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The moments of partitions, II*

by

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5. In this section we consider the distributions $p_A(n, m)$, the number of partitions of n into exactly m summands from a given set $A = \{a_1, a_2, \ldots\}$ of monotonically increasing integers. For certain A's we determine the asymptotic behaviour of the kth moments $t_A^{(k)}(n)$ defined by

$$t_A^k(n) = \sum_{m=1}^{n} m^k p_A(n, m).$$

We are especially interested in the case when A is the set of primes. This case is considered in § 7.

Let $\zeta_A(t)$ be defined by

$$\zeta_A(t) = \sum_{a \in A} a^{-t}.$$

Let σ_0 denote the abscissa of convergence of $\zeta_A(t)$. Since the a's are integers, we have $1 \geqslant \sigma_0 \geqslant 0$. Let $0 < \eta < 1$ be some fixed constant. We shall consider in this section those A's for which either of the following two conditions are satisfied.

(I) $\sigma_0 < 1-\eta$.

(II) $\zeta_A(t)$ is regular for $t \geqslant 1-\eta$ except at the point $t = 1$ where it has a pole of order one. In (I) and (II) the estimate

$$\zeta_A(t) = O\{|t|^c\}$$

holds uniformly in $\operatorname{Re} t \geqslant 1-\eta$ as $|t| \to \infty$, where c is a positive constant.

If (II) holds we define the constants u_i, $i = 1, 2$, by

$$\zeta_A(t)\zeta(t) = \frac{u_2}{(t-1)^2} + \frac{u_1}{(t-1)} + f(t)$$

where $f(t)$ is regular at $t = 1$.

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* The first part of this paper appeared in Acta Arith. 26 (1975), pp. 411–425. The numbers in brackets referee to the list of papers quoted there.

Let

$$P(x) = \begin{cases} u_1, & \text{if} \quad u_2 = 0, \\ x - u_2\gamma + u_1, & \text{if} \quad u_2 \neq 0, \end{cases} \tag{5.1}$$

where γ again denotes Euler's constant.

Now $p_A(n, m)$ has the generating function

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_A(n, m) x^n z^m = \prod_{a \in A} (1 - zx^a)^{-1} \equiv G_A(x, z)$$

which converges for $|x| < 1$ and all z. Again let θ be the operator

$$\theta \equiv z \frac{\partial}{\partial z}.$$

Let $S_A(x, z)$ be defined by

$$S_A(x, z) \equiv \sum_{a \in A} \frac{zx^a}{1 - zx^a}.$$

Then we obtain

$$t_A^{(k)}(n) = \frac{e^{\alpha n} G_A(e^{-\alpha}, 1)}{2\pi} S_A^{(k)}(e^{-\alpha}) \int_{-\pi}^{\pi} \frac{G_A(e^{-\alpha+i\theta}, 1)}{G_A(e^{-\alpha}, 1)} \frac{S_A^{(k)}(e^{-\alpha+i\theta})}{S_A^{(k)}(e^{-\alpha})} e^{-in\theta} d\theta, \tag{5.2}$$

where α is throughout this section defined by

$$n = \sum_{a \in A} \frac{a}{e^{\alpha a} - 1} - \left(S_A^{(k)}(e^{-\alpha})\right)^{-1} \frac{dS_A^{(k)}(e^{-\alpha})}{d\alpha} \tag{5.3}$$

and

$$S_A^{(k)}(e^{-\alpha}) = \sum (\theta^0 S_A, \theta^1 S_A, \ldots, \theta^{k-1} S_A)_{z=1}$$

the summation being that of (2.10).

Furthermore

$$\theta^{s-1} S(x, z) = \sum_{a \in A} \frac{\sum_{j=1}^{s} a_j^{(s)} z^j x^{ja}}{(1 - zx^a)^s},$$

where $a_j^{(s)}$ is defined by (2.1). Also we obtain

$$\theta^{s-1} S_A(e^{-\omega}, 1) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \omega^{-t} \zeta_A(t) \Gamma(t) \sum_{j=1}^{s} a_j^{(s)} \zeta_{j,s}(t)\, dt \tag{5.4}$$

$$\sigma > s, \quad |\arg\omega| \leqslant \frac{\pi}{2} - \delta.$$

Thus we obtain as before:

LEMMA 5.1. *Let $s \geqslant 1$. Then*

$$\theta^s S_A(e^{-\alpha}, 1) = \frac{2(s!)^2}{\alpha^{s+1}} \zeta_A(s+1) + O\{\alpha^{-s-\varepsilon}\}.$$

LEMMA 5.2. *If condition* (I) *holds*

$$S_A(e^{-\alpha}, 1) = \frac{\zeta_A(1)}{\alpha} + O\{\alpha^{-1+\eta}\}.$$

If condition (II) *holds*

$$S_A(e^{-\alpha}, 1) = \frac{P\left(\log \frac{1}{\alpha}\right)}{\alpha} + O\{\alpha^{-1+\eta}\}$$

where $P(x)$ is defined by (5.1).

