2.3) that for $\sigma > 2$ and $z \in \mathscr{H}$,

(3.1) $$G(z, s;\ \chi_1, \chi_2;\ r_1, r_2) = \frac{G(\chi_2)(-2\pi i/k)^s}{\Gamma(s)} H(z, s;\ \chi_1, \bar\chi_2;\ r_1, r_2) + \chi_1(-r_1)\mathscr{L}_+(s, \chi_2, r_2),$$

which gives the analytic continuation of $G(z, s;\ \chi_1, \chi_2;\ r_1, r_2)$ into the entire complex s-plane.

Let $Q = \{z = x+iy:\ x > -d/c,\ y > 0\}$. Define $R_1 = ar_1 + cr_2$, $R_2 = br_1 + dr_2$, and $\varrho = \varrho(R_1, R_2, c, d) = \{R_2\}c - \{R_1\}d$. For non-negative integers j, μ and ν, and for $z \in Q$, let

$$f(z, s; r_1, r_2; j, \mu, \nu) = \int_C u^{s-1} \frac{\exp\big(-((c\mu+j-\{R_1\})/ck)(cz+d)ku\big)}{\exp\big(-(cz+d)ku\big)-1} \times \frac{\exp\big(((\nu+\{(dj+\varrho)/c\})/k)ku\big)}{\exp(ku)-1}\, du.$$

Here, we choose the branch of u^s with $0 < \arg u < 2\pi$. Also, C is a loop beginning at $+\infty$, proceeding in $\mathscr{H}$, encircling the origin in the positive direction so that $u = 0$ is the only zero of $\big(\exp(-(cz+d)ku)-1\big) \times \big(\exp(ku)-1\big)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. If $s = -N$, where N is a non-negative integer, then, by (2.4) and the residue theorem, we get

(3.2) $$f(z, -N; r_1, r_2; j, \mu, \nu) = -2\pi i k^N \sum_{m+n=N+2} (-1)^m B_m\left(\frac{c\mu+j-\{R_1\}}{ck}\right) B_n\left(\frac{\nu+\{(dj+\varrho)/c\}}{k}\right) \times \frac{(cz+d)^{m-1}}{m!\,n!}.$$

With the above representation, $f(z, -N; r_1, r_2; j, \mu, \nu)$ can be analytically continued in z to all of $\mathscr{H}$.

We are now at last able to state the transformation formulae for $H(z, s; \chi_1, \chi_2; r_1, r_2)$, or equivalently, for $G(z, s; \chi_1, \chi_2; r_1, r_2)$.

Theorem 1. (i) *Suppose that $a \equiv d \equiv 0 \pmod k$. Then for $z \in Q$ and all s,*

(3.3) $$(cz+d)^{-s}(-2\pi i/k)^s G(\chi_2) H(Vz, s; \chi_1, \bar\chi_2; r_1, r_2) + \chi_1(-r_1)(cz+d)^{-s}\Gamma(s)\mathscr{L}_+(s, \chi_2, r_2)$$
$$= \chi_1(-c)\chi_2(-b)\Big\{(-2\pi i/k)^s G(\chi_1) H(z, s; \chi_2, \bar\chi_1; R_1, R_2) + \chi_2(-R_1)\Gamma(s)\mathscr{L}_-(s, \chi_1, R_2) + \chi_1(-1)\chi_2(-1)e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} \chi_1\big([R_2 + d(j-\{R_1\})/c] - \nu\big) \times \chi_2(c\mu + j + [R_1]) f(z, s; r_1, r_2; j, \mu, \nu)\Big\}.$$

(ii) *Suppose that* $b \equiv c \equiv 0 \pmod{k}$. *Then for* $z \in Q$ *and all* s

$$(3.4)\quad (cz+d)^{-s}(-2\pi i/k)^s G(\chi_2) H(Vz, s; \chi_1, \bar{\chi}_2; r_1, r_2) + \chi_1(-r_1)(cz+d)^{-s}\Gamma(s)\mathscr{L}_+(s, \chi_2, r_2)$$
$$= \chi_1(d)\chi_2(a)\Big\{(-2\pi i/k)^s G(\chi_2) H(z, s; \chi_1, \chi_2; R_1, R_2) + \chi_1(-R_1)\Gamma(s)\mathscr{L}_-(s, \chi_2, R_2) + \chi_1(-1)\chi_2(-1)e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\chi_1(j+[R_1]) \times \chi_2\big([R_2+d(j-\{R_1\})/c]+d\mu-\nu\big) f(z, s; r_1, r_2; j, \mu, \nu)\Big\}.$$

Furthermore, if $s = -N$ *is a non-positive integer,* (3.3) *and* (3.4) *are valid for* $z \in \mathscr{H}$ *upon the evaluation of* $f(z, -N; r_1, r_2; j, \mu, \nu)$ *by* (3.2).

Proof. The proof follows exactly along the same lines as that of Theorem 2 in [7], where $\chi_1 = \chi$ and $\chi_2 = \bar{\chi}$. (In [7], p. 16, line 8b, "u" has been omitted after the last parenthesis before du.) Upon obtaining the transformation formulae for $G(Vz, s; \chi_1, \chi_2; r_1, r_2)$, use (3.1) to routinely convert the transformation formulae into the desired results involving $H(Vz, s; \chi_1, \chi_2; r_1, r_2)$.

One could assume in the above work that χ_1 and χ_2 do not necessarily have the same moduli, but the resulting formulae would be more cumbersome.

Formulas (3.3) and (3.4) yield a multitude of interesting formulas for L-functions when at least one of the values $\chi_1(r_1)$, $\chi_1(R_1)$, and $\chi_2(R_1)$ is not zero, and especially when $s = -N$. There are other interesting deductions. Examine (3.3) when $s = -N$, $Vz = -1/z$, and $z = i$. If, in addition, $\chi_1(r_1) = \chi_2(r_2) = 0$, we see from (3.2) that

$$(-k/2\pi)^N G(\chi_2) H(i, -N; \chi_1, \bar{\chi}_2; r_1, r_2) - \chi_2(-1)(-k/2\pi i)^N G(\chi_1) H(i, -N; \chi_2, \bar{\chi}_1; r_2, -r_1)$$

lies in the cyclotomic field over the rational numbers generated by i and the values of χ_1 and χ_2. If either $\chi_1(r_1)$ or $\chi_2(r_2)$ is not zero, then this is not necessarily the case. Indeed, it appears very unlikely that for all values of the parameters r_1 and r_2, the aforementioned expression does belong to the cyclotomic field generated by i and the values of χ_1 and χ_2. Analogous remarks can be made for the results obtained by taking derivatives with respect to z on both sides of (3.3). In particular, if $s = -N$ and $Vz = -1/z$, we find that after taking $M \geqslant N+1$ derivatives, for *all* values of the parameters r_1 and r_2, the Mth derivative of

$$(-kz/2\pi i)^N G(\chi_2) H(-1/z, -N; \chi_1, \bar{\chi}_2; r_1, r_2) - \chi_2(-1)(-k/2\pi i)^N G(\chi_1) H(z, -N; \chi_2, \bar{\chi}_1; r_2, -r_1)$$

evaluated at $z = i$ belongs to the cyclotomic field over the rationals generated by i and the values of χ_1 and χ_2.

