and, by Theorem 3, we have $\nu_{17} \leqslant 49.75$ ($\delta = 0.93$), $\nu_{18} \leqslant 52.41$ ($\delta = 0.925$) and $\nu_{19} \leqslant 55.06$ ($\delta = 0.92$). Using these results we easily check that any natural number h with (h_2') satisfies (h_1), if $g \leqslant 19$. (Note that $\frac{g}{k}\log\left(\frac{k}{g}\nu_g\right)$ is, for fixed g, a decreasing function of k.)

When $g \geqslant 20$,

$$g(\log(2g)+\gamma) \geqslant (g+1)\log(2.63g+5.15)$$

holds, which, in virtue of (3.17) and

$$\sum_{j=1}^{g}\frac{1}{j} \geqslant \log g+\gamma,$$

completes the proof of Theorem 2.

Remark. If we make use of the corollary and substitute (3.17) in (h_1), we get the following lower bound for h, which is simpler than (h_1)

$$h > k-1+\left(g+\frac{g}{k}\right)\log\left(2.63k+5.15\frac{k}{g}\right)-\frac{g}{k}\,\frac{k-g}{2.63g+5.15}. \tag{4.4}$$

If $g \geqslant 17$, (4.4) represents the best known approximation to H. For small g Halberstam and Richert, using Porter's tables on ν_g, carried out some computations (see [3] and [4]) that lead to better results.

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Kaplansky's radical and quadratic forms over non-real fields

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In [4], Kaplansky introduced the concept of a *radical* for a field F as the set R of all $a \in \dot{F}$ satisfying $[a, b] = 1$ for all $b \in \dot{F}$ where $[a, b]$ is the Hilbert symbol for a, b. That is, $[a, b] = 1$ if there are $x, y \in F$ such that $ax^2+by^2 = 1$, and $[a, b] = -1$ otherwise. It is easy to see that two other descriptions of R are 1) $a \in R$ if and only if the quadratic form x^2-ay^2 is universal over F and 2) $R = \bigcap_{a\in \dot{F}} G(1, a)$ where $(1, a)$ is the quadratic form x^2+ay^2 and $G(1, a)$ is the non-zero elements of F represented by $(1, a)$. Thus R is a subgroup of $\dot{F}$ containing $\dot{F}^2$. Kaplansky showed that a field whose radical had index 2 on $\dot{F}$ was an ordered field in which every positive element was the sum of two squares and R coincided with the positive elements. He then found the relationship between the radical and the quadratic form structure over generalized Hilbert fields (see [4]).

In this paper the radical for non-real fields (characteristic not 2 is always assumed) is investigated. It turns out that in many cases, results which held in terms of $\dot{F}^2$ can be strengthened by replacing $\dot{F}^2$ with R. For example, Kneser's lemma states that if $\varphi \oplus (a)$ is anisotropic, then $G(\varphi) \nsubseteq G[\varphi \oplus (a)]$. It is immediate from this that if φ is an anisotropic n-ary quadratic form, then φ represents at least n square classes. We will show φ represents at least n cosets of R. We also look at when the radical's index is 4 and what happens to R under quadratic extensions. Finally, an example is given illustrating a field with a radical which is neither $\dot{F}$ or $\dot{F}^2$.

In the remainder of the paper F is always a non-real field of characteristic not 2. Any quadratic form over F represents whole cosets of $\dot{F}^2$. The following proposition shows any φ with $\dim\varphi > 1$ represents cosets of R.

Proposition 1. *Let φ be a quadratic form with diagonalization $(a_1, \ldots, a_n)$, $n \geqslant 2$ and let $r_1, \ldots, r_n \in R$. Then*

$$G(\varphi) = G(r_1a_1, \ldots, r_na_n).$$

Proof. Clearly $G(1, -a) = \{b \mid [a, b] = 1\}$. So since $[r, b] = 1$ for all $b \in K$ and since $[r, b] = [a, b] = 1$ implies $[ra, b] = 1$,

$$G(1, -a) = G(1, -ra) \quad \text{for} \quad a \in \dot{K},\ r \in R.$$

The case for $n = 2$ now follows and those for $n > 2$ are immediate by induction.

