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Sieve methods and polynomial sequences

by

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1. Introduction. A famous problem in prime number theory is the "prime twins" conjecture, i.e. the question of whether there are infinitely many primes p such that $p+2$ is again a prime. In 1958 Schinzel [9] suggested the following generalization:

HYPOTHESIS H. Let $f_1, \ldots, f_g$ be distinct irreducible polynomials in $Z[x]$ (with positive leading coefficients) and suppose that $f_1 \ldots f_g$ has no fixed prime divisors. Then there exist infinitely many integers n such that each $f_i(n)$ $(i = 1, \ldots, g)$ is prime.

Let $F_g := f_1 \ldots f_g$, and let P_r denote an almost prime of order r, that is, a number having at most r prime factors when these are counted according to multiplicity. Then H asserts (subject to the stated conditions on F_g) that

$$F_g(n) = P_g \quad \text{infinitely often.} \tag{1.1}$$

When $g = 1$, and $f_1(n) = an+b$, with $(a, b) = 1$, (1.1) reduces to the assertion that there exist infinitely many primes $p \equiv b \bmod a$; this is the famous theorem proved by Dirichlet and the only case where H is known to be true.

2. The distribution of polynomial sequences. Buchstab, Selberg, Miech, Halberstam and Richert, and other authors proved approximations to H, i.e. they found lower bounds for h such that

$$F_g(n) = P_h \quad \text{infinitely often,}$$

where $h = h(g, k)$ and k denotes the degree of F_g.

When $g = 1$, the best known results are stated in [2] and [8]. For the case $g > 1$ we find the following results [3] based on three different types of weighted sieves:

THEOREM 1. *Let $f_1(n), \ldots, f_g(n)$ $(g > 1)$ be distinct irreducible polynomials with integral coefficients, write $F_g(n)$ for the product $f_1(n) \ldots f_g(n)$, and let $k_1, \ldots, k_g, k$ denote the degrees of $f_1, \ldots, f_g$ and F_g respectively.*

Suppose that F_g has no fixed prime divisors. Then

$$F_g(n) = P_h \quad \textit{infinitely often} \tag{2.1}$$

for any natural number h satisfying one of the following three conditions

$$h > k-1+\left(g+\frac{g}{k}\right)\log\left(\frac{k}{g}\nu_g\right)-\frac{g}{k}\,\frac{k-g}{\nu_g} \tag{h_1}$$

where ν_g is defined in Chapter 3, or

$$h > k-1+g\sum_{j=1}^{g}\frac{1}{j}+g\log\left(\frac{2k}{g}+\frac{1}{g+1}\right) \tag{h_2}$$

or

$$h > k-1+g\sum_{j=1}^{g}\frac{1}{j}+g\log\frac{2}{\vartheta}+\sum_{j=1}^{g}[\vartheta k_j], \tag{h_3}$$

where ϑ is an arbitrary number with $0<\vartheta<1$.

In fact, Halberstam and Richert (see [3], Theorem 4 and Theorem C) stated (2.1) in a quantitative form, but we are mainly interested in a comparison of the conditions (h_1), (h_2) and (h_3). To be precise, we shall prove:

THEOREM 2. *Let $g>1$, $k_1, \ldots, k_g$, k be natural numbers with $\sum_{i=1}^{'} k_i = k$. Then any natural number h with (h_2) or (h_3) satisfies (h_1) as well.*

The conditions (h_1), (h_2) and (h_3) correspond to the weights of Richert [8], Selberg [10] and Miech [6], and Kuhn [5] respectively. So we may read Theorem 2 as follows: the choice of weights in [8] appears to be the best known. However one must mention that Buchstab's new weighted sieve method [2], which is exceedingly complicated, gives similar or possibly better results.

Halberstam and Richert [3] formulated a companion conjecture H* that, roughly speaking, $F_g(p) = P_g$ infinitely often, where p denotes a prime. If we compare the results (see [3], Theorem 6 and Theorem D) that correspond to Theorem 1, we may derive from Theorem 2 that the weights, introduced in [8], lead to the best known approximations of H* as well.