Before stating our next result, we require some definitions.

Let $A(u)$ denote the number of elements of A which are $\leqslant u$.

We define the function f_A for real $x > 0$ by

$$f_A(x) = \sum_{a \in A} e^{-xa}.$$

f_A shall denote this function throughout this paper.

We say that A *has property* (III) if with $\varepsilon > 0$ an arbitrary constant, and μ a fixed integer > 0,

$$\sum_{a \in A} (xa)^{\mu} e^{-xa} = O\{f_A^{1+\varepsilon}(x)\}$$

and

$$f_A(x) / f_A\left(x\left(1 - f_A^{(-1+\varepsilon)/3}(x)\right)\right) = O\{1\}$$

as $x \to 0$.

We say that A *has property* (IV) if there exists some constant δ_0 with $1 > \delta_0 > 0$ and some constant η_0 with $\frac{1}{3} > \eta_0 > 0$ such that

$$A\left(x^{-1} f_A^{-\delta_0}(x)\right) / \log f_A(x) \to \infty$$

and

$$A(x^{-1}) > f_A^{2/3+\eta_0}(x)$$

as $x \to 0$.

It is proven in [10] that A has properties (III) and (IV) when either

$$\left(\overline{\lim_{\nu \to \infty}} \log a_\nu / \log \nu\right) / \underline{\lim_{\nu \to \infty}} \log a_\nu / \log \nu < \tfrac{3}{2};$$

or

$$A(2u) = O\{A(u)\} \quad \text{as} \quad u \to \infty;$$

or

$$s = \lim_{\nu\to\infty} \frac{\log\log a_\nu}{\log \nu} \quad \text{exists and} \quad s > 0.$$

We say that A is a *P-sequence* if there does not exist a number p such that $p|a_\nu$ for all sufficiently large a_ν.

Let $A_\mu(\alpha)$ be defined by

$$(5.5)\qquad A_\mu(\alpha) = \sum_{a\in A} a^\mu g_\mu(e^{-a\alpha})(e^{a\alpha}-1)^{-\mu}$$

where $g_\mu(x)$ is a certain polynomial of degree less than μ. In particular $g_1(x) = 1$ and $g_2(x) = x$.

Let $D_\varrho(\alpha)$ be defined by

$$(5.6)\qquad D_\varrho = A_2^{-6\varrho} \sum_{\mu_1=2}^{\infty} \dots \sum_{\mu_{5\varrho}=2}^{\infty} d_{\mu_1\dots\mu_{5\varrho}} A_{\mu_1} \dots A_{\mu_{5\varrho}}$$

where the d's are certain numerical constants and

$$\mu_1+\mu_2+\dots+\mu_{5\varrho} = 12\varrho.$$

LEMMA 5.3. *For n sufficiently large, equation* (5.3) *has a unique solution. Let $k \geqslant 1$. If condition* (I) *holds*

$$n = \sum_{a\in A} \frac{a}{e^{a\alpha}-1} + \frac{1}{\alpha} + O\{\alpha^{-1+\eta}\};$$

if conditon (II) *holds*

$$n = \sum_{a\in A} \frac{a}{a^{a\alpha}-1} + \frac{1}{\alpha} + O\left\{\frac{1}{\alpha}\log\frac{1}{\alpha}\right\}.$$

Our first result is

THEOREM 5.1. *Suppose A satisfies conditions* (III) *and* (IV). *Suppose furthermore that either A is a P-sequence or that*

$$\lim_{x\to 0} \log f_A(x)/\log x = 0.$$

Suppose furthermore that

$$\overline{\lim} \log\log a_\nu/\log \nu < \infty.$$

Then if in addition condition (I) *holds*

$$t_A^k(n) = (2\pi A_2)^{-1/2} \exp\left[\alpha n - \sum_{a\in A} \log\{1-e^{-a\alpha}\}\right] \times$$
$$\times \alpha^{-k} \sum \left(\zeta_A(1), 2\zeta_A(2), \dots, 2\left((k-1)!\right)^2\zeta_A(k)\right)[1+O\{\alpha^\eta\}+D_1+O\{f_A^{-1}(\alpha)\}];$$

if in addition condition (II) *holds, then $\zeta_A(1)$ is replaced by $P\left(\log\frac{1}{\alpha}\right)$ in the result for condition* (I). *A_2, α, D_1, and $P\left(\log\frac{1}{\alpha}\right)$ are defined by* (5.5), (5.3), (5.6), *and* (5.1) *respectively. The summation is that of* (2.10).