COROLLARY. *$(a_1, \ldots, a_n)$, $n \geqslant 3$, is isotropic if and only if $(r_1 a_1, \ldots, r_n a_n)$ is isotropic.*

LEMMA. *If $R \neq \dot{F}$, then $G(1, -b) \neq R$ for any $b \in \dot{F}$.*

Proof. Suppose, in fact, that $G(1, -b) = R$. Clearly $b \notin R$ and so $b \notin G(1, -b)$. Hence $(1, -b, -b, 1)$ is anisotropic ([5], Hilfsatz 5). By Proposition 1, $G(1, -b) = G(r, r') = R$ for $r, r' \in R$. Thus

$$G(1, -b, -b, 1) = \bigcup_{\alpha, \beta \in G(1, -b)} G(\alpha, \beta) = R.$$

But this contradicts Kneser's lemma.

We are now in a position to strengthen Kneser's result. First some notation. If E is a subset of $\dot{F}$ consisting of cosets of $\dot{F}^2$, then denote the number of those cosets by $V(E)$. That is, $V(E) = |E/\dot{F}^2|$. Every quadratic form φ over F represents cosets of R and if $R \neq \dot{F}$ every binary form represents at least two cosets of R. The theorem below now follows by Kneser's lemma and by induction on $\dim\varphi$.

THEOREM 1. *Let φ be an n-ary anisotropic form over a non-real field with $R \neq \dot{F}^2$. Then for $n \geqslant 2$,*

$$V[G(\varphi)] \geqslant nV(R).$$

If we denote the order of $\dot{F}/\dot{F}^2$ by q and the maximum dimension of all anisotropic forms over F by u, we have the immediate corollary.

COROLLARY. *If $q < \infty$, then $u \leqslant q/V(R)$ providing $R \neq \dot{F}^2$.*

In [1] fields with $u = q$ were characterized. As might be expected, fields satisfying $u = q/V(R)$ behave similarly in general as the case $V(R) = 1$. Before proceeding, however, a little more information is needed.

Pfister [7] showed that s, the level of a field (smallest number of squares in a field of which -1 is the sum), when finite, is always a power of two. Scharlau ([9], p. 72) pointed out that if $-a \in R$ and if s_a is the smallest number of squares of which a is the sum, then s_a is also a power of two.

LEMMA. *If $s \geqslant 2$ and $a \notin \dot{F}^2$, then $s_a = s$ for all $-a \in R$.*

Proof. Clearly $s_a \leqslant s+1$ and since $s \geqslant 2$ and s_a is a 2-power, we have $s_a \leqslant s$. Let $\overset{k}{(b)}$ denote the form $\sum_{i=1}^{k} b x_i^2$. Then $a \in G(\overset{s_a}{(1)}) = G(\overset{s_a}{(-a)})$ by Proposition 1. But $G(\overset{s_a}{(-a)}) = -aG(\overset{s_a}{(1)})$. So $a, -a \in G(\overset{s_a}{(1)})$ and $G(\overset{s_a}{(1)})$ being a subgroup of $\dot{F}$ imply $-1 \in G(\overset{s_a}{(1)})$. Hence, $s \leqslant s_a$, and the proof is complete.

From Pfister's proof of Satz 18(d) [8], it is seen that $V[G(1, 1)] \geqslant s$. With a similar argument, we can make the lower bound $sV(R)$.

THEOREM 2. *If F is a non-real field, with $R \neq \dot{F}^2$, then $V[G(1, 1)] \geqslant sV(R)$.*

Proof. If $s = 1$, the result is obvious; and if $s = 2$, it follows from Theorem 1. So assume $s \geqslant 4$ and let $-1 = \sum_{i=1}^{s} x_i^2$, $x_i \in \dot{F}$. Furthermore, let $y_j = x_{2j-1}^2 + x_{2j}^2$, $1 \leqslant j \leqslant s/2$. Suppose $y_1 y_2 \in R$. Then from $-1 = \sum x_i^2$, we get

$$-y_1 y_2 = y_1^2 + y_1 + \sum_{i=3}^{s/2} y_1 y_i.$$

But $y_i \in G(1, 1)$ implies $y_1 y_i \in G(1, 1)$ and so $-y_1 y_2$ is the sum of less than s squares. This contradicts the lemma. So we see that no two distinct y_j lie in the same coset of R. In the same way, no y_i is actually in R. Therefore,

$$|G(1, 1)/R| \geqslant 1 + \tfrac{1}{2}s.$$

Now $G(1, 1)/R$ is a 2-power so the right side is at least s and the theorem is proved.