The result of Grosswald [11] on $G_n(i, \chi)$, to which we referred in the Introduction, is a special case of the above considerations. First, in (3.3), put $s = -2N$, where N is a non-negative integer, $Vz = -1/z$, $z = i$, $\chi_1 = \chi_2 = \chi$, and $r_1 = r_2 = 0$. Now,

$$H(i, -2N; \chi, \bar{\chi}; 0, 0) = 2A(i, -2N; \chi, \bar{\chi}; 0, 0) = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\chi(m)\bar{\chi}(n)e^{-2\pi mn/k}n^{-2N-1} = 2\sum_{r=1}^{\infty}\chi(r)\sigma_{-2N-1}(r, \chi)e^{-2\pi r/k},$$

where

$$\sigma_\nu(r, \chi) = \sum_{d|r} d^\nu \bar{\chi}^2(d).$$

Thus, with the help of (3.2), equation (3.3) becomes

$$(3.5)\quad (k/2\pi)^{2N} G(\chi)\sum_{r=1}^{\infty}\chi(r)\sigma_{-2N-1}(r, \chi)e^{-2\pi r/k}$$
$$= \chi(-1)(-1)^N (k/2\pi)^{2N} G(\chi)\sum_{r=1}^{\infty}\chi(r)\sigma_{-2N-1}(r, \chi)e^{-2\pi r/k} - \frac{\pi i k^{2N}}{(2N+2)!}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\sum_{m=0}^{2N+2}(-1)^m\chi(\mu+1)\chi(\nu)\binom{2N+2}{m}B_m\left(\frac{\mu+1}{k}\right) \times B_{2N+2-m}\left(\frac{\nu}{k}\right)i^{m-1}.$$

From (2.6),

$$(3.6)\quad \sum_{h=0}^{k-1}\chi(h)B_n(h/k) = k^{-n+1}B_n(\bar{\chi}).$$

Suppose that $\chi(-1)(-1)^N = -1$. Applying (3.6) with $n = m$ and with $n = 2N+2-m$, we find that we may write (3.5) as

$$(3.7)\quad \sum_{r=1}^{\infty}\chi(r)\sigma_{-2N-1}(r, \chi)e^{-2\pi r/k} = -\frac{\chi(-1)G(\bar{\chi})(2\pi/k)^{2N+1}}{4(2N+2)!}\sum_{m=0}^{2N+2}(-1)^m\binom{2N+2}{m}B_m(\bar{\chi})B_{2N+2-m}(\bar{\chi})i^m,$$

upon the use of (2.1).

Since ([6], Corollary 3.4)

(3.8) $$B_{2n+1}(\chi)=0 \quad (\chi \text{ even}, \ n\geqslant 0)$$

and

(3.9) $$B_{2n}(\chi)=0 \quad (\chi \text{ odd}, \ n\geqslant 0),$$

equation (3.7) may be further simplified.

Secondly, in (3.3), put $s=-2N$, where N is a non-negative integer, $Vz=-1/z$, $\chi_1=\chi_2=\chi$, and $r_1=r_2=0$. Differentiate both sides with respect to z and then put $z=i$. Supposing now that $\chi(-1)(-1)^N=+1$, we get after using (3.6) and simplifying

(3.10) $$\sum_{r=1}^{\infty}\chi(r)\sigma_{-2N-1}(r,\chi)\,(N+2\pi r/k)\,e^{-2\pi r/k}$$
$$=\frac{\chi(-1)G(\bar{\chi})\,(2\pi/k)^{2N+1}}{4(2N+2)!}\sum_{m=0}^{2N+2}(-1)^m\binom{2N+2}{m}B_m(\bar{\chi})B_{2N+2-m}(\bar{\chi})\,(m-1)\,i^{m-2}.$$

By using (3.8) and (3.9), the right side of (3.10) may be simplified.

The left sides of (3.7) and (3.10) are rational integral multiples of $G_{2N+1}(i,\chi)$ in Grosswald's notation. Put $a=1$ if χ is even and $a=i$ if χ is odd. Thus, we have shown Grosswald's result.

THEOREM 2 (Grosswald [11], p. 227). *We have*

$$aG_{2N+1}(i,\chi)=\pi^{2N+1}G(\bar{\chi})r(2N+1,\chi),$$

where $r(2N+1,\chi)$ is an algebraic number in the cyclotomic field over the rational field generated by the values of χ.

As we have seen, the above is just one of an infinitude of similar conclusions that one can infer from Theorem 1.

4. Transformation formulae for the second class of Eisenstein series. As before, let χ be a primitive character of modulus k, and let r_1 and r_2 denote arbitrary real numbers. For $\sigma>2$ and $z\in\mathscr{H}$, define

$$G_1(z,s;\chi;r_1,r_2)=\sum_{m,n=-\infty}^{\infty}{}'\frac{\chi(m)}{\big((m+r_1)z+n+r_2\big)^s}$$

and

$$G_2(z,s;\chi;r_1,r_2)=\sum_{m,n=-\infty}^{\infty}{}'\frac{\chi(n)}{\big((m+r_1)z+n+r_2\big)^s},$$

where the dash $'$ on the summation sign has the same meaning as in the previous section. For $z\in\mathscr{H}$ and s complex, define

$$A_1(z,s;\chi;r_1,r_2)=\sum_{m>-r_1}\chi(m)\sum_{n=1}^{\infty}e\Big(n\big((m+r_1)z+r_2\big)\Big)n^{s-1},$$

$$A_2(z,s;\chi;r_1,r_2)=\sum_{m>-r_1}\sum_{n=1}^{\infty}\chi(n)e\Big(n\big((m+r_1)z+r_2\big)/k\Big)n^{s-1},$$

(4.1) $$H_1(z,s;\chi;r_1,r_2)$$
$$=A_1(z,s;\chi;r_1,r_2)+\chi(-1)e(s/2)A_1(z,s;\chi;-r_1,-r_2),$$

and

(4.2) $$H_2(z,s;\chi;r_1,r_2)$$
$$=A_2(z,s;\chi;r_1,r_2)+\chi(-1)e(s/2)A_2(z,s;\chi;-r_1,-r_2).$$

Furthermore, for a real and $\sigma>1$, let

$$Z(s,a)=\sum_{n>-a}(n+a)^{-s}$$

and

$$Z_{\pm}(s,a)=Z(s,a)+e(\pm s/2)Z(s,-a).$$

Both $Z(s,a)$ and $Z_{\pm}(s,a)$ have analytic continuations into the entire complex plane, for they are easily expressed in terms of the Hurwitz zeta-function.