3. An upper bound for $\nu_\varkappa$. For each $\varkappa>1$ let $\sigma_\varkappa(u)$ denote the continuous solution of the differential-difference equation

$$\begin{aligned} \sigma_\varkappa(u) &= \frac{2^{-\varkappa}e^{-\varkappa\gamma}}{\Gamma(\varkappa+1)}\,u^\varkappa \qquad (0\leqslant u<2),\\ \left(u^{-\varkappa}\sigma_\varkappa(u)\right)' &= -\varkappa u^{-\varkappa-1}\sigma_\varkappa(u-2) \qquad (u\geqslant 2) \end{aligned} \tag{3.1}$$

where γ is Euler's constant.

Let $\nu_\varkappa$ denote the unique and positive solution x of the equation

$$\eta_\varkappa(x) \equiv \varkappa x^{-\varkappa}\int_x^\infty\left(\frac{1}{\sigma_\varkappa(t-1)}-1\right)t^{\varkappa-1}dt = 1. \tag{3.2}$$

From [1] we quote the following properties of the function (1)

$$f_\varkappa(u) \equiv 1-\sigma_\varkappa(2u) \qquad (u\geqslant 0). \tag{3.3}$$

LEMMA 1. (a) *$f_\varkappa(u)$ is monotonically decreasing towards 0.*

$$\text{(b)}\quad -f'_\varkappa(u) = \varkappa u^{-1}\{f_\varkappa(u-1)-f_\varkappa(u)\} \qquad (u\geqslant 1). \tag{3.4}$$

(c) *$f_\varkappa(u)$ is convex from below for $u\geqslant\varkappa$.*

Ankeny and Onishi [1] deduced results on $\nu_\varkappa$ as well:

LEMMA 2. *Let $\varkappa>1$:*

$$\nu_\varkappa \leqslant 2(e-1)\varkappa+1+2\log\frac{e-1}{e-2}, \tag{3.5}$$

$$\lim_{\varkappa\to\infty}\frac{\nu_\varkappa}{\varkappa} = 2\exp\left\{\int_0^{\log 2}\frac{e^t-1}{t}\,dt-\log\log 2-1\right\} = 2.44\ldots \tag{3.6}$$

Halberstam and Richert [3] conjecture that $\nu_\varkappa/\varkappa$ increases with $\varkappa$; then (3.6) would imply $\nu_\varkappa \leqslant 2.44\ldots\varkappa$, and (3.5) would appear to be rather a weak result. This conjecture is quite difficult to prove, but we can improve (3.5) essentially.

We shall need the following tools:

LEMMA 3. (a) *$f_\varkappa(u)e^{2u}u^{-2\varkappa}$ decreases monotonically for $u\geqslant\varkappa+1$.*

(b) *For every δ, $0<\delta<2$, there exists a $\xi=\xi(\varkappa,\delta)$ such that $\xi<(e^\delta-1)\delta^{-1}\varkappa$ and*

$$f_\varkappa(u)e^{\delta u}\leqslant f_\varkappa(\xi)e^{\delta\xi} \qquad (u\geqslant 0). \tag{3.7}$$

Proof. Using (3.4) and integrating by parts we infer

$$\int_u^\infty f_\varkappa(t)\,dt = -uf_\varkappa(u)-\int_u^\infty tf'_\varkappa(t)\,dt = -uf_\varkappa(u)+\varkappa\int_{u-1}^{u}f_\varkappa(t)\,dt$$

or

$$uf_\varkappa(u) < \varkappa\int_{u-1}^{u}f_\varkappa(t)\,dt \qquad (u\geqslant 1). \tag{3.8}$$

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(1) Note that $f_\varkappa(u) = F_\varkappa(u)/(\Gamma(\varkappa)e^{\varkappa\gamma})$, and $\nu_\varkappa = 2\zeta_\varkappa$ in the notation of [1].