Using Satz 18(d) again, we obtain this immediate consequence

COROLLARY. *If $s = 2^{s_0}$, $q \geqslant 2^{\frac{s_0(s_0+1)}{2}} V(R)$ providing $R \neq \dot{F}^2$.*

THEOREM 3. *Let F be a non-real field with $q < \infty$ and denote the number of anisotropic forms of dimension k by N_k and let $t = q/V(R)$. Then*

(i) $N_2 \leqslant q/t - 1 + q/t\binom{t}{2}$, $N_k \leqslant q/t\binom{t}{k}$ *for* $k \neq 2$.

(ii) $N_k = q/t\binom{t}{k}$ *for* $k \geqslant 3$ *if and only if*

$$V[G(\varphi)] = (\dim\varphi)\, V(R) \quad \textit{for all anisotropic } \varphi,\ \dim\varphi \geqslant 3.$$

(iii) $N_k = q/t\binom{t}{k}$ *for any particular* $k \geqslant 3$ *if and only if*

$$N_2 = q/t - 1 + q/t\binom{t}{2}.$$

Proof. This theorem is the R-analog of Theorem 3.3 in [1]. The proof is virtually the same. In (i), the case $k = 1$ is obvious. For $k = 2$, consider first the universal forms. They all look like $(1, -a)$, $a \in R$. Discounting $a \in \dot{F}^2$ since it is not anisotropic, we see there are $q/t - 1$ universal, anisotropic binary forms. By Theorem 1, each binary form represents at least 2 cosets of R. So for each of the possible $q-q/t$ determinant values not in $-R$, there are at most $t/2$ non-equivalent binary forms. Hence, there are at most $\frac{t}{2}(q-q/t) = q/t\binom{t}{2}$ non-universal binary forms; and $N_2 \leqslant q/t-1+\frac{q}{t}\binom{t}{2}$. The remainder of the proof is identical to Theorem 3.3 of [1] except cosets of R are considered instead of $\dot{F}^2$.

We now are in a position to start characterizing fields with $u = q/V(R)$. The situation is easier if $V(R) = 1$ since (iii) of the last theorem can be applied immediately. Here a proposition is needed first.

Proposition 2. *Let F be a field with $q < \infty$ and $u = q/V(R)$. Then every anisotropic n-ary form, $n \geqslant 3$, represents exactly n cosets of R. Moreover, every non-universal binary form represents exactly two cosets of R and $s \leqslant 2$.*

Proof. Suppose $-a \notin R$ and let φ be u-dimensional and anisotropic. Then $1 \in G(\varphi)$ and so $\varphi = (1)\oplus\psi$. By Theorem 1, $V[G(\psi)] \geqslant (u-1)V(R)$ and equality follows immediately. Thus ψ represents all cosets of R except $-R$, and we may write $\psi = (a)\oplus\psi'$, $\varphi = (1, a)\oplus\psi'$. φ anisotropic means $G(1, a)\cap -G(\psi') = \emptyset$. Using this and Theorem 1 again, we obtain

$$V[G(\psi')] = (u-2)V(R), \qquad V[G(1, a)] = 2.$$

The binary case now follows. Combining the fact that

$$G(a_1, \ldots, a_n) = \bigcup_{a \in G(a_1, \ldots, a_{n-1})} G(a, a_n)$$

with Theorem 1 and an induction argument yields the result for $n \geqslant 3$. $s \leqslant 2$ is an immediate consequence of the above and Theorem 2.

By Proposition 2 it is possible to determine modulo R all possible forms which represent a given set of cosets of R. For example, if $s = 1$ or $s = 2$ and $-1 \in R$ and $\{a_1R, \ldots, a_nR\}$ is such a set, then the forms representing it are $(a_1r_1, \ldots, a_nr_n)$, $r_i \in R$. If $s = 2$ and $-1 \notin R$ and $\{\pm a_1R, \ldots, \pm a_nR, b_1R, \ldots, b_mR\}$ is such a set (where $b_i \notin -b_jR$, $i \neq j$), then all the forms representing it are $(a_1r_1, a_1\bar{r}_1, \ldots, a_nr_n, a_n\bar{r}_n, r_1'b_1, \ldots, r_m'b_m)$ with $r_i, \bar{r}_i, r_j' \in R$.

If $u = q/V(R)$, Proposition 2 shows that (ii) and (iii) of Theorem 3 apply; and the following theorem is an immediate consequence.