Proceeding as in [7], pp. 12, 13, but using (2.2) rather than (2.3), we deduce that

(4.3) $$G_1(z,s;\chi;r_1,r_2)=\frac{(-2\pi i)^s}{\Gamma(s)}H_1(z,s;\chi;r_1,r_2)+\chi(-r_1)Z_+(s,r_2).$$

Let $\lambda(r)$ denote the characteristic function of the integers. Proceeding as in [7], pp. 12, 13, we find with the use of (2.3) that

(4.4) $$G_2(z,s;\chi;r_1,r_2)$$
$$=\frac{G(\chi)\,(-2\pi i/k)^s}{\Gamma(s)}H_2(z,s;\bar{\chi};r_1,r_2)+\lambda(r_1)\mathscr{L}_+(s,\chi,r_2).$$

Formulas (4.3) and (4.4) provide analytic continuations of $G_1(z,s;\chi;r_1,r_2)$ and $G_2(z,s;\chi;r_1,r_2)$, respectively, into the whole complex s-plane.

For non-negative integers j and μ and for $z\in Q$, define

$$f^*(z,s;r_1,r_2;j,\mu)$$
$$=\int_C u^{s-1}\frac{\exp\big(-\big((c\mu+j-\{R_1\})/ck\big)(cz+d)ku\big)}{\exp\big(-(cz+d)ku\big)-1}\,\frac{\exp\big(\{(dj+\varrho)/c\}u\big)}{\exp(u)-1}\,du.$$

Again, we choose the branch of u^s with $0 < \arg u < 2\pi$, and C is a loop beginning at $+\infty$, proceeding in $\mathscr{H}$, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$\left(\exp\left(-(cz+d)ku\right)-1\right)\left(\exp(u)-1\right)$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. If $s = -N$ is a non-positive integer, then by (2.4) and the residue theorem, we have

$$(4.5)\quad f^*(z, -N; r_1, r_2; j, \mu)$$
$$= -2\pi i \sum_{m+n=N+2} (-1)^m B_m\left(\frac{c\mu+j-\{R_1\}}{ck}\right) B_n\left(\{(dj+\varrho)/c\}\right) k^{m-1}\frac{(cz+d)^{m-1}}{m!\,n!}.$$

With the above representation, $f^*(z, -N; r_1, r_2; j, \mu)$ can be analytically continued in z to all of $\mathscr{H}$.

We now state the transformation formulae involving $H_1(z, s; \chi; r_1, r_2)$ and $H_2(z, s; \chi; r_1, r_2)$.

THEOREM 3. *Let $z \in Q$ and suppose that s is an arbitrary complex number.*

(i) *If $a \equiv 0 \pmod{k}$, then*

$$(4.6)\quad (cz+d)^{-s}(-2\pi i/k)^s G(\chi) H_2(Vz, s; \bar{\chi}; r_1, r_2) + \lambda(r_1)(cz+d)^{-s}\Gamma(s)\mathscr{L}_+(s, \chi, r_2)$$
$$= \chi(-b)(-2\pi i)^s H_1(z, s; \chi; R_1, R_2) + \chi(bR_1)\Gamma(s) Z_-(s, R_2) + \chi(b) e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\chi(c\mu+j+[R_1]) f^*(z, s; r_1, r_2; j, \mu).$$

(ii) *If $b \equiv 0 \pmod{k}$, then*

$$(4.7)\quad (cz+d)^{-s}(-2\pi i/k)^s G(\chi) H_2(Vz, s; \bar{\chi}; r_1, r_2) + \lambda(r_1)(cz+d)^{-s}\Gamma(s)\mathscr{L}_+(s, \chi, r_2)$$
$$= \chi(a)(-2\pi i/k)^s G(\chi) H_2(z, s; \bar{\chi}; R_1, R_2) + \lambda(R_1)\chi(a)\Gamma(s)\mathscr{L}_-(s, \chi, R_2) + \chi(-a) e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\chi\left([R_2+d(j-\{R_1\})/c]+d\mu-\nu\right)\times f(z, s; r_1, r_2; j, \mu, \nu).$$

(iii) *If $c \equiv 0 \pmod{k}$, then*

$$(4.8)\quad (cz+d)^{-s}(-2\pi i)^s H_1(Vz, s; \chi; r_1, r_2) + \chi(-r_1)(cz+d)^{-s}\Gamma(s) Z_+(s, r_2)$$
$$= \chi(d)(-2\pi i)^s H_1(z, s; \chi; R_1, R_2) + \chi(-dR_1)\Gamma(s) Z_-(s, R_2) + \chi(-d) e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\chi(c\mu+j+[R_1]) f^*(z, s; r_1, r_2; j, \mu).$$

(iv) *If $d \equiv 0 \pmod{k}$, then*

$$(4.9)\quad (cz+d)^{-s}(-2\pi i)^s H_1(Vz, s; \chi; r_1, r_2) + \chi(-r_1)(cz+d)^{-s}\Gamma(s) Z_+(s, r_2)$$
$$= \chi(-c)(-2\pi i/k)^s G(\chi) H_2(z, s; \bar{\chi}; R_1, R_2) + \lambda(R_1)\chi(-c)\Gamma(s)\mathscr{L}_-(s, \chi, R_2) + \chi(c) e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\chi\left([R_2+d(j-\{R_1\})/c]-\nu\right) f(z, s; r_1, r_2; j, \mu, \nu).$$

Furthermore, if $s = -N$ is a non-positive integer, upon the evaluation of $f(z, -N; r_1, r_2; j, \mu, \nu)$ and $f^*(z, -N; r_1, r_2; j, \mu)$ by (3.2) and (4.5), respectively, (4.6)–(4.9) are valid for $z \in \mathscr{H}$.

Proof. For $z \in \mathscr{H}$, $\sigma > 2$, $M = ma + nc$, and $N = mb + nd$, we have

$$G_2(Vz, s; \chi; r_1, r_2) = \sum_{m,n=-\infty}^{\infty}{}' \chi(n)\left\{\frac{(M+R_1)z+N+R_2}{cz+d}\right\}^{-s}$$
$$= \sum_{M,N=-\infty}^{\infty}{}' \chi(Na-Mb)\left\{\frac{(M+R_1)z+N+R_2}{cz+d}\right\}^{-s}$$
$$= \chi(-b)\sum_{m,n=-\infty}^{\infty}{}' \chi(m)\left\{\frac{(m+R_1)z+n+R_2}{cz+d}\right\}^{-s} \qquad (a \equiv 0 \pmod{k})$$
$$= \chi(a)\sum_{m,n=-\infty}^{\infty}{}' \chi(n)\left\{\frac{(m+R_1)z+n+R_2}{cz+d}\right\}^{-s} \qquad (b \equiv 0 \pmod{k}).$$

To prove (4.6) and (4.7), we follow precisely the method of proof of Theorem 2 in [7], except for one difference. For the proof of (4.6), we make one less change of index of summation. To be precise, in [7], p. 16, line 6b, we put $n' = nk+\nu$, $0 \le n < \infty$, $0 \le \nu \le k-1$. To prove (4.6), the introduction of n' is unnecessary.