Proof. Let

$$\theta_0 = \alpha f_A^{(-1+\eta_0)/3}(\alpha).$$

Then in [10] it is shown that for any constant $M > 0$

$$(5.7)\qquad G_A(e^{-\alpha+i\theta})/G(e^{-\alpha}) = O\{\alpha f_A^{-M}(\alpha)\}$$

for $\theta_0 < |\theta| \leqslant \pi$.

As in the proof of Lemma 2.6 it follows from (5.4) that for $i = 1, 2, 3$ and $|\theta| \leqslant \theta_0$

$$\frac{d^i\{S_A^{(k)}(e^{-\alpha+i\theta})/S_A^{(k)}(e^{-\alpha})\}}{d\theta^i} = O\{\alpha^{-i}\}.$$

Thus the proof of Lemma 2.6 shows that

$$(5.8)\qquad \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} \frac{G_A(e^{-\alpha+i\theta},1)}{G_A(e^{-\alpha},1)} \frac{S_A^{(k)}(e^{-\alpha+i\theta})}{S_A(e^{-\alpha})} e^{in\theta}\,d\theta = \frac{1}{\sqrt{2\pi A_2}}[1+O\{f_A^{-1}(\alpha)\}]$$

(since $A_2 > c\alpha^{-2}f_A(\alpha)$ for some constant $c > 0$). Now the theorem follows from (5.7), (5.8) and (5.2).

THEOREM 5.2. *Under the assumptions of Theorem* 5.1

$$(2\pi A_2)^{-1/2}\exp\left[\alpha n - \sum_{a\in A}\log(1-e^{-a\alpha})\right] = t_A^0(n)[1+O\{f_A^{-1}(\alpha)\}].$$

Proof. Due to the saddle-point condition, the term

$$\exp\left[\alpha n - \sum_{a\in A}\log(1-e^{-a\alpha})\right]$$

is insensitive to small changes in α. That is if $\alpha_1 = \alpha+\Delta\alpha$, where $\Delta\alpha > 0$

$$(5.9)'\qquad \Delta\exp\left[\alpha n - \sum_{a\in A}\log(1-e^{-a\alpha})\right]$$
$$= \left[\left(n - \sum_{a\in A}\frac{a}{e^{a\alpha}-1}\right)\Delta\alpha + O\{A_2(\alpha)(\Delta\alpha)^2\}\right]\exp\left\{\alpha n - \sum_{a\in A}\log(1-e^{-a\alpha})\right\}.$$

Let α be the solution of

$$n = \sum_{a\in A}\frac{a}{e^{a\alpha}-1},$$

and α_1 be the solution of

$$n=\sum_{a\in A}\frac{a}{e^{\alpha_1 a}-1}+\frac{1}{\alpha_1}+O\left\{\frac{1}{\alpha_1}\log^{-1}\frac{1}{\alpha_1}\right\}.$$

First of all, clearly $\alpha_1=\alpha+\Delta\alpha$, where $\Delta\alpha$ is positive. Also

$$\sum_{a\in A}\frac{a}{e^{(\alpha+\Delta\alpha)a}-1}=\sum_{a\in A}\frac{a}{e^{\alpha a}-1}-A_2\Delta\alpha+O\left\{\frac{dA_2}{d\alpha}(\Delta\alpha)^2\right\}.$$

Therefore

$$\frac{1}{\alpha}=A_2\Delta\alpha+O\left\{\frac{1}{\alpha}\log^{-1}\frac{1}{\alpha}+\frac{dA_2}{d\alpha}(\Delta\alpha)^2+\frac{\Delta\alpha}{\alpha^2}\right\}.$$

From this it follows that

(5.10) $$\Delta\alpha\sim(A_2\alpha)^{-1}.$$

Now

$$n-\sum\frac{a}{e^{\alpha a}-1}=0,$$

hence from (5.9) and (5.10) it follows that

$$\Delta\exp\left[\alpha n-\sum\log(1-e^{-\alpha a})\right]=O\left\{\exp\left[\alpha n-\sum\log(1-e^{-\alpha a})\right]f_A^{-1}(\alpha)\right\},$$

since $A_2>C\alpha^{-2}f_A(\alpha)$ for some $C>0$. Now the theorem easily follows.

