The moments of partitions, II*

by

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5. In this section we consider the distributions $p_d(n, m)$, the number of partitions of $n$ into exactly $m$ summands from a given set $A = \{a_1, a_2, \ldots\}$ of monotone increasing integers. For certain $A$'s we determine the asymptotic behaviour of the $k$th moments $\zeta_d^k(n)$ defined by

$$\zeta_d^k(n) = \sum_{m=1}^{n} m^k p_d(n, m).$$

We are especially interested in the case when $A$ is the set of primes. This case is considered in § 7.

Let $\zeta_d(t)$ be defined by

$$\zeta_d(t) = \sum_{a \in A} a^{-t}.$$

Let $\sigma_0$ denote the abscissa of convergence of $\zeta_d(t)$. Since the $a$'s are integers, we have $1 > \sigma_0 > 0$. Let $0 < \eta < 1$ be some fixed constant. We shall consider in this section those $A$'s for which either of the following two conditions are satisfied.

(I) $\sigma_0 < 1 - \eta$.

(II) $\zeta_d(t)$ is regular for $t > 1 - \eta$ except at the point $t = 1$ where it has a pole of order one. In (I) and (II) the estimate

$$\zeta_d(t) = O\left(\left|t\right|^{\sigma_0}\right)$$

holds uniformly in $\text{Re} t > 1 - \eta$ as $|t| \to \infty$, where $c$ is a positive constant.

If (II) holds we define the constants $u_\ell, \ell = 1, 2$, by

$$\zeta_d(t) \zeta_d(\ell t) = \frac{u_1}{(t-1)^2} + \frac{u_2}{(t-1)} + f(t)$$

where $f(t)$ is regular at $t = 1$.

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Let

\[ P(w) = \begin{cases} w_1, & \text{if } w_2 = 0, \\ x - w_2 y + w_1, & \text{if } w_2 \neq 0, \end{cases} \tag{5.1} \]

where \( y \) again denotes Euler's constant.

Now \( p_d(n, m) \) has the generating function

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_d(n, m) \omega^n \omega^m = \prod_{\alpha \in d} (1 - \omega^a)^{-1} = G_d(x, z) \]

which converges for \( |x| < 1 \) and all \( z \). Again let \( \theta \) be the operator

\[ \theta = z \frac{\partial}{\partial z}. \]

Let \( S_d(x, z) \) be defined by

\[ S_d(x, z) = \sum_{\alpha \in d} \frac{x \omega^d}{1 - z \omega^d}. \]

Then we obtain

\[ \theta^k S_d(x, z) = \frac{\theta^k G_d(e^{-a}, 1)}{2\pi} \int \frac{\Gamma(1 + \theta)}{\Gamma(\theta)} \frac{S_d(e^{-a}, 1)}{S_d(e^{-a}, 1)} e^{-a \theta} d\theta, \tag{5.2} \]

where \( a \) is throughout this section defined by

\[ a = \sum_{\alpha \in d} \frac{a}{e^{a} - 1} - (S_d(e^{-a}) - 1) \frac{d S_d(e^{-a})}{d a} \tag{5.3} \]

and

\[ S_d(x, z) = \sum_{i=0}^{\infty} \theta^i S_d(x, z). \]

Furthermore

\[ \theta^{a-1} S_d(x, z) = \sum_{\alpha \in d} \frac{x \omega^d}{(1 - z \omega^d)^a} \]

where \( \theta^a \) is defined by (2.1). Also we obtain

\[ \theta^{a-1} S_d(x, z) = \frac{1}{2\pi j} \int_{\sigma = \infty}^{\sigma = 0} \int_{t = 1}^{t = \infty} e^{-t} \xi_d(t) \Gamma(t) \sum_{j=1}^{\infty} \psi(j) \xi_d(t) dt \]

\[ \sigma > \delta, \ |\arg \omega| < \frac{\pi}{2} - \delta. \tag{5.4} \]

Thus we obtain as before:

**Lemma 5.1.** Let \( s \geq 1 \). Then

\[ \theta^s S_d(e^{-a}, 1) = \frac{\theta^s}{\theta^s - 1} \xi_d(\theta + 1) + O(a^{-\alpha}). \]

**Lemma 5.2.** If condition (1) holds

\[ S_d(e^{-a}, 1) = \frac{\xi_d(1)}{e^{a}} + O(a^{-1+\alpha}). \]

If condition (II) holds

\[ S_d(e^{-a}, 1) = \frac{F(0)}{e^{a}} + O(a^{-1+\alpha}) \]

where \( F(x) \) is defined by (5.1).