To prove (4.8) and (4.9), we follow the method outlined above, but we begin by examining $G_1(Vz, s; \chi; r_1, r_2)$ instead of $G_2(Vz, s; \chi; r_1, r_2)$. The proof of (4.8), like that of (4.6), does not need the introduction of n' mentioned above.

Upon obtaining the transformation formulae involving G_1 and G_2, we now use (4.3) and (4.4) to convert the transformation formulae to the desired formulae containing H_1 and H_2.

By letting $s = -N$ be a non-positive integer in Theorem 3, we can obtain various interesting formulae for L-functions or curious arithmetical results. Comments analogous to those made after Theorem 1 can be made here. Thus, for example, if $s = -N$, r_1 is not an integer, $\chi(br_2) = 0$, $Vz = -1/z$, and $z = i$, we conclude from (4.6) that

$$(-k/2\pi)^N G(\chi) H_2(i, -N; \bar{\chi}; r_1, r_2) - (-2\pi i)^{-N} H_1(i, -N; \chi; r_2, -r_1)$$

lies in the cyclotomic field over the rationals generated by i and the values of χ. In the next section we shall see that some formulae for L-functions that are analogous to Ramanujan's formula (1.1) for $\zeta(2n+1)$ and that are due to Katayama [13] are particular instances of Theorem 3.

5. Character analogues of Ramanujan's formula for $\zeta(2N+1)$. If $\chi(-1)(-1)^N = +1$, we have from (4.1) and (4.2), for $j=1,2$,

$$H_j(z,-N;\chi;0,0) = 2A_j(z,-N;\chi;0,0) = 2A_j(z,-N;\chi), \tag{5.1}$$

say. We remark that $A_j(z,-N;\chi)$, and more generally, $A_j(z,s;\chi;r_1,r_2)$, are easily written in terms of Lambert series. In passing, we might observe from (4.8) that for $c\equiv 0 \pmod k$,

$$(cz+d)^N(-2\pi i)^{-N}A_1(Vz,-N;\chi)-\chi(d)(-2\pi i)^{-N}A_1(z,-N;\chi)$$

is always a polynomial in $(cz+d)$.

THEOREM 4 (Katayama [13]). *Let N denote a non-negative integer and let a be an arbitrary positive number.*

If $N\geqslant 0$ and χ is even, then

$$L(2N+1,\chi) = \frac{2}{k}(-1)^N a^{2N}G(\chi)A_1(i/ka,-2N;\bar\chi)- -2A_2(ika,-2N;\chi)+\frac{2}{\pi}\sum_{m=0}^{N}(-1)^{m+1}\zeta(2m)L(2N+2-2m,\chi)a^{2m-1}. \tag{5.2}$$

If $N\geqslant 1$ and χ is odd, then

$$L(2N,\chi) = -\frac{2i}{k}(-1)^N a^{2N-1}G(\chi)A_1(i/ka,-2N+1;\bar\chi)- -2A_2(ika,-2N+1;\chi)+\frac{2}{\pi}\sum_{m=0}^{N}(-1)^{m+1}\zeta(2m)L(2N+1-2m,\chi)a^{2m-1}. \tag{5.3}$$

Proof. From the functional equation of $L(s,\chi)$ ([2], p. 371), we have

$$\lim_{s\to -N}\Gamma(s)\mathscr{L}_+(s,\chi,0) = \lim_{s\to -N}\Gamma(s)L(s,\chi)\big(1+\chi(-1)e(s/2)\big) = \chi(-1)e^{-\pi iN/2}G(\chi)(k/2\pi)^N L(N+1,\bar\chi). \tag{5.4}$$

Similarly, we find that

$$\lim_{s\to -N}\Gamma(s)\mathscr{L}_-(s,\chi,0) = e^{\pi iN/2}G(\chi)(k/2\pi)^N L(N+1,\bar\chi). \tag{5.5}$$

We find from Theorem 3, (4.5), and (5.4) that for $a\equiv 0 \pmod k$,

$$2(cz+d)^N(-2\pi i/k)^{-N}G(\chi)A_2(Vz,-N;\bar\chi)+ +(cz+d)^N(k/2\pi i)^N\chi(-1)G(\chi)L(N+1,\bar\chi) = 2\chi(-b)(-2\pi i)^{-N}A_1(z,-N;\chi)- -\frac{2\pi i\chi(b)(-1)^N}{(N+2)!}\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{m=0}^{N+2}(-1)^m\binom{N+2}{m}\chi(c\mu+j)\times \times B_m\left(\frac{c\mu+j}{ck}\right)B_{N+2-m}(\{dj/c\})k^{m-1}(cz+d)^{m-1}. \tag{5.6}$$

We deduce from Theorem 3, (3.2), and (5.5) that for $d\equiv 0 \pmod k$,

$$2(cz+d)^N(-2\pi i)^{-N}A_1(Vz,-N;\chi) = 2\chi(-c)(-2\pi i/k)^{-N}G(\chi)A_2(z,-N;\bar\chi)+ +\chi(-c)(-2\pi i/k)^{-N}G(\chi)L(N+1,\bar\chi)- -\frac{2\pi i\chi(c)(-k)^N}{(N+2)!}\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\sum_{m=0}^{N+2}(-1)^m\binom{N+2}{m}\chi([dj/c]-\nu)\times \times B_m\left(\frac{c\mu+j}{ck}\right)B_{N+2-m}\left(\frac{\nu+\{dj/c\}}{k}\right)(cz+d)^{m-1}. \tag{5.7}$$

Suppose first that χ is even. Then because $\chi(-1)(-1)^N = +1$, N is even. Hence, in (5.6), replace N by $2N$, replace χ by $\bar\chi$, let $Vz = -1/z$, and put $z = i/ka$, where $a>0$. Using (3.6), we find that (5.6) yields

$$2(2\pi a)^{-2N}G(\bar\chi)A_2(ika,-2N;\chi)+(2\pi a)^{-2N}G(\bar\chi)L(2N+1,\chi) = 2(-1)^N(2\pi)^{-2N}A_1(i/ka,-2N;\bar\chi)- -\frac{2\pi i}{(N+2)!}\sum_{m=0}^{2N+2}(-1)^m\binom{2N+2}{m}B_m(\chi)B_{2N+2-m}(i/ka)^{m-1}. \tag{5.8}$$