At this point let us consider a particular set, namely the set of kth powers.

THEOREM 5.3. *Let* $K=\{1,2^k,3^k,\ldots\}$ *be the set of k-th powers where* $k\geqslant 2$. *Let*

$$\overline{m}_K(n)=t_K^1(n)/t_K^0(n),$$

$$\sigma_K^2(n)=\frac{t_K^2(n)}{t_K^0(n)}-\left(\frac{t_K^1(n)}{t_K^0(n)}\right)^2.$$

Then

$$\overline{m}_K(n)=\zeta(k)\left(\frac{n}{\zeta\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{1}{k}\right)}\right)^{\frac{k}{k+1}}[1+O\{n^{-\frac{1}{k+1}}\}],$$

$$\sigma_K^2(n)=(2\zeta(k))\left[\frac{n}{\zeta\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{1}{k}\right)}\right]^{\frac{2k}{k+1}}[1+O\{n^{-\frac{1}{k+1}}\}].$$

Proof. It is only necessary to note that $\zeta_K(t)=\zeta(kt)$ and that (5.3) may be rewritten as

$$n=\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}\Gamma(t)\zeta\big(k(t-1)\big)\zeta(t)\,dt+\frac{1}{\alpha}+O\{\alpha^{-\frac{1}{k+1}}\}$$

from which it follows that

$$\alpha=\left(\frac{\zeta\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{1}{k}\right)}{n}\right)^{\frac{k}{k+1}}[1+O\{n^{-\frac{1}{k+1}}\}].$$

COROLLARY. *Let h be any number* >0. *Then the number of partitions of n into k-th powers having between*

$$\left[\frac{n}{\zeta\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{1}{k}\right)}\right]^{k/(k+1)}[\zeta(k)\pm(h+f(n))(2\zeta(2k))^{1/2}]$$

summands is

$$\geqslant\left(1-\frac{1}{h^2}\right)t_k^0(n)$$

where $f(n)$ is any function of n such that $f(n)n^{1/(k+1)}\to\infty$ *as* $n\to\infty$.

Hence almost all partitions of n into k-th powers have fewer than $g(n)n^{k/(k+1)}$ *summands where $g(n)$ is any function of n which tends to infinity with n.*

Let us consider the variance for general sets A.

THEOREM 5.4. *Let*

$$\sigma_A^2(n)=\frac{t_A^2(n)}{t_A^0(n)}-\left(\frac{t_A^1(n)}{t_A^0(n)}\right)^2.$$

Then under the assumptions of Theorem 5.1

$$\sigma_A^2(n)=\frac{\zeta_A(2)}{\alpha^2}[1+O\{\alpha^\eta\}+O\{f_A^{-1}(\alpha)\}].$$

Proof. From Theorems 5.1 and 5.2 we obtain that

$$\sigma_A^2(n)=\left([\theta^0S(e^{-\alpha})]^2+\frac{\theta^1S(e^{-\alpha})}{2}-[\theta^0S(e^{-\alpha_1})]^2\right)(1+O\{f_A^{-1}(\alpha)\})$$

where α_1 is defined by (5.3) with $k=0$ and α with $k=1$. Here we use equation (5.10) and the fact that A has property (III). The theorem now follows from Lemmas 5.1 and 5.2 and equation (5.10).

6. In this section we consider the distributions $q_A(n, m)$, the number of paritions of n into exactly m distinct summands from $A = \{a_1, a_2, \ldots\}$. We examine the asymptotic behaviour of the kth moments $u_A^k(n)$, defined by

$$u_A^k(n) = \sum_{m=1}^{n} m^k q_A(n, m),$$

as $n \to \infty$.

We suppose that one of the following two conditions holds:

Let $0 < \eta \leqslant 1$ be a real constant.

(I′) $\zeta_A(t)$ is regular for $\operatorname{Re} t > \sigma_0 - \eta$ except at $t = 0 < \sigma_0 < 1$, where it has a pole of finite order with residue R and where the estimate

$$\zeta_A(t) = O\{|t|^c\}$$

holds uniformly in $\operatorname{Re} t \geqslant \sigma_0 - \eta$ as $|t| \to \infty$, where c is a positive constant.

(II′) $\zeta_A(t)$ is regular for $t \geqslant 1 - \eta$ except at $t = 1$ where it has a pole of finite order with residue R and where the estimate

$$\zeta_A(t) = O\{|t|^c\}$$

holds uniformly in $\operatorname{Re} t \geqslant 1 - \eta$ as $|t| \to \infty$, where c is some positive constant.