Theorem 4. *Let F be a field with $q < \infty$ and denote $q/V(R)$ by t. Then the following statements are equivalent:*

(i) $u = t$.

(ii) *The Witt group for F has order $(q/t)2^t$.*

(iii) $V[G(\varphi)] = 2V(R)$ *for all non-universal binary forms* φ.

As has been noted already, Kaplansky showed there are no non-real fields F whose radical has index two in $\dot{F}$. Consider the case when the index is four. By the corollary to Theorem 1, $u \leqslant 4$. Since u is never 3 and since $R = \dot{F}$ when $u = 2$, we must have $u = 4$, and F satisfies the statements in the last theorem. Since $N_3 = \frac{q}{4}\binom{4}{3} = q$, the number of non-split quaternion algebras over F is 1 by Theorem 3.8 of [1]. The Witt group structure can be calculated as in Theorem 4.5 of [1]. Furthermore, F must be a non-real generalized Hilbert field (see [4]). From this (or directly from Proposition 2), it is clear that every anisotropic ternary form is determined by its determinant and represents all but one coset of R. In fact, for non-real fields, this property is equivalent to there being only two quaternion algebras. Before showing this, we need a lemma.

Lemma. *Let V be a vector space of dimension r over $GF(q)$ and let $\{H_i\}_{i=1}^n$ be a set of k-dimensional proper subspaces of V satisfying $H_i+H_j = V$ for all $i \neq j$. Then*

$$n \leqslant \frac{q^r-1}{q^{r-k}-1}.$$

Proof. Denote the dual space of V by V' and let $K_i \subseteq V$ be the annihilator of H_i, $1 \leqslant i \leqslant n$. Then $H_i+H_j = V$ implies $K_i \cap K_j = \{0\}$. Since each K_i has dimension $r-k$, $\bigcup_{i=1}^n K_i$ must contain $n(q^{r-k}-1)$ distinct non-zero elements. From $\dim V' = r$, we get $n(q^{r-k}-1) \leqslant q^r-1$.

From Kaplansky's work in [4], it can be seen that if the index of $G(1, a)$ in $\dot{F}$ is at most two for all $a \in F$ (and is two for at least one a), then there are exactly two quaternion algebras over F. Another characterization of this property is given in the next theorem.

Theorem 5. *Let F be a non-real field with $u > 2$, $q < \infty$. Suppose every anisotropic ternary form over F represents all but one coset of R. Then there are exactly two quaternion algebras over F and hence F is a generalized Hilbert field.*

Proof. Clearly $u = 4$. If φ is a quaternion form ($\dim\varphi = 4$, $\det\varphi = 1$), then $\varphi = (1)\oplus\psi$ where ψ is an anisotropic ternary form. By hypothesis $G(\psi)$ must represent everything not in $-R$. So for $a \notin -R$, $\varphi = (1, a, b, ab)$. Since $u = 4$, the number of anisotropic quaternary forms of determinant d is the number of quaternion algebras minus the order of $\dot{F}/G(1, -d)$. Since isotropic quaternion forms are unique, the above shows the number of quaternion algebras is the index of $G(1, a)$ in $\dot{F}$. This is true for every $a \notin -R$ and so $V[G(1, a)]$ is constant. Moreover, the above formula shows

$(1, a, -x, -bx)$ is isotropic if $ab \notin R$. Consequently, every coset of $G(1, a)$ has a non-empty intersection with every coset of $G(1, b)$ providing $ab \notin R$. Pick a set of representatives $\{a_i R\}_{i=1}^{2^t}$ with $a_1 \in -R$ for $\dot{F}/R$ and consider the Z_2-vector space $\dot{F}/R$. If $H_i = G(1, a_i)/R$, then the set $\{H_i\}_{i=2}^{2^t}$ satisfies $(a+H_i)\cap(b+H_i) \neq \emptyset$ for all $a, b \in \dot{F}/R$ if $i \neq j$. But this is equivalent to $H_i + H_j = \dot{F}/R$ for all $i \neq j$ (since char $\dot{F}/R = 2$). So $\{H_i\}_{i=2}^{2^t}$ satisfy the conditions of the lemma and we may conclude $2^t - 1 \leqslant \dfrac{2^t-1}{2^{t-k}-1}$ where $k = \dim H_i$, $i \geqslant 2$. Thus $k = t-1$ and this means $V[G(1, a_i)] = q/2$ for $i \geqslant 2$. The index of every non-universal binary form is two, but this index is the number of quaternion algebras.