Now for $n\geqslant 1$ and χ even ([6], equation (4.7)),

$$B_{2n}(\chi) = \frac{2(-1)^{n-1}G(\bar\chi)(2n)!}{k(2\pi/k)^{2n}}L(2n,\chi). \tag{5.9}$$

Using Euler's formula for $\zeta(2n)$ and the fact that $\zeta(0) = -1/2$, we have for $n\geqslant 0$,

$$B_{2n} = \frac{2(-1)^{n-1}(2n)!}{(2\pi)^{2n}}\zeta(2n). \tag{5.10}$$

By (3.8), we can replace m by $2m$ on the right side of (5.8). Also, note that, trivially, $B_0(\chi) = 0$ for all χ. Using (2.1), (5.9) and (5.10), we find that (5.8) becomes

$$2A_2(ika,\ -2N;\chi)+L(2N+1,\chi)$$
$$=\frac{2}{k}(-1)^N\alpha^{2N}G(\chi)A_1(i/ka,\ -2N;\bar{\chi})+$$
$$+\frac{2(-1)^N}{\pi}\sum_{m=1}^{N+1}(-1)^m L(2m,\chi)\zeta(2N+2-2m)\alpha^{2N-2m+1},$$

which is equivalent to (5.2).

Alternatively, we can derive (5.2) by letting $Vz=-1/z$, putting $z=ika$, replacing N by $2N$, and replacing χ by $\bar{\chi}$ in (5.7). The only essential difference in the calculation is that now (2.5) must be also employed.

Secondly, suppose that χ is odd. Then N is odd. Thus, in (5.6) replace N by $2N-1$, replace χ by $\bar{\chi}$, let $Vz=-1/z$, and put $z=i/ka$, where $a>0$. Using (3.6), we obtain from (5.6)

(5.11) $$-2(2\pi a)^{-2N+1}G(\bar{\chi})A_2(ika,\ -2N+1;\chi)-(2\pi a)^{-2N+1}G(\bar{\chi})L(2N,\chi)$$
$$=-2i(-1)^N(2\pi)^{-2N+1}A_1(i/ka,\ -2N+1;\bar{\chi})-$$
$$-\frac{2\pi i}{(2N+1)!}\sum_{m=0}^{2N+1}(-1)^m\binom{2N+1}{m}B_m(\chi)B_{2N+1-m}(i/ka)^{m-1}.$$

For odd χ and $n\geqslant 1$ ([6], equation (4.8)),

(5.12) $$B_{2n-1}(\chi)=\frac{2(-1)^{n-1}iG(\bar{\chi})(2n-1)!}{k(2\pi/k)^{2n-1}}L(2n-1,\chi).$$

By (3.9), we may replace m by $2m+1$ on the right side of (5.11). Using (2.1), (5.10) and (5.12), we deduce from (5.11) that

$$-2A_2(ika,\ -2N+1;\chi)-L(2N,\chi)$$
$$=\frac{2i}{k}(-1)^N\alpha^{2N-1}G(\chi)A_1(i/ka,\ -2N+1;\bar{\chi})+$$
$$+\frac{2(-1)^N}{\pi}\sum_{m=0}^{N}(-1)^m L(2m+1,\chi)\zeta(2N-2m)\alpha^{2N-2m-1},$$

which is equivalent to (5.3).

Alternatively, we can derive (5.3) from (5.7).

Theorem 4 is just one of an infinite class of such formulae that can be deduced from Theorem 3 when $s=-N$ and $r_1=r_2=0$. Similar formulae may be derived by applying any modular transformation with one of the entries congruent to zero modulo k and then specifying z. The most interesting results arise from those V which are elliptic and by then

letting z be a fixed point of V. We examine one more such example. A corresponding result for the Riemann zeta-function is indicated by Smart [17].

THEOREM 5. *Let N denote a non-negative integer and let $\varrho=(-1+i\sqrt{3})/2$. Then, if $N\geqslant 0$ and χ is even,*

(5.13) $$L(2N+1,\chi)=\frac{2}{k}(\varrho/k)^{2N}G(\chi)A_1(\varrho,\ -2N;\bar{\chi})-2A_2(\varrho,\ -2N;\chi)-$$
$$-\frac{2i}{\pi}\sum_{m=0}^{N}\zeta(2m)L(2N+2-2m,\chi)(\varrho/k)^{2m-1}.$$

If $N\geqslant 1$ and χ is odd, then

(5.14) $$L(2N,\chi)=\frac{2}{k}(\varrho/k)^{2N-1}G(\chi)A_1(\varrho,\ -2N+1;\bar{\chi})-$$
$$-2A_2(\varrho,-2N+1;\chi)+\frac{2i}{\pi}\sum_{m=0}^{N}\zeta(2m)L(2N+1-2m,\chi)(\varrho/k)^{2m-1}.$$

Proof. In (5.7), let $Vz=-(z+1)/z$ and $z=\varrho$. Observe that ϱ is fixed by V.

Suppose first that χ is even. Replace N by $2N$ and χ by $\bar{\chi}$ to find that

$$2(-1)^N(2\pi/\varrho)^{-2N}A_1(\varrho,\ -2N;\bar{\chi})$$
$$=2(-1)^N(2\pi/k)^{-2N}G(\bar{\chi})A_2(\varrho,-2N;\chi)+(-1)^N(2\pi/k)^{-2N}G(\bar{\chi})L(2N+1,\chi)-$$
$$-\frac{2\pi ik^{2N}}{(2N+2)!}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\sum_{m=0}^{2N+2}(-1)^m\binom{2N+2}{m}\bar{\chi}(\nu)B_m\left(\frac{\mu+1}{k}\right)B_{2N+2-m}\left(\frac{\nu}{k}\right)\varrho^{m-1}.$$

Using (2.1), (2.5), (3.6) and (3.8), we obtain

$$\frac{2}{k}(\varrho/k)^{2N}G(\chi)A_1(\varrho,\ -2N;\bar{\chi})=2A_2(\varrho,\ -2N;\chi)+L(2N+1,\chi)-$$
$$-\frac{2\pi i(-1)^N(2\pi/k)^{2N}}{(2N+2)!G(\bar{\chi})}\sum_{m=0}^{N}\binom{2N+2}{2m}B_{2m}B_{2N+2-2m}(\chi)\varrho^{2m-1}.$$

Using (5.9) and (5.10) in the above, we deduce (5.13) forthwith.