We define β by

$$n = \sum_{a \in A} \frac{a}{e^{\beta a} + 1} - \frac{dT_A^{(k)}(e^{-\beta})}{d\beta} \left(T_A^{(k)}(e^{-\beta})\right)^{-1} \tag{6.1}$$

where

$$T_A^{(k)}(e^{-\beta}) = \sum (\theta^0 T, \theta^1 T, \ldots, \theta^{k-1} T)_{z=1}$$

the summation being that of (2.10) and

$$T(x, z) = \sum_{a \in A} \frac{zx^a - z^2 x^{2a}}{1 - z^2 x^{2a}}.$$

We define $B_\mu(\beta)$ by

$$B_\mu = \sum_{a \in A} a^\mu g_\mu^*(e^{\beta a})(e^{\beta a} - 1)^{-\mu} \tag{6.2}$$

where $g_\mu^*(x)$ is a certain polynominal of degree $\leqslant \mu - 1$ and in particular $g_1^*(x) = 1$, $g_2^*(x) = x$.

We define $D_1^*(\beta)$ by

$$D_1^* = A_2^{-6\varrho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{5\varrho}=2}^{\infty} d^*_{\mu_1 \mu_2 \ldots \mu_{5\varrho}} A_{\mu_1} A_{\mu_2} \ldots A_{\mu_{5\varrho}}, \tag{6.3}$$

where the d^*'s are certain numerical constants and

$$\mu_1 + \mu_2 + \ldots + \mu_{5\varrho} = 12\varrho.$$

We now state conditions (V) and (VI).

(V) $S = \lim_{k \to \infty} \dfrac{\log a_k}{\log k}$ exists.

(VI) $J_k = \inf \left\{ (\log k)^{-1} \sum_{\nu=1}^{\infty} \|a_\nu \beta\|^2 \right\} \to \infty$ as $k \to \infty$ where the lower bound is taken over those a_ν satisfying $\frac{1}{2} a_\nu^{-1} < \beta \leqslant \frac{1}{2}$ and $\|\theta\|$ denotes the distance of θ from the nearest integer.

In [6] it is shown that these conditions hold for rather general A. The proof of the following theorem is very similar to the proof of Theorems 3.1 and 5.1, the main difference being that one refers to [6] instead of [10].

Theorem 6.1. *Suppose A satisfies* (V) *and* (VI). *Then: if in addition* (I′) *holds*

$$u_A^k(n) = (2\pi\beta_2)^{-1/2} \exp\left\{\beta n + \sum_{a \in A} \log(1 + e^{-\beta a})\right\} \times$$
$$\times \sum \left(R\beta^{-\sigma_0} \Gamma(\sigma_0) \zeta(\sigma_0)(1 - 2^{-\sigma_0 + 1}),\right.$$
$$\left.\tfrac{3}{4}\beta^{-1} \zeta_A(1), \ldots, \beta^{-k+1} 2^{-k+1} C_{k-1,k-2} \Gamma(k-1) \zeta_A(k-1)\right) \times$$
$$\times [1 + D_1^* + O\{\beta^\eta\} + O\{f_A^{-1+\varepsilon}(\beta)\}]$$

$C_{k,k-1}$ is defined by Lemma 3.1. *If in addition* (II′) *holds*

$$u_A^k(n) = (2\pi B_2)^{-1/2} \exp\left\{\beta n + \sum_{a \in A} \log(1 + e^{-\beta a})\right\} \beta^{-k} (R \log 2)^k \times$$
$$\times [1 + D_1^* + O\{\beta^\eta\} + O\{f_A^{-1+\varepsilon}(\beta)\}].$$

Here β, B_2, and D_1 are defined by (6.1), (6.2), *and* (6.3). *f_A is the same function as in* § 5.

Corresponding to Theorem 5.2 we have

Theorem 6.2. *Under the assumptions of Theorem* 6.1,

$$(2\pi B_2)^{-1/2} \exp\left\{\beta n + \sum_{a \in A} \log(1 + e^{-\beta a})\right\} = u_A^0(n)[1 + O\{\beta^\eta\} + O\{f^{-1+\varepsilon}(\beta)\}].$$

Corresponding to Theorem 5.3 we have

Theorem 6.3. *Let $K = \{1, 2^k, 3^k, \ldots\}$ be the set of k-th powers. Let*

$$m_K(n) = u_K^1(n)/u_K^0(n),$$

$$\sigma_K^2(n) = \frac{u_K^2(n)}{u_K^0(n)} - \left(\frac{u_K^1(n)}{u_K^0(n)}\right)^2.$$