Before stating our next result, we require some definitions.

Let \( A(u) \) denote the number of elements of \( A \) which are \( \leq u \).

We define the function \( f_d \) for real \( x > 0 \) by

\[ f_d(x) = \sum_{a \in d} e^{-a x}. \]

\( f_d \) shall denote this function throughout this paper.

We say that \( A \) has property (III) if with \( \epsilon > 0 \) an arbitrary constant, and \( \mu \) a fixed integer > 0,

\[ \sum_{a \in d}(e^{a})^{\mu} e^{-a x} = O(f_{d}^{\mu}(x)) \]

and

\[ f_d(x) f_d(x(1 - f_{d}^{-1+\alpha/2}(x))) = O(1) \]

as \( x \to 0 \).

We say that \( A \) has property (IV) if there exists some constant \( \delta_0 \) with

\[ 1 > \delta_0 > 0 \] and some constant \( \eta_0 \) with

\[ \frac{1}{2} > \eta_0 > 0 \] such that

\[ A(x - f_{d}^{-\alpha}(x))/\log f_{d}(x) \to \infty \]

and

\[ A(x^{-1}) > f_{d}^{\eta}(x) \]

as \( x \to 0 \).

It is proven in [10] that \( A \) has properties (III) and (IV) when either

\[ \frac{\lim \log a_i/\log y}{\lim \log a_i/\log y < \frac{1}{2}} \]

or

\[ A(2u) = O(A(u)) \]

as \( u \to \infty \).
\[ y = \lim_{x \to \infty} \frac{\log \log a}{\log v} \quad \text{exists and} \quad s > 0. \]

We say that \( A \) is a \( P \)-sequence if there does not exist a number \( \rho \) such that \( p_{\rho a} \), for all sufficiently large \( a \).

Let \( A_\rho(a) \) be defined by
\[
\tag{5.5}
A_\rho(a) = \sum_{\alpha \in \mathcal{D}} a^{\rho} g_\rho(e^{-\alpha a})(e^{-\alpha} - 1)^{-\mu}
\]
where \( g_\rho(w) \) is a certain polynomial of degree less than \( \rho \). In particular \( g_1(w) = 1 \) and \( g_\rho(w) = w \).

Let \( D_\rho(a) \) be defined by
\[
\tag{5.6}
D_\rho = \sum_{\alpha_1 = 1}^\infty \sum_{\alpha_2 = 1}^\infty \ldots \sum_{\alpha_{n_\rho} = 1}^\infty \bigg( \prod_{i=1}^{n_\rho} a_{\alpha_i} \bigg) A_{\alpha_1} \ldots A_{\alpha_{n_\rho}}
\]
where the \( \alpha_i \)'s are certain numerical constants and
\[
\alpha_1 + \alpha_2 + \ldots + \alpha_{n_\rho} = 12 \rho.
\]

**Lemma 5.3.** For \( n \) sufficiently large, equation (5.3) has a unique solution. Let \( k \geq 1 \). If condition (I) holds
\[
n = \sum_{\alpha \in \mathcal{D}} - \frac{\rho}{e^{-\alpha} - 1} + \frac{1}{\alpha} + O\left( a^{-1+\eta} \right);
\]
if condition (II) holds
\[
n = \sum_{\alpha \in \mathcal{D}} - \frac{\rho}{e^{-\alpha} - 1} + \frac{1}{\alpha} + O\left( \frac{1}{\alpha} \log \frac{1}{\alpha} \right).
\]

Our first result is

**Theorem 5.1.** Suppose \( A \) satisfies conditions (III) and (IV). Suppose furthermore that either \( A \) is a \( P \)-sequence or that
\[
\lim_{x \to \infty} \log f_\rho(x)/\log x = 0.
\]

Suppose furthermore that
\[
\lim_{x \to \infty} \log \log a_{\rho}/\log x < \infty.
\]

Then if in addition condition (I) holds
\[
\eta_\rho(n) = (2\pi A_\rho)^{-1/2} \exp \left[ \psi - \sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \right] \times a^{-\frac{\rho}{2}} \sum_{k=0}^{\infty} \left( \left( \left( \frac{1}{e^{-\alpha}} - 1 \right)^{a} \right)^{k+1} \right) \left( \left( \left( \frac{1}{e^{-\alpha}} - 1 \right)^{a} \right)^{k} \right) \left( 1 + O\left( a^{\rho} \right) \right) + D_\rho + O\left( f_\rho(x) \right);
\]
if in addition condition (II) holds, then \( \eta_\rho(n) \) is replaced by \( P\left( \log \frac{1}{\alpha} \right) \) in the result for condition (I). \( A_\rho, \alpha, D_\rho, \) and \( P\left( \log \frac{1}{\alpha} \right) \) are defined by (5.5), (5.3), (5.6), and (5.1) respectively. The summation is that of (2.10).