Assume next that χ is odd. Replacing N by $2N-1$ and χ by $\bar{\chi}$ in (5.7), we find, with the aid of (2.5) and (3.6), that

$$-2i(-1)^N(2\pi/\varrho)^{-2N+1}A_1(\varrho,\ -2N+1;\bar{\chi})$$
$$=2i(-1)^N(2\pi/k)^{-2N+1}G(\bar{\chi})A_2(\varrho,\ -2N+1;\chi)+$$
$$+i(-1)^N(2\pi/k)^{-2N+1}G(\bar{\chi})L(2N,\chi)+$$
$$+\frac{2\pi i}{(2N+1)!}\sum_{m=0}^{2N+1}(-1)^m\binom{2N+1}{m}B_mB_{2N+1-m}(\chi)\varrho^{m-1}.$$

Replacing m by $2m$ and using (5.10) and (5.12), we get (5.14) after some routine manipulation.

We remark that Theorem 3 also yields the values of $L(N+1,\chi)$ when $(-1)^N\chi(-1)=-1$. Put $r_1=r_2=0$ and $s=-N$ in (4.1) and (4.2). Then if $(-1)^N\chi(-1)=-1$,

$$H_1(z,-N;\chi;0,0)=0=H_2(z,-N;\chi;0,0).$$

Thus, with the aid of (4.5) and (5.4), equation (4.6) reduces to

$$(5.15)\quad (cz+d)^N\chi(-1)(k/2\pi i)^N G(\chi)L(N+1,\bar\chi) = -\frac{2\pi i\chi(b)(-1)^N}{(N+2)!}\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{m=0}^{N+2}(-1)^m\chi(c\mu+j)\binom{N+2}{m}\times \times B_m\left(\frac{c\mu+j}{ck}\right)B_{N+2-m}(\{dj/c\})k^{m-1}(cz+d)^{m-1}.$$

Although the above is valid under the provision that $a\equiv 0 \pmod k$, a does not explicitly appear in (5.15). Similar formulae for $L(N+1,\bar\chi)$ may be deduced from (4.7) and (4.9). Thus, we obtain three infinite classes of closed form expressions for $L(N+1,\bar\chi)$. It is rather interesting that, upon the division of both sides of (5.15) by $(cz+d)^N$, the right side of (5.15) is independent of z.

Formulas (5.9) and (5.12) result by letting $Vz=-1/z$ and $z=i$ in (5.15). If $N=2n-1$, $n\geqslant 1$, and χ is even, then (5.15) yields (5.9) after a brief calculation; if $N=2n-2$, $n\geqslant 1$, and χ is odd, then (5.15) easily gives (5.12). Thus, we have another, albeit not the most straightforward, proof of the familiar closed form evaluations of $L(n,\chi)$ when $(-1)^n\chi(-1)=1$.

6. Dedekind sums with characters. Let χ be an even, primitive character of modulus k, and let c and d be coprime, positive integers. Then the two Dedekind sums with characters $S_1(d,c;\chi)$ and $S_2(d,c;\chi)$ are defined by

$$(6.1)\qquad S_1(d,c;\chi)=\sum_{n \bmod ck}\chi(n)\mathscr{B}_1(n/ck)\mathscr{B}_1(dn/c)$$

and

$$(6.2)\qquad S_2(d,c;\chi)=\sum_{n \bmod ck}\mathscr{B}_1(n/ck)\mathscr{B}_1(dn/c,\chi).$$

If we formally let $\chi(n)\equiv 1$ in (6.1), so that $k=1$, then $S_1(d,c;\chi)=s(d,c)$, the classical Dedekind sum. If χ were odd, then $S_j(d,c;\chi)\equiv 0$, $j=1,2$. The objective of this section is to prove two reciprocity theorems for $S_1(d,c;\chi)$ and $S_2(d,c;\chi)$. We first transcribe Theorem 3 for the case $s=r_1=r_2=0$.

THEOREM 6. *Let χ be an even, primitive character of modulus k. Put*

$$A_1(z;\chi)=A_1(z,0;\chi)\quad and \quad A_2(z;\chi)=A_2(z,0;\chi).$$

Assume that $z\in\mathscr{H}$.

(i) *If $a\equiv 0 \pmod k$, then*

$$(6.3)\quad G(\chi)A_2(Vz;\bar\chi)=\chi(b)A_1(z;\chi)-\tfrac12 G(\chi)L(1,\bar\chi)-\tfrac12\chi(b)\pi i(z+d/c)B_2(\bar\chi)+\chi(b)\pi iS_1(d,c;\chi).$$

(ii) *If $b\equiv 0 \pmod k$, then*

$$(6.4)\quad G(\chi)A_2(Vz;\bar\chi)=\chi(a)G(\chi)A_2(z;\bar\chi)-\tfrac12 G(\chi)\dot L(1,\bar\chi)+\tfrac12\chi(a)G(\chi)L(1,\bar\chi)+\chi(a)\pi iS_2(d,c;\bar\chi).$$

(iii) *If $c\equiv 0 \pmod k$, then*

$$(6.5)\quad A_1(Vz;\chi)=\chi(d)A_1(z;\chi)-\tfrac12\chi(d)\pi i(z+d/c)B_2(\bar\chi)-\frac{\pi i}{2c(cz+d)}B_2(\bar\chi)+\chi(d)\pi iS_1(d,c;\chi).$$

(iv) *If $d\equiv 0 \pmod k$, then*

$$(6.6)\quad A_1(Vz;\chi)=\chi(c)G(\chi)A_2(z;\bar\chi)+\tfrac12\chi(c)G(\chi)L(1,\bar\chi)-\frac{\pi i}{2c(cz+d)}B_2(\bar\chi)+\chi(c)\pi iS_2(d,c;\bar\chi).$$

Proof of (i). From (4.5) and (4.6), we see that we must calculate

$$(6.7)\quad T_1=\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\chi(c\mu+j)f^*(z,0;0,0;j,\mu) = -2\pi i\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\chi(c\mu+j)\left\{\frac{(cz+d)k}{2}B_2\left(\frac{c\mu+j}{ck}\right)+\frac{1}{2k(cz+d)}B_2(\{dj/c\})-B_1\left(\frac{c\mu+j}{ck}\right)B_1(\{dj/c\})\right\}.$$

First, putting $c\mu+j=n$, then letting $n=\nu k+r$, $0\leqslant\nu\leqslant c-1$, $0\leqslant r\leqslant k-1$, and lastly using (2.5) and (2.6), we get

$$(6.8)\quad \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\chi(c\mu+j)B_2\left(\frac{c\mu+j}{ck}\right)=\sum_{n=1}^{ck}\chi(n)B_2(n/ck) =\sum_{r=0}^{k-1}\chi(r)\sum_{\nu=0}^{c-1}B_2(\nu/c+r/ck)=c^{-1}\sum_{r=0}^{k-1}\chi(r)B_2(r/k)=(ck)^{-1}B_2(\bar\chi).$$

Secondly, since $(a, c) = 1$ and $a \equiv 0 \pmod{k}$, we have $(c, k) = 1$. Thus, by summing on μ first, we see that

$$\sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \chi(c\mu+j) B_2(\{dj/c\}) = 0. \tag{6.9}$$

(In [7], p. 19, lines 12–14, the sentence beginning with "By summing" should read "By summing on ν first, we observe that the contribution of the first expression in $f(z, 0; 0, 0)$ is zero; by summing on μ first, we observe that the contribution of the second expression in $f(z, 0; 0, 0)$ is zero.")