Then

$$m_K(n) = n^{1/(k+1)}(1-2^{1-1/k})\zeta\left(\frac{1}{k}\right)\Gamma\left(\frac{1}{k}\right)k^{-k/(k+1)}\times$$
$$\times\left(\left((1-2^{-1/k})\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{-1/(k+1)}\right)[1+O\{n^{-1/(k+1)}\}],$$

$$\sigma_K^2(n) = n^{1/(k+1)}k^{-k/(k+1)}\left((1-2^{-1/k})\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)}[1+O\{n^{-1/(k+1)}\}].$$

The conclusions corresponding to the corollary of Theorem 5.3 concerning the distribution of the number of distinct summands that can be deduced from this theorem and Tchebycheff's inequality are somewhat stronger than those obtained by Erdös and Turán ([13], p. 56). This is different from the Corollary to Theorem 5.3 which may indicate the true state of affairs ([13], Section 4). Unless the η in the following theorem is relatively large as above the results will not be asymptotic. However the standard deviation is significantly smaller than the mean and under the conditions below gives a slight improvement on the results of Erdös and Turán in Theorem III of [13]. It seems difficult to compare the strength of asymptotic assumptions directly and the main goals are different in their work and ours.

Corresponding to Theorem 5.4 we have

THEOREM 6.4. *Let*

$$\sigma_A^2(n) = \frac{u_A^2(n)}{u_A^0(n)} - \left(\frac{u_A^1(n)}{u_A^0(n)}\right)^2.$$

Then under the assumptions of Theorem 6.1: *if condition* (I′) *holds*

$$\sigma_A^2(n) = \frac{\Gamma(\sigma_0)R(1-2^{-\sigma_0})\xi(1+\sigma_0)}{2\beta^{\sigma_0}}[1+O\{\beta^{\eta-\sigma_0}\}+O\{f_A^{-1+\varepsilon}(\beta)\}];$$

if condition (II′) *holds*

$$\sigma_A^2(n) = \frac{R\pi^2}{24\beta}[1+O\{\beta^{\eta-1}\}+O\{f_A^{-1+\varepsilon}\}].$$

Of course unless $\eta > 1$ the last relation is not an asymptotic relation.

7. In this section we let $P = \{p_1, p_2, \ldots\}$ where p_ν denotes the νth prime. While $\zeta_P(t)$ may not satisfy one of conditions (I), (II), (I′) or (II′), the other conditions of Theorems 5.1 and 6.1 are satisfied (see [6]). The only significant difference arises in the determination of the asymptotic behaviour of the sums

$$S_1(x) = \sum_p \frac{x^p}{1-x^p} \quad \text{and} \quad S_2(x) = \sum_p \frac{x^p}{1+x^p}.$$

Let us consider S_1. We have

$$S_1(e^{-\omega}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \omega^{-t}\zeta_P(t)\zeta(t)\Gamma(t)\,dt, \quad \sigma > 1,\ |\arg\omega| < \frac{\pi}{2}-\delta.$$

It is well-known that

$$\zeta_P(t) = \log\zeta(t) - \sum_{m\geqslant 2}\sum_p \frac{1}{mp^{mt}} = \log\zeta(t) - h(t)$$

and $h(t)$ has no singularities in $\operatorname{Re} t > \frac{1}{2}$. Thus, letting α be defined by (5.3) with $A = P$,

$$S_1(e^{-\alpha}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\Gamma(t)\log\zeta(t)\zeta(t)\,dt -$$
$$-\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\Gamma(t)\log h(t)\zeta(t)\,dt, \quad \sigma > 1,$$
$$= I_n' - I_n''$$

We may estimate these integrals much as in [11], pp. 77–81. We deform the path of integration to the following intervals of integration:

There exists a constant $c > 0$ such that $\zeta(t)$ has no zeros in the region ([12], p. 114)

$$\operatorname{Re} t \geqslant 1 - c\log^{-3/4}|t|\log\log^{-3/4}|t|.$$

Let $\mathscr{C}$ denote the curve $\operatorname{Re} t = 1 - c\log^{-3/4}|t|\log\log^{-3/4}|t|$. Let $I_2 \cup I_6 \equiv$ the segment of

$$\operatorname{Re} t_\alpha = 1 - c\log^{-3/4}\frac{1}{\alpha}\log\log^{-3/4}\frac{1}{\alpha}$$

contained between the upper and lower parts of $\mathscr{C}$.