**Proof.** Let
\[
\theta_\rho = 2\pi A_\rho^{-1/2} \log(1 - e^{-\alpha}) \log x.
\]

Then in [10] it is shown that for any constant \( M > 0 \)
\[
G_\rho(e^{-\alpha})/G(\alpha) = O\left( a^{-1+\mu} \right)
\]
for \( \theta_\rho < |\theta| < \pi \).

As in the proof of Lemma 2.6 it follows from (5.4) that for \( \varphi = 1, 2, 3 \) and \( |\theta| < \theta_\rho \)
\[
\frac{\delta(\theta)\delta(\varphi)/(\theta^2 \varphi^2)}{
\sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \log x}
= O\left( a^{-1} \right).
\]

Thus the proof of Lemma 2.6 shows that
\[
\frac{1}{2\pi} \int_{-\theta_\rho}^{\theta_\rho} G_\rho(e^{-\alpha})/G(\alpha) \frac{\delta(\theta)\delta(\varphi)/(\theta^2 \varphi^2)}{
\sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \log x} d\theta = \frac{1}{2\pi A_\rho} \left[ 1 + O\left( f_\rho(x) \right) \right]
\]
(since \( A_\rho > \rho a^{-\rho} f_\rho(a) \) for some constant \( c > 0 \)). Now the theorem follows from (5.7), (5.8) and (5.2).

**Theorem 5.2.** Under the assumptions of Theorem 5.1
\[
(2\pi A_\rho)^{-1/2} \exp \left[ \psi - \sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \right] = \eta_\rho(n) \left[ 1 + O\left( f_\rho(x) \right) \right].
\]

**Proof.** Due to the saddle-point condition, the term
\[
\exp \left[ \psi - \sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \right]
\]
is insensitive to small changes in \( \alpha \). That is if \( \alpha_1 = \alpha + \Delta \alpha \), where \( \Delta \alpha > 0 \)
\[
(5.9) \quad \exp \left[ \psi - \sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \right]
= \left[ \left( 1 - \frac{\alpha}{e^{-\alpha} - 1} \right) \Delta \alpha \right] + O\left( \alpha_1(a)(\Delta \alpha)^2 \right) \exp \left[ \psi - \sum_{\alpha \in \mathcal{D}} \log(1 - e^{-\alpha}) \right].
\]

Let \( a \) be the solution of
\[
n = \sum_{\alpha \in \mathcal{D}} \frac{\alpha}{e^{-\alpha} - 1}.
and \( a_1 \) be the solution of
\[
\eta = \sum_{\alpha \in A} \frac{a}{e^{a_1^2}} - 1 + O\left(\frac{1}{a_1} \log^{-1} \frac{1}{a_1}\right).
\]
First of all, clearly \( a_1 = a + \Delta a \), where \( \Delta a \) is positive. Also
\[
\sum_{\alpha \in A} \frac{a}{e^{a_1^2}} - 1 - \Delta a \Delta a + O\left(\frac{\Delta a^3}{a_1} (\Delta a)^3\right).
\]
Therefore
\[
\frac{1}{\alpha} = a_1 \Delta a + O\left(\frac{1}{a} \log^{-1} \frac{1}{a} + \frac{\Delta a}{a_1} (\Delta a)^2 + \frac{\Delta a}{a^2}\right).
\]
From this it follows that
\[
\Delta a \sim (\Delta a)^{-1}. \tag{5.10}
\]
Now
\[
n - \frac{\sum_{\alpha \in A} a}{e^{a_1} - 1} = 0,
\]
hence from (5.9) and (5.10) it follows that
\[
A \exp \left[ a_1 - \sum_{\alpha \in A} \log(1 - e^{\alpha_1}) \right] = O\left(\exp \left[ a_1 - \sum_{\alpha \in A} \log(1 - e^{\alpha_1}) \right] f_A^{\lambda}(a)\right),
\]
since \( A > Ca_1^2 f_A(a) \) for some \( C > 0 \). Now the theorem easily follows.