Thirdly, since $B_1(0) = -1/2$, and since

$$\sum_{h=0}^{k-1} \chi(h) B_1(h/k) = 0, \tag{6.10}$$

we have

$$\sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \chi(c\mu+j) B_1\left(\frac{c\mu+j}{ck}\right) B_1(\{dj/c\}) \tag{6.11}$$
$$= \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \chi(c\mu+j) \mathscr{B}_1\left(\frac{c\mu+j}{ck}\right) \mathscr{B}_1(dj/c) - \frac{1}{2}\chi(c) \sum_{\mu=0}^{k-1} \chi(\mu+1) \mathscr{B}_1\left(\frac{\mu+1}{k}\right)$$
$$= \sum_{n \bmod ck} \chi(n) \mathscr{B}_1(n/ck) \mathscr{B}_1(dn/c) = S_1(d, c; \chi),$$

by (6.1).

Putting (6.8), (6.9) and (6.11) in (6.7), we get

$$T_1 = -\pi i(z+d/c) B_2(\bar{\chi}) + 2\pi i S_1(d, c; \chi). \tag{6.12}$$

Using (5.4) and (6.12) in (4.6), we arrive at (6.3).

Proof of (ii). From (3.2) and (4.7), we must calculate

$$T_2 = \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi([dj/c]+d\mu-\nu) f(z, 0; 0, 0; j, \mu, \nu) \tag{6.13}$$
$$= -2\pi i \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi([dj/c]+d\mu-\nu) \left\{\frac{cz+d}{2} B_2\left(\frac{c\mu+j}{ck}\right) + \right.$$
$$\left. + \frac{1}{2(cz+d)} B_2\left(\frac{\nu+\{dj/c\}}{k}\right) - B_1\left(\frac{c\mu+j}{ck}\right) B_1\left(\frac{\nu+\{dj/c\}}{k}\right)\right\}.$$

By summing on ν first, we see that the contribution of the first expression in curly brackets is zero. Since $b \equiv 0 \pmod{k}$ and $(b, d) = 1$, we have $(d, k) = 1$. Hence, by summing on μ first, we see that the contribution of the second expression in curly brackets is zero. Next, using (6.10) twice and (2.6), we have

$$\sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi([dj/c]+d\mu-\nu) B_1\left(\frac{c\mu+j}{ck}\right) B_1\left(\frac{\nu+\{dj/c\}}{k}\right) \tag{6.14}$$
$$= \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \mathscr{B}_1\left(\frac{c\mu+j}{ck}\right) \sum_{\nu=0}^{k-1} \chi(-\nu) \mathscr{B}_1\left(\frac{\nu+d\mu+dj/c}{k}\right)$$
$$= \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \mathscr{B}_1\left(\frac{c\mu+j}{ck}\right) \mathscr{B}_1(d\mu+dj/c, \bar{\chi})$$
$$= \sum_{n=1}^{ck} \mathscr{B}_1(n/ck) \mathscr{B}_1(dn/c, \bar{\chi}) = S_2(d, c; \bar{\chi}),$$

by (6.2).

Putting (6.14) into (6.13) and then (6.13) into (4.7), we obtain (6.4) with the aid of (5.4) and (5.5).

Proof of (iii). From (4.8) and (4.5), we see that we must calculate (6.7) again. The only difference from the previous calculation of (6.7) is the calculation of

$$\sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \chi(c\mu+j) \mathscr{B}_2(dj/c) = k \sum_{j=1}^{c} \chi(j) \mathscr{B}_2(dj/c)$$
$$= \bar{\chi}(d) k \sum_{j=1}^{c} \chi(dj) \mathscr{B}_2(dj/c) = \bar{\chi}(d) k \sum_{j=1}^{c} \chi(j) \mathscr{B}_2(j/c),$$

where twice we used the fact that $c \equiv 0 \pmod{k}$. In the last step, we also used the fact that $(c, d) = 1$. Now put $c = mk$ and $j = \mu k + \nu$, $0 \leqslant \mu \leqslant m-1$, $0 \leqslant \nu \leqslant k-1$. Using (2.5) and (2.6), we find that the above becomes

$$\bar{\chi}(d) k \sum_{\nu=0}^{k-1} \chi(\nu) \sum_{\mu=0}^{m-1} \mathscr{B}_2\left(\frac{\mu k+\nu}{mk}\right) = \frac{\bar{\chi}(d) k}{m} \sum_{\nu=0}^{k-1} \chi(\nu) \mathscr{B}_2(\nu/k) = \frac{\bar{\chi}(d) k}{c} B_2(\bar{\chi}).$$

Using (6.8), (6.11) and the calculation above, we find that (6.7) yields

$$T_1 = -\pi i(z+d/c) B_2(\bar{\chi}) - \frac{\pi i \bar{\chi}(d)}{c(cz+d)} B_2(\bar{\chi}) + 2\pi i S_1(d, c; \chi). \tag{6.15}$$

If we substitute (6.15) into (4.8), we arrive at (6.5).

Proof of (iv). By (4.9) and (3.2), we must calculate (6.13) again. The contributions of the first and third expressions in curly brackets on the right side of (6.13) are the same as before. Putting $d = mk$, using the facts that $(c, m) = (c, k) = 1$, and employing (2.5) and (2.6), we deduce that

$$\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\chi([dj/c]+d\mu-\nu)B_2\left(\frac{\nu+\{dj/c\}}{k}\right)$$
$$= k\sum_{j=1}^{c}\sum_{\nu=0}^{k-1}\chi(\nu)\mathscr{B}_2(\nu/k+mj/c) = k\sum_{\nu=0}^{k-1}\chi(\nu)\sum_{j=1}^{c}\mathscr{B}_2(\nu/k+j/c)$$
$$= \frac{k}{c}\sum_{\nu=0}^{k-1}\chi(\nu)\mathscr{B}_2(c\nu/k) = \frac{k\bar{\chi}(c)}{c}\sum_{\nu=0}^{k-1}\chi(\nu)\mathscr{B}_2(\nu/k) = \frac{\bar{\chi}(c)}{c}B_2(\bar{\chi}).$$

Hence, with (6.14) and the above calculation, we find from (6.13) that

$$T_2 = -\frac{\pi i\bar{\chi}(c)}{c(cz+d)}B_2(\bar{\chi})+2\pi iS_2(d, c; \bar{\chi}).$$

Using the above in (4.9) and employing (5.5), we arrive at (6.6).