Lam and Elman [3] demonstrated that $u \leqslant q/2$ if $s \geqslant 4$ and that u could not lie strictly between $q/2$ and q if $s = 1, 2$. Upon substituting R for $\dot{F}^2$ in their proofs of these results and using Proposition 1 and Theorem 1, we can get an R-analog.

THEOREM 6. *Let F be a non-real field with $q < \infty$. If $u \neq q/V(R)$, then $u \leqslant q/2V(R)$.*

By Proposition 2, we note that $u = q/V(R)$ only if $s = 1$ or 2. Also by the corollary to Theorem 1, $u \leqslant q/V(R)$ always.

We now have completed our work on "strengthening" previous results by viewing them in terms of R instead of $\dot{F}^2$. Next we wish to look at what happens to R and some quadratic forms in quadratic extensions of F. The results here are incomplete as the problem of determining quadratic form structure after a quadratic extension appears to be a difficult one. From now on, let $K = F(\sqrt{a})$ where $a \in \dot{F}$.

PROPOSITION 3. *If φ is a multiplicative and universal form over F, then φ is universal over $K = F(\sqrt{a})$.*

Proof. We use Scharlau's [9] method of transfer. Let $s\colon K \to F$ be any non-zero linear functional. For any $x \in \dot{K}$, $s^*(x\varphi) = \varphi \oplus s^*(x)$ is isotropic over F since φ is universal. If s is defined by $s(1) = 0$, $s(\sqrt{a}) = 1$, $s^*(x\varphi)$ being isotropic implies there is a vector whose value under the quadratic form $x\varphi$ is $e \in \dot{F}$. Since φ is multiplicative, $x\varphi \cong e\varphi$. So $\varphi \oplus x\varphi$ is isotropic over K since $\varphi \oplus e\varphi$ is isotropic over F. But this means $-x \in G(\varphi)$ over K for all $x \in \dot{K}$. Thus, φ is universal over K.

Applying the proposition to the form $(1, -b)$ when $b \in R(F)$ gives the next result.

COROLLARY. $R(F) \subseteq R(K)$.

Suppose $b \in \dot{F}$ and $-b \in R(K)$. By Scharlau's norm principle ([9], Theorem 2.2.6), $N_{K/F}(x) \in G(1, b)$ over F for all $x \in \dot{K}$. But $N_{K/F}(\dot{K})$

$= G(1, -a)$. So $(1, b)$, $b \in \dot{F}$, is universal over K only if $G(1, -a) \subseteq G(1, b)$ over F. We can use this remark to show that if $u(F) > 2$, then $\dot{F} \nsubseteq R(K)$ and so $u(K) > 2$. If $\dot{F} \subseteq R(K)$,

$$G(1, -a) \subseteq \bigcap_{b \in \dot{F}} G(1, b) = R(F).$$

But this contradicts the lemma preceding Theorem 1. Consequently, if F is a field with non-trivial radical (ie $-R \neq \dot{F}^2, \dot{F}$), then so is $K = F(\sqrt{a})$ providing $R(F) \nsubseteq \{1, a\}\dot{F}^2$. Furthermore, by Gross and Fischer [2], $q(K) = \frac{1}{2} q_F V[G(1, -a)]$; and since $V[G(1, -a)] \geqslant 2V(R) \geqslant 4$, $q(K) > q(F)$ providing $q(F) < \infty$. Adjoining square roots then is a method for generating these fields where R is non-trivial.

Using the norm principle again, we can obtain quickly a partial converse to Proposition 3.

PROPOSITION 4. *If φ is a multiplicative form over F and is universal over $K = F(\sqrt{a})$ where $a \in R(F)$, then φ is universal over F.*

This proposition does not necessarily hold if $a \notin R(F)$. For example, let $\varphi = (1, -a)$. Even for binary φ, it is a hard question to discover when φ is universal over K. It is possible for $(1, b)$ to represent $\dot{F}$ over K but $-b \notin R(K)$ — this happens in certain extensions of the 2-adic numbers. We do, however, have the following result which also is a consequence of the norm principle.

PROPOSITION 5. $-x \in R(K) \Rightarrow -N_{K/F}(x) \in R(F)$.

The converse is false though. Let F be the 2-adic numbers and $K = F(\sqrt{-1})$. Then $-(-1) \in R(F)$ but $-\sqrt{-1} \notin R(K)$.