Let I_2 denote the part of $I_1 \cup I_6$ below the real axis and the integration is in the upward direction.

Let $I_1 \equiv$ the part of $\mathscr{C}$ asymptotic to the line $\operatorname{Re} t = 1$ and meeting I_2. Then let I_7 be the reflection of I_1 about the real axis.

Let I_3 = the segment of the real axis from I_2 to $1-\exp\left(-\log^{3/4}\frac{1}{a}\right)$.

Let I_4 = circle with centre at $t=1$ and radius $\exp\left(-\log^{3/4}\frac{1}{a}\right)$ in the counterclockwise direction.

In view of (2.7) it is easily verified that the integrations over I_1 and I_7 are

$$O\left\{a^{-1}\exp\left(-c_1\log^{4/7}\frac{1}{a}\log\log^{-3/7}\frac{1}{a}\right)\right\}$$

for some constant $c_1>0$.

It is also easily verified that the integrations over I_2 and I_6 are

$$O\left\{a^{-1}\exp\left(-c_2\log^{4/7}\frac{1}{a}\log\log^{-3/7}\frac{1}{a}\right)\right\}$$

for some constant $c_2>0$.

Now

$$\log\zeta(t) = \log\frac{1}{t-1}+\gamma(t-1)+\ldots,$$

hence we may replace $\log\zeta(t)$ in the integrals over I_3, I_4, and I_5 by $-\log(t-1)$. Also on I_3, I_4, and I_5

$$\Gamma(t)\zeta(t) = \frac{1}{t-1}+\sum_{\nu=0}^{\infty}\frac{b_\nu}{(t-1)^\nu}. \tag{7.1}$$

It can also be shown as in [11], pp. 77–81, that

$$\left(\int_{I_3}+\int_{I_4}+\int_{I_5}\right)a^{-t}(t-1)^{-1}\log\frac{1}{t-1}\,dt$$
$$= a^{-1}\left(\log\log\frac{1}{a}+\gamma+O\left\{\exp\left[-c_3\log^{4/7}\frac{1}{a}\log\log^{-3/7}\frac{1}{a}\right]\right\}\right).$$

Furthermore let $F\left(\log\frac{1}{a}\right)$ be defined by

$$F\left(\log\frac{1}{a}\right) \tag{7.2}$$
$$= \int_0^\infty e^{-u}\left\{\Gamma\left(1+\frac{u}{\log\frac{1}{a}}\right)\zeta\left(1+\frac{u}{\log\frac{1}{a}}\right)-\frac{\log\frac{1}{a}}{u}\right\}\left(\log\log\frac{1}{a}-\log u\right)du;$$

then we obtain much as in [11], pp. 77–81, that

$$\left(\int_{I_3}+\int_{I_4}+\int_{I_5}\right)a^{-t}\left(\zeta(t)\Gamma(t)-\frac{1}{t-1}\right)\log\frac{1}{t-1}\,dt$$
$$= F\left(\log\frac{1}{a}\right)+O\left\{\exp\left[-c_4\log^{4/7}\frac{1}{a}\log\log^{-3/7}\frac{1}{a}\right]\right\}$$

and that

$$F\left(\log\frac{1}{a}\right) = a^{-1}\sum_{\nu=1}^{V}\frac{b_\nu(-1)^{\nu+1}\Gamma(\nu+1)}{\log^{\nu+1}\frac{1}{a}}+O\left\{a^{-1}\log^{-V-2}\frac{1}{a}\right\} \tag{7.3}$$

where b_ν is defined by (7.1).

We therefore conclude:

THEOREM 7.1. *There exists a constant $c>0$ such that*

$$\frac{t_P^k(n)}{t_P^0(n)} = a^{-k}\sum\left(\log\log\frac{1}{a}+\gamma-\sum_{m=2}^{\infty}\frac{1}{mp^m}+F\left(\log\frac{1}{a}\right)+\right.$$
$$\left.+O\left\{\exp\left[-c\log^{4/7}\frac{1}{a}\left(\log\log\frac{1}{a}\right)^{-3/7}\right]\right\},\right.$$
$$\left.2\zeta_p(2),\ldots,2\left((k-1)!\right)^2\zeta_p(k)\right)[1+O\{a\}],$$

where a is defined by (5.3), $F\left(\log\frac{1}{a}\right)$ is defined by (7.2) and has the asymptotic expansion in (7.3).