Using Theorem 6, we shall derive the aforementioned reciprocity laws.

THEOREM 7. *Let χ be even and let c and d be positive, coprime integers.*

(i) *If $(c, k) = (d, k) = 1$, then*

$$S_1(d, c; \chi)+S_2(c, d; \bar{\chi}) = \frac{d}{2c}B_2(\bar{\chi}). \tag{6.16}$$

(ii) *If $c \equiv 0 \pmod{k}$, then*

$$S_1(d, c; \chi)+S_2(c, d; \bar{\chi}) = \frac{d}{2c}B_2(\bar{\chi})+\frac{\bar{\chi}(d)}{2cd}B_2(\bar{\chi}). \tag{6.17}$$

Proof of (i). For brevity, we write $Tz = -1/z$. Since $(c, d) = (c, k) = 1$, there exists a modular transformation V with $a \equiv 0 \pmod{k}$. Apply (6.3) with z replaced by Tz. Letting $V^*z = (bz-a)/(dz-c)$, we have

$$G(\chi)A_2(V^*z; \bar{\chi}) = \chi(b)A_1(Tz; \chi)-\tfrac{1}{2}G(\chi)L(1, \bar{\chi})- -\tfrac{1}{2}\chi(b)\pi i(Tz+d/c)B_2(\bar{\chi})+\chi(b)\pi iS_1(d, c; \chi). \tag{6.18}$$

Now apply (6.4) to V^* to obtain

$$G(\chi)A_2(V^*z; \bar{\chi}) = \chi(b)G(\chi)A_2(z; \bar{\chi})-\tfrac{1}{2}G(\chi)L(1, \bar{\chi})+ +\tfrac{1}{2}\chi(b)G(\chi)L(1, \bar{\chi})+\chi(b)\pi iS_2(-c, d; \bar{\chi}). \tag{6.19}$$

Lastly, apply (6.6) to T and get

$$A_1(Tz; \chi) = G(\chi)A_2(z; \bar{\chi})+\frac{1}{2}G(\chi)L(1, \bar{\chi})-\frac{\pi i}{2z}B_2(\bar{\chi})+ +\pi iS_2(0, 1; \bar{\chi}). \tag{6.20}$$

It is very easy to show that

$$S_j(0, 1; \chi) = 0 \quad (j = 1, 2) \tag{6.21}$$

and that

$$S_j(-c, d; \chi) = -S_j(c, d; \chi) \quad (j = 1, 2). \tag{6.22}$$

Now multiply (6.20) by $\chi(b)$ and add this to the equation that one gets by subtracting equation (6.19) from equation (6.18). Using (6.21) and (6.22), we find that

$$\chi(b)\pi iS_1(d, c; \chi)+\chi(b)\pi iS_2(c, d; \bar{\chi})-\chi(b)\pi i\frac{d}{2c}B_2(\bar{\chi}) = 0,$$

from whence (6.16) is immediate since $\chi(b) \neq 0$.

Alternatively, we could proceed as follows. Since $(c, d) = (d, k) = 1$, there exists a modular transformation V such that $b \equiv 0 \pmod{k}$. Then apply (6.4) to V with z replaced by Tz, apply (6.3) to V^*, and lastly apply (6.3) to T. Combining the results together in a manner like that above and using (6.21) and (6.22), we arrive at (6.16), but with the roles of c and d interchanged.

Proof of (ii). Let V be a modular transformation with $c \equiv 0 \pmod{k}$. Let $V^*z = (bz-a)/(dz-c)$. Apply (6.5) with z replaced by Tz to get

$$A_1(V^*z; \chi) = \chi(d)A_1(Tz; \chi)-\tfrac{1}{2}\chi(d)\pi i(Tz+d/c)B_2(\bar{\chi})- -\frac{\pi iz}{2c(dz-c)}B_2(\bar{\chi})+\chi(d)\pi iS_1(d, c; \chi). \tag{6.23}$$

Apply (6.6) to V^* and obtain

$$A_1(V^*z; \chi) = \chi(d)G(\chi)A_2(z; \bar{\chi})+\tfrac{1}{2}\chi(d)G(\chi)L(1, \bar{\chi})- -\frac{\pi i}{2d(dz-c)}B_2(\bar{\chi})-\chi(d)\pi iS_2(c, d; \bar{\chi}), \tag{6.24}$$

by (6.22). Lastly, apply (6.6) to T and obtain

$$A_1(Tz; \chi) = G(\chi)A_2(z; \bar{\chi})+\frac{1}{2}G(\chi)L(1, \bar{\chi})-\frac{\pi i}{2z}B_2(\bar{\chi}), \tag{6.25}$$

by (6.21). If we multiply (6.25) by $\chi(d)$ and add the result to the equation obtained by subtracting (6.24) from (6.23), we arrive at (6.17) at once.

Alternatively, we could have proved (ii), with c and d interchanged, by applying (6.6) to V with z replaced by Tz, applying (6.5) to V^*, and then applying (6.3) to T.

Many of the remarks made in this paper have analogies to those that can be made of $\zeta(2n+1)$ [8].

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Quantitative versions of a result of Hecke in the theory of uniform distribution mod 1

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1. Introduction. Let a be an irrational number. Then the sequence (na), $n = 0, 1, \ldots$, is uniformly distributed mod 1, and so we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(\{na\}) = \int_0^1 f(t)\,dt$$

for every Riemann-integrable function f on $[0,1]$, where $\{x\}$ denotes the fractional part of the real number x. Since the Abel summation method includes the summation method of arithmetic means, it follows that

$$\lim_{r\to 1-0}(1-r)\sum_{n=0}^{\infty} f(\{na\})r^n = \int_0^1 f(t)\,dt. \tag{1}$$

From this observation, Hecke [3] deduced easily that the power series $\sum_{n=0}^{\infty}\{na\}z^n$ cannot be continued analytically across the unit circle. More generally, one can show by Hecke's method that the power series $\sum_{n=0}^{\infty} g(\{na\})z^n$ has the unit circle as its natural boundary whenever g is a Riemann-integrable function for which all but finitely many of the integrals $\int_0^1 g(t)e^{2\pi imt}\,dt$, $m\in \mathbf{Z}$, are nonzero (see [6], Ch. 1, Theorem 2.4). For other results on noncontinuable power series of the above type, see [6], Ch. 1, Sect. 2, and the survey article of Schwarz [17].

We remark that in the argument leading to (1), the sequence (na) may, of course, be replaced by any sequence (x_n), $n = 0, 1, \ldots$, of real numbers that is uniformly distributed mod 1. Evidently, an analogous

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