At this point let us consider a particular set, namely the set of \( k \)-th powers.

**Theorem 5.3.** Let \( K = \{1, 2^k, 3^k, \ldots\} \) be the set of \( k \)-th powers where \( k \gg 2 \). Let
\[
\bar{m}_k(n) = \frac{\zeta_k(n)}{\zeta_k^2(n)},
\]
\[
\sigma_k^2(n) = \frac{\zeta_k^2(n)}{\zeta_k^2(n)} - \left(\frac{\zeta_k(n)}{\zeta_k(n)}\right)^2.
\]
Then
\[
\bar{m}_k(n) = \zeta(k) \left[ \frac{\zeta_k(n)}{\zeta(k)^2(1 + \frac{1}{k}) \Gamma\left(1 + \frac{1}{k}\right)} \right]^k \left[1 + O\left(n^{-\frac{1}{k+1}}\right)\right],
\]
\[
\sigma_k^2(n) = (2\zeta(k)) \left[ \frac{\zeta_k(n)}{\zeta(k)^2(1 + \frac{1}{k}) \Gamma\left(1 + \frac{1}{k}\right)} \right]^\frac{k}{k+1} \left[1 + O\left(n^{-\frac{1}{k+1}}\right)\right].
\]

**Proof.** It is only necessary to note that \( \zeta_k(t) = \zeta(kt) \) and that (5.3) may be rewritten as
\[
n = \int_{\alpha - \infty}^{\alpha + \infty} e^{-t} f(t) \zeta(\alpha-t) \zeta(\alpha) dt + \frac{1}{\alpha} + O\left(e^{-\frac{1}{k+1}}\right),
\]
from which it follows that
\[
n = \left( 1 + \frac{1}{k} \right) \Gamma\left(1 + \frac{1}{k}\right) \left[1 + O\left(n^{-\frac{1}{k+1}}\right)\right].
\]

**Corollary.** Let \( h \) be any number \( > 0 \). Then the number of partitions of \( n \) into \( k \)-th powers having between
\[
\left[ \frac{n}{\zeta\left(1 + \frac{1}{k}\right) \Gamma\left(1 + \frac{1}{k}\right)} \right]^{k+1} \left[ \zeta(k) \pm n + f(n) \right]^{2k} (2k)^{1/2}
\]
summations is
\[
\geq \left(1 - \frac{1}{k}\right)^{\zeta_k(n)},
\]
where \( f(n) \) is any function of \( n \) such that \( f(n) n^{1/(k+1)} \to \infty \) as \( n \to \infty \).

Hence almost all partitions of \( n \) into \( k \)-th powers have fewer than
\( g(n)n^{1/(k+1)} \) summations where \( g(n) \) is any function of \( n \) which tends to infinity with \( n \).

Let us consider the variance for general sets \( A \).

**Theorem 5.4.** Let
\[
\sigma_A^2(n) = \frac{\zeta_A(n)}{\zeta_A^2(n)} - \left(\frac{\zeta_A(n)}{\zeta_A(n)}\right)^2.
\]
Then under the assumptions of Theorem 5.1
\[
\sigma_A^2(n) = \frac{\zeta_A^2(n)}{\zeta_A^2(n)} \left[1 + O\left(n^{-\frac{1}{k+1}}\right)\right].
\]

**Proof.** From Theorems 5.1 and 5.2 we obtain that
\[
\sigma_A^2(n) = \left(\frac{\Theta^0 S(e^{-n})^2}{2} - \frac{\Theta^1 S(e^{-n})^2}{2} \right) \left[1 + O\left(n^{-\frac{1}{k+1}}\right)\right],
\]
where \( \alpha_i \) is defined by (5.3) with \( k = 0 \) and \( \alpha \) with \( k = 1 \). Here we use equation (5.10) and the fact that \( A \) has property (III). The theorem now follows from Lemmas 5.1 and 5.2 and equation (5.10).
6. In this section we consider the distributions $g_A(n, m)$, the number of partitions of $n$ into exactly $m$ distinct summands from $A = \{a_1, a_2, \ldots \}$. We examine the asymptotic behaviour of the $k$th moments $u_A^k(n)$, defined by

$$u_A^k(n) = \sum_{m=1}^{n} m^{k} g_A(n, m),$$

as $n \to \infty$.