Thus, by Lemma 1(c)

$$uf_\varkappa(u) < \frac{\varkappa}{2}\big(f_\varkappa(u-1)+f_\varkappa(u)\big) \qquad (u \geqslant \varkappa+1)$$

or

$$-f'_\varkappa(u) = \varkappa u^{-1}\big(f_\varkappa(u-1)-f_\varkappa(u)\big) > \varkappa u^{-1} f_\varkappa(u)\,(2u\varkappa^{-1}-2),$$

$$0 > f'_\varkappa(u)+f_\varkappa(u)\,(2-2\varkappa u^{-1}) \tag{3.9}$$

or

$$0 > \big(f_\varkappa(u)\,e^{2u-2\varkappa\log u}\big)' \qquad (u \geqslant \varkappa+1),$$

which completes the proof of part (a).

If $u > 2\varkappa(2-\delta)^{-1}$, we note by (3.9) that

$$-f'_\varkappa(u) f_\varkappa^{-1}(u) > \delta \qquad (u \geqslant \varkappa+1).$$

If $0 < u < 1$, by (3.1),

$$-f'_\varkappa(u) f_\varkappa^{-1}(u) = \frac{\varkappa u^{\varkappa-1}}{\Gamma(\varkappa+1)e^{\varkappa\gamma}-u^\varkappa} < \delta$$

for sufficiently small $u > 0$.

Let u_1, u_2 be respectively the smallest and the largest root of

$$-f'_\varkappa(u) f_\varkappa^{-1}(u) = \delta.$$

Let $\xi = \xi(\varkappa, \delta)$ be the real number in the interval $[u_1, u_2]$ such that

$$f_\varkappa(\xi)\,e^{\delta\xi} = \operatorname*{Max}_{u_1 \leqslant u \leqslant u_2} \big(f_\varkappa(u)\,e^{\delta u}\big). \tag{3.10}$$

If $u < u_1$, by $-f'_\varkappa(u) f_\varkappa^{-1}(u) < \delta$, we infer that $f_\varkappa(u)\,e^{\delta u}$ is increasing, and $f_\varkappa(u)\,e^{\delta u}$ decreases monotonically for $u > u_2$. Considering (3.10), this completes the proof of (3.7). Combining (3.8) and (3.7) we find

$$\xi f_\varkappa(\xi) < \varkappa \int_{\xi-1}^{\xi} f_\varkappa(t)\,dt \leqslant \varkappa f_\varkappa(\xi) \int_{\xi-1}^{\xi} e^{\delta(\xi-t)}\,dt$$

or

$$\xi < (e^\delta-1)\,\delta^{-1}\varkappa.$$

Next we shall prove the main result of this chapter:

THEOREM 3. *For any* δ, $\delta_0 < \delta < 2$, *we have*

$$v_\varkappa \leqslant 2\,\mathrm{Max}\left(\varkappa+1, \frac{\varkappa}{\delta}\log M(\delta)\right)+1+\frac{2}{\delta}\log\frac{\log M(\delta)}{\log M(\delta)-1}, \tag{3.11}$$

where $M(\delta) := \left(\dfrac{e^\delta-1}{\delta}\right)^2 \exp\{2+(\delta-2)\,\delta^{-1}(e^\delta-1)\}$ *and* δ_0 *denotes the unique root of* $M(\delta) = e$.

Remark 1. $\delta_0 < 0.8$, in fact $\delta_0 = 0.79089\ldots$

Remark 2. As $\delta^{-1}\log M(\delta)$ increases with δ, we have

$$\delta^{-1}\log M(\delta) \geqslant \delta_0^{-1},$$

and

$$\mathrm{Max}\left(\varkappa+1, \frac{\varkappa}{\delta}\log M(\delta)\right) = \frac{\varkappa}{\delta}\log M(\delta) \qquad \text{for all } \varkappa \geqslant \delta_0(1-\delta_0)^{-1},$$

at least for all numbers $\varkappa \geqslant 4$.

