by Fubini's theorem. Replacing b_j $(1 \leqslant j \leqslant n)$ by 0 and letting c_j $(1 \leqslant j \leqslant n)$ to be ∞ in (6), we can infer

$$\lim_{\substack{\lambda_j \to \infty \\ (1 \leqslant j \leqslant n)}} \int_0^\infty \dots \int_0^\infty \{f(t_1+x_1, \dots, t_n+x_n) - f(x_1, \dots, x_n)\} \times \times \chi_{\lambda_1}(t_1) \dots \chi_{\lambda_n}(t_n)\, dt_1 \dots dt_n = 0$$

for the function defined as (5). Thus we get

THEOREM 6. *Let $f(t_1, \dots, t_n)$ be a function defined as* (5) *over the whole space T. Then the Fourier inversion formula*

$$(15) \qquad \lim_{\substack{\lambda_j \to \infty \\ (1 \leqslant j \leqslant n)}} \int_{-\lambda_1}^{\lambda_1} \dots \int_{-\lambda_n}^{\lambda_n} \hat{f}(t_1, \dots, t_n) e^{2\pi i (x_1 t_1 + \dots + x_n t_n)} dt_1 \dots dt_n = f(x_1, \dots, x_n)$$

holds.

Noting the relation (9), we also have

THEOREM 7. *Let $f(t_1, \dots, t_n)$ be a function such that*

$$\frac{\partial^{p_1+\dots+p_n}}{\partial t_1^{p_1} \dots \partial t_n^{p_n}} f(t_1, \dots, t_n) \qquad (p_j = 0 \text{ or } 1)$$

are continuous and integrable over T. Then the Fourier inversion formula (15) *also holds.*

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The Diophantine equation $y^2 = Dx^4+1$, II

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Ljunggren [4] has shown by a deep and complicated method that the equation of the title, where D is a positive integer not a square, has at most two solutions in positive integers x, y. Elementary methods have been employed for special values of D which specify the solutions more closely ([1], [3]).

Conditions of a simple type have been found under which there are no such solutions ([2], [5], [6]). We prove

THEOREM. *Let $D \neq 2$ be an integer such that $u^2 - Dv^2 = 2\varepsilon$ has solutions where $\varepsilon = \pm 1$. Then neither $y^2 = Dx^4+1$ nor $y^2 = 4Dx^4+1$ has a solution with $x > 0$ unless at least one of the equations $X^4 - DY^4 = 2\varepsilon$ or $4X^4 - DY^4 = 2\varepsilon$ has solutions.*

We shall require the following result, which is due to Nagell ([7], Theorems 8, 11).

LEMMA. *Let $D > 2$ be an integer, not a square. Then*

(i) *if the equation $u^2 - Dv^2 = 2$ has solutions in integers u, v then there is exactly one class of solutions, which is therefore ambiguous; if $\alpha = U + VD^{1/2}$ is the fundamental solution, then $\frac{1}{2}\alpha^2$ is the fundamental solution of $u^2 - Dv^2 = 1$;*

(ii) *ditto for the equation $u^2 - Dv^2 = -2$;*

(iii) *at most one of the equations $u^2 - Dv^2 = -1, 2$ and -2 has solutions in integers.*

Proof of the theorem. Let $\alpha = U + VD^{1/2}$ be the fundamental solution of $u^2 - Dv^2 = 2\varepsilon$ and let $\beta = \frac{1}{2}\alpha^2$. If either $y^2 = Dx^4+1$ or $y^2 = 4Dx^4+1$ has any solution with $x > 0$, let x be the smallest positive integer which provides a solution of either of them. Then

$$y + x^2 D^{1/2} \text{ or } y + 2x^2 D^{1/2} = \beta^n, \qquad n \geqslant 1,$$

i.e.

$$x^2 \text{ or } 2x^2 = \frac{\beta^n - \beta'^n}{2D^{1/2}}.$$

If n were even, say $n = 2m$, then

$$x^2 \text{ or } 2x^2 = 2\frac{\beta^m + \beta'^m}{2} \cdot \frac{\beta^m - \beta'^m}{2D^{1/2}} = 2hk,$$

say.

Now clearly both h and k are integers and $h^2 - Dk^2 = 1$, whence $(h, k) = 1$. Thus we must have $k = x_1^2$ or $2x_1^2$, whence $h^2 = Dx_1^4 + 1$ or $4Dx_1^4 + 1$. But this contradicts the assumption that $x > 0$ was the least integer for which either of these equations had solutions, since it is easily seen that $0 < x_1 < x$. Thus this case does not arise.

Thus $n = 2m+1$, say, where $m \geq 0$. Then

$$x^2 \text{ or } 2x^2 = \frac{\beta^{2m+1} - \beta'^{2m+1}}{2D^{1/2}} = \frac{\alpha^{4m+2} - \alpha'^{4m+2}}{2^{2m+2}D^{1/2}}$$
$$= \frac{\alpha^{2m+1} + \alpha'^{2m+1}}{2^{m+1}} \cdot \frac{\alpha^{2m+1} - \alpha'^{2m+1}}{2^{m+1}D^{1/2}} = HK,$$

say.

Again, since $\alpha^2 = 2\beta$, it is easily seen that both H and K are integers, and

$$H^2 - DK^2 = 4 \cdot 2^{-2m-2}(\alpha\alpha')^{2m+1} = 2\varepsilon.$$

Thus K must be odd and $(H, K) = 1$. Thus $K = Y^2$ and $H = X^2$ or $2X^2$, and so $X^4 - DY^4 = 2\varepsilon$ or $4X^4 - DY^4 = 2\varepsilon$, which concludes the proof.

COROLLARY. *Neither of the equations $y^2 = Dx^4 + 1$, $y^2 = 4Dx^4 + 1$ has any solutions in positive integers if*

(i) *$u^2 - Dv^2 = 2$ has solutions in integers u, v and either D has any prime factor of which 2 is a biquadratic non-residue, or $D \not\equiv 2$, 14, 18, 62, 63, 78 or 79 (mod 80);*

nor if

(ii) *$u^2 - Dv^2 = -2$ has solutions in integers u, v and either D has any prime factor of which -2 is a biquadratic non-residue, or $D \not\equiv 2$, 3, 6, 18, 22, 66 or 67 (mod 80).*

This follows immediately from the theorem on consideration of residues modulo 5 and 16.

As an illustration we observe that since $13^2 - 19 \cdot 3^2 = -2$ it follows that neither $y^2 = 19x^4 + 1$ nor $y^2 = 76x^4 + 1$ possesses solutions in positive integers. Neither of these equations is covered by [2], [5] or [6].

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