We suppose that one of the following two conditions holds:

Let $0 < \eta \leq 1$ be a real constant.

(I) $\xi_A(t)$ is regular for $\text{Re } t > \sigma_0 - \eta$ except at $t = 0 < \sigma_0 < 1$, where it has a pole of finite order with residue $\mathcal{R}$ and where the estimate

$$\xi_A(t) = O(|t|^\sigma)$$

holds uniformly in $\text{Re } t \geq \sigma_0 - \eta$ as $|t| \to \infty$, where $\sigma$ is a positive constant.

(II) $\xi_A(t)$ is regular for $t \geq 1 - \eta$ except at $t = 1$ where it has a pole of finite order with residue $\mathcal{R}$ and where the estimate

$$\xi_A(t) = O(|t|^{\sigma})$$

holds uniformly in $\text{Re } t \geq 1 - \eta$ as $|t| \to \infty$, where $\sigma$ is some positive constant.

We define $\beta$ by

$$\beta = \sum_{a \in c} \frac{a}{\phi(a) + 1} \frac{dT_{\phi}(\theta^{-\beta})}{d\theta} \bigg|_{\theta = 1} \bigg( T_{\phi}(\theta^{-\beta}) \bigg)^{-1}$$

where

$$T_{\phi}(\theta^{-\beta}) = \sum_{i=1}^{\infty} (\theta^i T, \theta^{i+1} T, \ldots, \theta^{i+k} T)_{i=1}$$

the summation being that of (2.10) and

$$T(a, x) = \sum_{a \in c} \frac{a x^a - x^{a+1}}{1 - x^{a+1}}.$$

We define $B_\beta$ by

$$B_\beta = \sum_{a \in c} \phi(a) \bigg|_{\theta = 1} \bigg( \frac{\phi(a) + 1}{\phi(a) + 2} \bigg)^{-1}$$

where $g_a^\beta(x)$ is a certain polynomial of degree $\leq \mu - 1$ and in particular

$$g_1^\beta(x) = 1, \quad g_2^\beta(x) = x.$$

We define $D_k^\beta$ by

$$D_k^\beta = A_{1}^{\beta_{k-1}} \cdots A_{\mu_1} A_{\mu_2} \cdots A_{\mu_k},$$

where the $\mu_i$'s are certain numerical constants and

$$\mu_1 + \mu_2 + \cdots + \mu_k = 12g.$$

We now state conditions (V) and (VI).

(V) $S = \lim_{k \to \infty} \frac{\log a_k}{\log k}$ exists.

(VI) $J_k = \inf \{ (\log k)^{-1} \sum_{i=1}^{\infty} |a_i \beta|^2 \} \to \infty$ as $k \to \infty$ where the lower bound is taken over those $\alpha$ satisfying $\frac{1}{2} a_i \beta < \beta \leq \frac{1}{2}$ and $|\beta|$ denotes the distance of $\beta$ from the nearest integer.

In ([6]) it is shown that these conditions hold for rather general $A$. The proof of the following theorem is very similar to the proof of Theorems 3.1 and 5.1, the main difference being that one refers to [6] instead of [10].

**Theorem 6.1.** Suppose $A$ satisfies (V) and (VI). Then: if in addition (I) holds

$$w_A^\beta(n) = (2\pi B_\beta)^{-\frac{1}{2}} \exp \left( \beta n + \sum_{a \in c} \log(1 + e^{-a \eta}) \right) \times$$

$$\times \left\{ 2\beta^{-\eta} \Gamma(\sigma_0) \xi(\sigma_0)(1 - 2^{-\eta + 1}) \right\}$$

$$\times \sum_{k=1}^{\infty} \left[ B_\beta^2 \Gamma(\sigma_0) \xi(\sigma_0)(1 - 2^{-\eta + 1}) \right]$$

$$\times \left[ 1 + D_k^\beta \right.$$

$$\left. + O(\beta^2) + O(f_{\eta}^{-\frac{1}{2}}(\beta)) \right]$$

$C_k, k = 1$ is defined by Lemma 3.1. If in addition (II) holds

$$w_A^\beta(n) = (2\pi B_\beta)^{-\frac{1}{2}} \exp \left( \beta n + \sum_{a \in c} \log(1 + e^{-a \eta}) \right) \times$$