Finally we want to present an example of a non-real field F with $R \neq \dot{F}$ or $\dot{F}^2$. Such fields seem to be unknown in the literature. And even for our example it is not known whether q is finite. Non-real fields with $q \leqslant 8$ were classified in [1], but for $q = 8$ there are still three cases whose existence has not been resolved. These all have non-trivial radicals. In fact, they all satisfy $u = q/V(R)$. This forthcoming example is due to an idea of A. Pfister.

Let Q_3 be the 3-adic numbers and $K_0 = K$. Assume the chain $K_0 \subset K_1 \subset \ldots \subset K_n \subset Q_3$ is constructed through n with the properties that $(1, 2)$ is anisotropic over each K_i and $K_i = K_{i-1}(\sqrt{c})$, $c \in K_{i-1}$, $1 \leqslant i \leqslant n$. Enumerate the algebraic integers in Q_3 and let a be the first one which is in K_n but not in $G(1, 2)$ over K_n. There is such an a since otherwise $(1, 2)$ would be universal over K_n and this can happen for binary forms over algebraic number fields only if they are isotropic. We will now find a $d \in K_n$ such that $a - 2d^2 \in \dot{Q}_3^2 - (-2\dot{K}_n^2)$. This will ensure $(1, 2)$ remains anisotropic over $K_{n+1} = K_n(\sqrt{\dot{a}-2d^2}) \subset Q_3$ and also represents a. Suppose $a = \sum_{i=0}^{\infty} a_i 3^i$ where $a_i \in \{0, 1, 2\}$. If $b = \sum_{i=0}^{\infty} b_i 3^i$, we can make $a-2b^2 \in \dot{Q}_3^2$ by doing the following: let $b_0 = 1$ if $a_0 = 0$, let $b_0 = 0$ if $a_0 = 1$, and let $b_0 = 1$, $b_1 = a_1$, $b_2 = a_2 - a_1 + a_1^2 - 1$ if $a_0 = 2$. Fix b to be such a rational integer. We note that if $c \equiv 1 \pmod{27}$, then $a-2(bc)^2 \in \dot{Q}_3^2$ also.

Let $\mathscr{L}$ be any finite prime of K_n for which 2, 3, a, b are units. Denote the residue class field of K_n with respect to $\mathscr{L}$ by $\bar{K}_n$ and the image of a unit $a \in K_n$ by $\bar{a}$. Since $\bar{K}_n$ is finite $(-\overline{2a}, \bar{1})$ is universal over $\bar{K}_n$. Hence, there exists a unit $x \in K_n$ such that $-\overline{2a} + \bar{x}^2 \notin \dot{\bar{K}}_n^2$. Choose a unit c such that $\bar{x} = \overline{2bc}$. Then $-2[\dot{a}-2(bc)^2]$ is a non-square mod $\mathscr{L}$.

Now since 3, $\mathscr{L}$ are distinct prime spots over K_n, by the Strong Approximation Theorem (see [6]), there is an integer $c_1 \in K_n$ such that

$$|c_1 - 1| < (\tfrac{1}{3})^4, \qquad |c_1 - c|_{\mathscr{L}} < 1.$$

So $c_1 \equiv 1 \pmod{27}$ and $c_1 \equiv c \pmod{\mathscr{L}}$. The d we were originally looking for is $d = bc_1$.

Continue the construction of the field chain by induction and let $K = \bigcup_{i=0}^{\infty} K_i$. Clearly $(1, 2)$ is anisotropic over K. Any element in K comes from some K_i and any such element is the quotient of two algebraic integers. By the construction each of these integers, and hence their quotient, is in $G(1, 2)$ over K_i. Therefore, $(1, 2)$ is universal over K. In particular, $-1 \in G(1, 2)$ implies $s \leqslant 2$ and K is non-real. Since $(1, 1, 3, 3)$ is anisotropic over Q_3, $u(K) \geqslant 4$. But by the Hasse–Minkowski theorem then, we see $u = 4$. K is thus a non-real field with non-trivial radical.

Unfortunately $q(K)$ is probably not finite so examples of Kneser fields ($s \leqslant q < \infty$) with non-trivial radicals (if they exist) are still missing. By using the remarks made just prior to Proposition 4, it is possible to build a chain of fields so that the union has a radical containing more than just the squares. One might then attempt to construct a field with finite q as in Gross and Fischer [2]. However, we were unable to show that when this was done that $R \neq \dot{F}$.

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