To determine the variance we proceed somewhat differently. We must estimate the difference between $S_1(e^{-a_1})$ and $S_1(e^{-a})$ where

$$n = \sum_p\frac{p}{e^{a_1p}-1}$$

and

$$n = \sum_p\frac{p}{e^{ap}-1}+\frac{1}{a}+O\left\{\frac{1}{a}\log\frac{1}{a}\right\}.$$

We have equation (5.10) holding however, and $f_P(a) = O\left\{a^{-1}\log\frac{1}{a}\right\}$. Hence on I_3, I_4 and I_5 we have

$$|a^{-t}-a_1^{-t}| = O\left\{a\log\frac{1}{a}\right\}$$

and it readily follows that

$$S_1(e^{-\alpha_1})-S_1(e^{-\alpha}) = O\left\{\alpha\log\frac{1}{\alpha}\right\}.$$

Hence we conclude:

THEOREM 7.2. *Let*

$$\sigma_P^2(n) = \frac{t_P^2(n)}{t_P^0(n)} - \left(\frac{t_P^1(n)}{t_P^0(n)}\right)^2.$$

Then there exists a constant $c>0$ *such that*

$$\sigma_P^2(n) = \frac{\zeta_P(2)}{\alpha^2}\left[1+O\left\{\exp\left[-c\log^{4/7}\frac{1}{\alpha}\log\log^{-3/7}\frac{1}{\alpha}\right]\right\}\right].$$

where α *is defined by* (5.3).

COROLLARY.

$$\frac{t_P^k(n)}{t_P^0(n)} = (3n)^{k/2}\left(\frac{\log n}{\pi}\right)^k\left(\log\log n+\gamma+\sum_{m=2}^{\infty}\sum_p\frac{1}{mp^m}\right)^k\left[1+O\left\{\frac{\log\log n}{\log n}\right\}\right].$$

Furthermore

$$\sigma_P^2(n) = \frac{3n\zeta_P(2)}{\pi^2}\log^2 n\left[1+O\{\log\log n/\log n\}\right]$$

Finally let $f(n)$ *be any function of* n *which goes to infinity with* n*. Almost all partitions of* n *into primes have between*

$$\pi^{-1}(3n)^{1/2}\log n\big(\log\log n\pm f(n)\big)$$

summands.

Remark 1. Let θ be the least upper bound of the real parts of the roots of the Riemann zeta function. Then

$$\sigma_P^2(n) = \frac{\zeta_P(2)}{\alpha^2}+O\{n^{\theta+\varepsilon}\}$$

for each constant $\varepsilon>0$.

In the case of partitions into distinct primes one obtains

THEOREM 7.3. *There exists a constant* $c>0$ *such that*

$$\frac{u_P^k(n)}{u_P^0(n)} = \beta^{-k}H^k\left(\log\frac{1}{\beta}\right)\left\{1+O\left\{\exp\left[-c\log^{4/7}\frac{1}{\beta}\log\log^{-3/7}\frac{1}{\beta}\right]\right\}\right\}$$

where

$$H\left(\log\frac{1}{\beta}\right)$$

$$= \int_0^{\infty} e^{-u}\Gamma\left(1+\frac{u}{\log\frac{1}{\beta}}\right)\left(1-2^{\frac{u}{\log\frac{1}{\beta}}}\right)\zeta\left(1+\frac{u}{\log\frac{1}{\beta}}\right)\left(\log\log\frac{1}{\beta}-\log u\right)du$$

$$= \sum_{l=0}^{L}\frac{d_l\Gamma(l+1)}{\log^{l+1}\frac{1}{\beta}}+O\left\{\log^{-L-2}\frac{1}{\beta}\right\}$$

where

$$\Gamma(t)\,(1-2^{-t+1})\,\zeta(t) = \sum_{l=0}^{\infty}d_l(t-1)^l.$$

Here β *is defined by equation* (6.2). *Furthermore there exists a constant* $c>0$ *such that with*

$$\bar\sigma_P^2(n) = \frac{u_P^2(n)}{u_P^0(n)}-\left(\frac{u_P^1(n)}{u_P^0(n)}\right)^2$$

$$\bar\sigma_P^2(n) = O\{n\exp(-c\log^{4/7}n\log\log^{-3/7}n)\}.$$

Finally let $f(n)$ *be any function of* n *which tends to infinity with* n*. Then almost all partitions of* n *into distinct primes have between*

$$(6n)^{1/2}\frac{2\log 2}{\pi}\left\{1\pm f(n)\frac{\log\log n}{\log n}\right\}$$

summands.

Remark 2. With θ as in Remark 1, one obtains $\bar\sigma_P^2(n) = O\{n^{\theta+\varepsilon}\}$ for every constant $\varepsilon>0$.

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