$$\times \left[ 1 + D_k^\beta \right.$$
Then

\[ m_k(n) = n^{1/(2-1/h)} k^{-h/(2-h-1)} \times \]

\[ \times \left( \left(1-2^{-1/h} \Gamma \left(1+\frac{1}{h} \right) \zeta \left(1+\frac{1}{h} \right) \right)^{1/(2-1/h)} \right) [1 + O \{ n^{-1/(2-1/h)} \}] , \]

\[ \sigma_k(n) = n^{1/(2-1/h)} k^{-h/(2-h-1)} \left( 1-2^{-1/h} \Gamma \left(1+\frac{1}{h} \right) \zeta \left(1+\frac{1}{h} \right) \right)^{1/(2-1/h)} [1 + O \{ n^{-1/(2-1/h)} \}] . \]

The conclusions corresponding to the Corollary of Theorem 5.3 concerning the distribution of the number of distinct summands that can be deduced from this theorem and Chebycheff's inequality are somewhat stronger than those obtained by Erdős and Turán ([13], p. 56). This is different from the Corollary to Theorem 5.3 which may indicate the true state of affairs ([13], Section 4). Unless the \( \eta \) in the following theorem is relatively large as above the results will not be asymptotic. However the standard deviation is significantly smaller than the mean and under the conditions below gives a slight improvement on the results of Erdős and Turán in Theorem III of [13]. It seems difficult to compare the strength of asymptotic assumptions directly and the main goals are different in their work and ours.

Corresponding to Theorem 5.4 we have

Theorem 6.4. Let

\[ \sigma_k^2(n) = \frac{w_k^2(n)}{w_k(n)} - \left( \frac{w_k(n)}{w_k^2(n)} \right)^2 . \]

Then under the assumptions of Theorem 6.1: if condition (I') holds

\[ \sigma_k^2(n) = \frac{\Gamma(\alpha) R(1-2^{-\gamma}) \zeta(1+\alpha)}{2 \pi^2} \left[ 1 + O(\beta^{1-\gamma}) \right]; \]

if condition (II') holds

\[ \sigma_k^2(n) = \frac{R \pi^2}{2 \pi^2} \left[ 1 + O(\beta^{1-\gamma}) \right] . \]

Of course unless \( \eta > 1 \) the last relation is not an asymptotic relation.

7. In this section we let \( P = \{ p_1, p_2, \ldots \} \) where \( p \) denotes the \( p \)th prime. While \( \zeta_p(t) \) may not satisfy one of conditions (I), (II), (I') or (II'),

the only conditions of Theorems 5.1 and 6.1 are satisfied (see [6]). The only significant difference arises in the determination of the asymptotic behaviour of the sums

\[ S_1(x) = \sum_p \frac{x^p}{1-x^p} \quad \text{and} \quad S_2(x) = \sum_p \frac{x^p}{1+a^p} . \]

Let us consider \( S_1 \). We have

\[ S_1(e^{-a}) = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{-t} \zeta_p(t) \Gamma(t) \, dt , \quad a > 1, \quad |\arg a| < \frac{\pi}{2} - \delta. \]

It is well-known that

\[ \zeta_p(t) = \log \zeta(t) - \sum_p \sum_{m=2}^{P} \frac{1}{m \pi \Gamma - \log \zeta(t) - \Lambda(t)} \]

and \( \Lambda(t) \) has no singularities in \( \text{Re } t > \frac{1}{2} \). Thus, letting \( a \) be defined by (5.3) with \( A = P \),

\[ S_1(e^{-a}) = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{-t} \Gamma(t) \log \zeta(t) \, dt - \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{-t} \Gamma(t) \log \Lambda(t) \, dt , \quad a > 1 , \]

\[ = \mathcal{I}_a - \mathcal{I}_\pi . \]

We may estimate these integrals much as in [11], pp. 77-81. We deform the path of integration to the following intervals of integration:

There exists a constant \( c > 0 \) such that \( \zeta(t) \) has no zeros in the region ([12], p. 114)

\[ \text{Re } t \geq 1 - \frac{1}{2} \log^{-1/2} |t| . \]

Let \( \mathcal{C} \) denote the curve \( \text{Re } t = 1 - \frac{1}{2} \log^{-1/2} |t| \). Let \( I_1 \cup I_4 = \mathcal{C} \)

\[ \text{ contained between the upper and lower parts of } \mathcal{C} . \]

Let \( I_2 \) denote the part of \( \mathcal{C} \) above the real axis and the integration is in the upward direction.