Remark 3. The best possible constant in the leading term would be $2\delta_0^{-1} = 2.52\ldots$, which should be compared with $2(e-1) = 3.43\ldots$ and $2.44\ldots$ respectively (see Lemma 2).

Proof. Let $x > 0.5$, then we have

$$\eta_\varkappa(2x) = \varkappa x^{-\varkappa}\int_x^\infty \left(\frac{1}{\sigma_\varkappa(2t-1)}-1\right)t^{\varkappa-1}\,dt$$

$$= \varkappa x^{-\varkappa}\int_x^\infty \frac{f_\varkappa(t-0.5)}{1-f_\varkappa(t-0.5)}\,t^{\varkappa-1}\,dt$$

$$\leqslant \varkappa x^{-\varkappa}\frac{f_\varkappa(\xi)\,e^{\delta\xi}}{1-f_\varkappa(x-0.5)}\int_x^\infty e^{-\delta(t-0.5)}t^{\varkappa-1}\,dt$$

in view of Lemma 1(a) and (3.7).

Noting that

$$\int_x^\infty e^{-\delta t}t^{\varkappa-1}\,dt < x^\varkappa e^{-\delta x}(\delta x+1-\varkappa)^{-1} \qquad \text{if} \quad \delta x > \varkappa,$$

and again using (3.7) we have

$$\eta_\varkappa(2x) < \frac{\varkappa}{\delta x+1-\varkappa}\;\frac{f_\varkappa(\xi)\,e^{\delta\xi+\delta(0.5-x)}}{1-f_\varkappa(\xi)\,e^{\delta\xi+\delta(0.5-x)}} \qquad (\delta x > \varkappa). \tag{3.12}$$

For fixed $\varkappa$ and fixed δ we treat the two cases (a) $\xi \geqslant \varkappa+1$ and (b) $\xi < \varkappa+1$.

In case (a) we apply Lemma 3(a):

$$f_\varkappa(\xi)\,e^{\delta\xi} \leqslant f_\varkappa(\varkappa+1)\,e^{2(\varkappa+1)}(\varkappa+1)^{-2\varkappa}e^{\delta\xi-2\xi}\xi^{2\varkappa}.$$

As $e^{(\delta-2)x}x^{2\varkappa}$ increases with x, provided that

$$x < \frac{2\varkappa}{2-\delta}, \qquad \text{and} \qquad \xi < (e^\delta-1)\,\delta^{-1}\varkappa \leqslant \frac{2}{2-\delta}\varkappa \qquad (0 < \delta < 2),$$

we conclude

$$f_\varkappa(\xi)\,e^{\delta\xi} \leqslant f_\varkappa(\varkappa+1)\,e^2\left(\frac{\varkappa}{\varkappa+1}\right)^{2\varkappa}M(\delta)^\varkappa.$$

By (3.8) we find that $f_\varkappa(\varkappa+1) < \frac{\varkappa}{\varkappa+1} f_\varkappa(\varkappa)$, hence

$$f_\varkappa(\xi)e^{\delta\xi} < f_\varkappa(\varkappa)\, M(\delta)^\varkappa, \tag{3.13}$$

using that $\left(1+\frac{1}{\varkappa}\right)^{2\varkappa+1} \geqslant e^2$ for $\varkappa > 1$.