Let \( I_4 = \mathcal{C} \) the part of \( \mathcal{C} \) below the real axis.

Then let \( I_4 \) be the reflection of \( I_1 \) about the real axis.
Let $I_3$ be the segment of the real axis from $I_4$ to $1 - \exp \left( -\log^{3/4} \frac{1}{a} \right)$.

Let $I_4$ = circle with centre at $t = 1$ and radius $\exp \left( -\log^{3/4} \frac{1}{a} \right)$ in the counterclockwise direction.

In view of (2.7) it is easily verified that the integrations over $I_1$ and $I_7$ are

$$O \left( a^{-1} \exp \left( -c_1 \log^{\frac{3}{4}} \frac{1}{a} \log^{-\frac{3}{4}} \frac{1}{a} \right) \right)$$

for some constant $c_1 > 0$.

It is also easily verified that the integrations over $I_5$ and $I_6$ are

$$O \left( a^{-1} \exp \left( -c_2 \log^{\frac{3}{4}} \frac{1}{a} \log^{-\frac{3}{4}} \frac{1}{a} \right) \right)$$

for some constant $c_2 > 0$.

Now

$$\log \zeta(t) = \log \frac{1}{t-1} + \gamma(t-1) + \ldots,$$

hence we may replace $\log \zeta(t)$ in the integrals over $I_3$, $I_4$, and $I_5$ by $-\log(t-1)$. Also on $I_3$, $I_4$, and $I_5$

$$(7.1) \quad \Gamma(t) \zeta(t) = \frac{1}{t-1} + \sum_{m=1}^{\infty} \frac{b_m}{(t-1)^m}.$$  

It can also be shown as in [11], pp. 77-81, that

$$\left( \int_{I_3} + \int_{I_4} + \int_{I_5} \right) a^{-1} (t-1)^{-1} \log \frac{1}{t-1} dt$$

$$= a^{-1} \left[ \log \log \frac{1}{a} + \gamma + O \left( \exp \left[ -c_1 \log^{\frac{3}{4}} \frac{1}{a} \log^{-\frac{3}{4}} \frac{1}{a} \right] \right) \right].$$

Furthermore let $F \left( \log \frac{1}{a} \right)$ be defined by

$$(7.2) \quad F \left( \log \frac{1}{a} \right)$$

$$= \int_0^\infty e^{-u} \left\{ \Gamma \left( 1 + \frac{u}{\log \frac{1}{a}} \right) \zeta \left( 1 + \frac{u}{\log \frac{1}{a}} \right) - \frac{\log \frac{1}{a}}{u} \left( \log \log \frac{1}{a} - \log u \right) \right\} du;$$

then we obtain much as in [11], pp. 77-81, that

$$\left( \int_{I_3} + \int_{I_4} + \int_{I_5} \right) a^{-1} \left( \zeta(t) \frac{1}{t-1} \right) \log \frac{1}{t-1} dt$$

$$= F \left( \log \frac{1}{a} \right) + O \left( \exp \left[ -c_1 \log^{\frac{3}{4}} \frac{1}{a} \log^{-\frac{3}{4}} \frac{1}{a} \right] \right)$$

and that

$$(7.3) \quad F \left( \log \frac{1}{a} \right) = a^{-1} \sum_{n=1}^{\infty} \frac{b_n \left( -1 \right)^{n+1} \Gamma(n+1)}{\log^{n} \frac{1}{a}} + O \left( \log^{-r-1} \frac{1}{a} \right)$$

where $b_n$ is defined by (7.1).

We therefore conclude:

**Theorem 7.1.** There exists a constant $c > 0$ such that

$$\frac{\zeta_p(n)}{\zeta_p(n)} = a^{-k} \sum \left( \log \log \frac{1}{a} + \gamma - \sum_{m=1}^{\infty} \frac{1}{mp^m} + F \left( \log \frac{1}{a} \right) + \right.$$  

$$+ O \left( \exp \left[ -c_1 \log^{\frac{3}{4}} \frac{1}{a} \log^{-\frac{3}{4}} \frac{1}{a} \right] \right),$$

$$2\zeta_p(2), \ldots, 2\left( k-1 \right)! \zeta_p(k) \left[ 1 + O \left( a \right) \right],$$

where $a$ is defined by (5.3), $F \left( \log \frac{1}{a} \right)$ is defined by (7.2) and has the asymptotic expansion in (7.3).

To determine the variance we proceed somewhat differently. We must estimate the difference between $S_1(a^{1/n})$ and $S_1(a^{-n})$ where

$$n = \sum_{p} \frac{1}{e^{ap} - 1}$$

and

$$n = \sum_{p} \frac{1}{e^{ap} - 1} + \frac{1}{a} + O \left( \frac{1}{a} \log \frac{1}{a} \right).$$

We have equation (5.10) holding however, and $f_p(a) = O \left( a^{-1} \log \frac{1}{a} \right)$. Hence on $I_3, I_4$ and $I_5$ we have

$$|a^{-1} - a^{-1}| = O \left( a \log \frac{1}{a} \right).$$
and it readily follows that

\[ S_1(\varepsilon^{-a}) - S_1(\varepsilon^{-a}) = O \left( a \log \frac{1}{a} \right) . \]

Hence we conclude:

**Theorem 7.2.** Let

\[ \sigma_p^2(n) = \frac{\varphi_p(n)}{\varphi_p(n)} - \frac{\left( \varphi_p(n) \right)^2}{\varphi_p(n)}, \]

Then there exists a constant \( c > 0 \) such that

\[ \sigma_p(n) = \frac{\zeta_p(2)}{\sigma_p} \left[ 1 + O \left( \exp \left[ -c \log \frac{1}{\sigma_p} \log \log \frac{1}{\sigma_p} \right] \right) \right]. \]

where \( \sigma_p \) is defined by (5.3).

**Corollary.**

\[ \varphi_p(n) = (3n)^{\log \frac{1}{\pi}} \left( \log \log n + \sum_{m=2}^{\infty} \frac{1}{m \log m} \right) + O \left( \log \log n \right) \]

Furthermore

\[ \sigma_p(n) = \frac{3n \zeta_p(2)}{\pi^2} \log n [1 + O(\log \log n)] \]

Finally let \( f(n) \) be any function of \( n \) which goes to infinity with \( n \). Almost all partitions of \( n \) into primes have between

\[ \pi^{-1}(3n)^{1/2} \log n \left( \log \log n \pm f(n) \right) \]

summands.

**Remark 1.** Let \( \theta \) be the least upper bound of the real parts of the roots of the Riemann zeta function. Then

\[ \sigma_p^2(n) = \frac{\zeta_p(2)}{\sigma_p} + O \left( n^{1/2} \right) \]

for each constant \( \varepsilon > 0 \).

In the case of partitions into distinct primes one obtains

**Theorem 7.3.** There exists a constant \( c > 0 \) such that

\[ \frac{\varphi_p(n)}{\sigma_p^2(n)} = \beta^{-1} \varphi^2 \left( \log \frac{1}{\beta} \right) \left[ 1 + O \left( \exp \left[ -c \log \frac{1}{\beta} \log \log \frac{1}{\beta} \right] \right) \right] \]

where

\[ \varphi = \int_0^1 e^{-u} \left( 1 + \frac{u}{\log \frac{1}{\beta}} \right) \left( 1 - \frac{u}{\log \frac{1}{\beta}} \right)^{1/2} \left( 1 + \frac{u}{\log \frac{1}{\beta}} \right) \left( \log \log \frac{1}{\beta} - \log \frac{1}{\beta} \right) \]

\[ = \sum_{i=0}^{\infty} \frac{i(i+1)}{\log^i \frac{1}{\beta}} + O \left( \log e^{-1} \frac{1}{\beta} \right) \]

where

\[ \Gamma(t) (1 - 2^{-t+1}) \zeta(t) = \sum_{i=0}^{\infty} d(i)(t-1)^i. \]

Here \( \beta \) is defined by equation (6.2). Furthermore there exists a constant \( c > 0 \) such that with

\[ \varphi_p^2(n) = \frac{\varphi_p^2(n)}{\varphi_p(n)} + O \left( n \exp \left( -c \log \log \log n \log \log \log \log n \right) \right) \]

Finally let \( f(n) \) be any function of \( n \) which tends to infinity with \( n \). Then almost all partitions of \( n \) into distinct primes have between

\[ (6n)^{1/2} \frac{2 \log 2}{\pi} \left( 1 \pm f(n) \log \log n \right) \]

summands.

**Remark 2.** With \( \beta \) as in Remark 1, one obtains \( \varphi_p^2(n) = O \left( n^{1/2} \right) \) for each constant \( c > 0 \).

**Bibliography**


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