Combining (3.13) and (3.12) we have

$$\eta_\varkappa(2x) < \frac{\varkappa}{\varkappa \log M(\delta)+1-\varkappa} \cdot \frac{f_\varkappa(\varkappa)h(\varkappa, \delta, x)}{f_\varkappa(\varkappa)-f_\varkappa(\varkappa)h(\varkappa, \delta, x)}, \tag{3.14}$$

if $\delta x > \varkappa \log M(\delta)$. Here we put

$$h(\varkappa, \delta, x) := M(\delta)^\varkappa e^{\delta(0.5-x)}.$$

If now

$$x > \frac{1}{2} + \frac{\varkappa}{\delta} \log M(\delta) + \frac{1}{\delta} \log \frac{\log M(\delta)}{\log M(\delta)-1}, \tag{3.15}$$

we obtain

$$M(\delta)^\varkappa e^{\delta(0.5-x)} < \frac{\log M(\delta)-1}{\log M(\delta)}$$

or

$$\frac{1}{\log M(\delta)-1} \frac{h(\varkappa, \delta, x)}{1-h(\varkappa, \delta, x)} < 1.$$

Together with (3.14) this yields, in particular, $\eta_\varkappa(2x) < 1$. As $\eta_\varkappa(x)$ decreases with x, this proves (3.11) in view of (3.2).

In case (b) we start with

$$x > \varkappa+1.5 + \frac{1}{\delta} \log \frac{\log M(\delta)}{\log M(\delta)-1}$$

and infer

$$\frac{1}{\log M(\delta)-1} \frac{e^{\delta(\varkappa+1.5-x)}}{1-e^{\delta(\varkappa+1.5-x)}} < 1. \tag{3.16}$$

If, moreover, x satisfies (3.15), we may derive from (3.12) that

$$\eta_\varkappa(2x) < \frac{\varkappa}{\varkappa \log M(\delta)+1-\varkappa} \cdot \frac{e^{\delta(\varkappa+1.5-x)}}{1-e^{\delta(\varkappa+1.5-x)}}$$

and, by (3.16), $\eta_\varkappa(2x) < 1$, which completes the proof of Theorem 3.

From Theorem 3 we easily get the following explicit upper bounds for $\nu_\varkappa$, choosing $\delta = 1$, $\delta = 0.9$ and $\delta = 0.8$ respectively.

COROLLARY.

$$\nu_\varkappa \leqslant \begin{cases} 2.73\varkappa+3.65 & (\varkappa \geqslant 3), \\ 2.63\varkappa+5.15 & (\varkappa \geqslant 4), \\ 2.54\varkappa+11.58 & (\varkappa \geqslant 4). \end{cases} \tag{3.17}$$

4. Proof of Theorem 2. Comparing the three conditions (h_1), (h_2) and (h_3), the latter one appears to be the most complicated, as the optimal choice of the free parameter ϑ is quite difficult. Considering the modified condition

$$h > k-1+g\sum_{j=1}^{g} \frac{1}{j} + g\log\frac{2k}{g} \tag{h_2'}$$

instead of (h_2), we shall prove that (h_3) implies (h_2').

Assume that $0 < \vartheta \leqslant g/k$, then we have

$$g\log\frac{2}{\vartheta} + \sum_{j=1}^{g} [\vartheta k_j] \geqslant g\log\frac{2k}{g}. \tag{4.1}$$

Now let $g/k < \vartheta < 1$. Then we infer

$$\log\frac{k\vartheta}{g} \leqslant \frac{k\vartheta}{g} - 1$$

or

$$\log\frac{2k}{g} \leqslant \log\frac{2}{\vartheta} + \frac{1}{g}(k\vartheta - g). \tag{4.2}$$

We notice that

$$\sum_{j=1}^{g} [\vartheta k_j] \geqslant \sum_{j=1}^{g} (\vartheta k_j - 1) = \vartheta k - g, \tag{4.3}$$

and (4.2) combined with (4.3), implies (4.1) for all ϑ, $0 < \vartheta < 1$; i.e. any natural number h with (h_3) or (h_2) satisfies (h_2') as well.

For $g \leqslant 16$ Porter [7] computed ν_g:

g	2	3	4	5	6
ν_g	4.42 ...	6.85 ...	9.32 ...	11.80 ...	14.28 ...

g	7	8	9	10	11
ν_g	16.77 ...	19.25 ...	21.74 ...	24.22 ...	26.70 ...

g	12	13	14	15	16
ν_g	29.21 ...	31.68 ...	34.15 ...	36.62 ...	39.09 ...

and, by Theorem 3, we have $\nu_{17} \leqslant 49.75$ ($\delta = 0.93$), $\nu_{18} \leqslant 52.41$ ($\delta = 0.925$) and $\nu_{19} \leqslant 55.06$ ($\delta = 0.92$). Using these results we easily check that any natural number h with (h_2') satisfies (h_1), if $g \leqslant 19$. (Note that $\frac{g}{k}\log\left(\frac{k}{g}\nu_g\right)$ is, for fixed g, a decreasing function of k.)

When $g \geqslant 20$,

$$g(\log(2g)+\gamma) \geqslant (g+1)\log(2.63g+5.15)$$

holds, which, in virtue of (3.17) and

$$\sum_{j=1}^{g}\frac{1}{j} \geqslant \log g+\gamma,$$

completes the proof of Theorem 2.

Remark. If we make use of the corollary and substitute (3.17) in (h_1), we get the following lower bound for h, which is simpler than (h_1)

$$(4.4)\quad h > k-1+\left(g+\frac{g}{k}\right)\log\left(2.63k+5.15\frac{k}{g}\right)-\frac{g}{k}\,\frac{k-g}{2.63g+5.15}.$$

If $g \geqslant 17$, (4.4) represents the best known approximation to H. For small g Halberstam and Richert, using Porter's tables on ν_g, carried out some computations (see [3] and [4]) that lead to better results.

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Kaplansky's radical and quadratic forms over non-real fields

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In [4], Kaplansky introduced the concept of a *radical* for a field F as the set R of all $a \in \dot F$ satisfying $[a, b] = 1$ for all $b \in \dot F$ where $[a, b]$ is the Hilbert symbol for a, b. That is, $[a, b] = 1$ if there are $x, y \in F$ such that $ax^2+by^2 = 1$, and $[a, b] = -1$ otherwise. It is easy to see that two other descriptions of R are 1) $a \in R$ if and only if the quadratic form x^2-ay^2 is universal over F and 2) $R = \bigcap_{a\in\dot F} G(1, a)$ where $(1, a)$ is the quadratic form x^2+ay^2 and $G(1, a)$ is the non-zero elements of F represented by $(1, a)$. Thus R is a subgroup of $\dot F$ containing $\dot F^2$. Kaplansky showed that a field whose radical had index 2 on $\dot F$ was an ordered field in which every positive element was the sum of two squares and R coincided with the positive elements. He then found the relationship between the radical and the quadratic form structure over generalized Hilbert fields (see [4]).

In this paper the radical for non-real fields (characteristic not 2 is always assumed) is investigated. It turns out that in many cases, results which held in terms of $\dot F^2$ can be strengthened by replacing $\dot F^2$ with R. For example, Kneser's lemma states that if $\varphi \oplus (a)$ is anisotropic, then $G(\varphi) \subsetneqq G[\varphi \oplus (a)]$. It is immediate from this that if φ is an anisotropic n-ary quadratic form, then φ represents at least n square classes. We will show φ represents at least n cosets of R. We also look at when the radical's index is 4 and what happens to R under quadratic extensions. Finally, an example is given illustrating a field with a radical which is neither $\dot F$ or $\dot F^2$.

In the remainder of the paper F is always a non-real field of characteristic not 2. Any quadratic form over F represents whole cosets of $\dot F^2$. The following proposition shows any φ with $\dim\varphi > 1$ represents cosets of R.

Proposition 1. *Let φ be a quadratic form with diagonalization $(a_1, \ldots, a_n)$, $n \geqslant 2$ and let $r_1, \ldots, r_n \in R$. Then*

$$G(\varphi) = G(r_1 a_1, \ldots, r_n a_n).$$