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Bounds on the number of integer valued monotone functions of k integer arguments*

by

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1. Introduction. Consider the set

(1) $$(0, n-1)^k = \{(x_1, x_2, \ldots, x_k),\ x_i \text{ integer, and } 0 \leqslant x_i \leqslant n-1\}$$

together with the partial order $\leqslant$ given by:

$$\boldsymbol{x} \leqslant \boldsymbol{y} \Leftrightarrow x_i \leqslant y_i\ \forall i, \quad 1 \leqslant i \leqslant k.$$

We say an integer valued function f is monotone on $(0, n-1)^k$ if:

$$\boldsymbol{x} \leqslant \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \leqslant f(\boldsymbol{y}).$$

The problem we shall be concerned with is to count the number (denoted by $L_k(N, n)$) of monotone functions $f\colon (0, n-1)^k \to (0, 1, 2, \ldots, N)$, to which we refer as N-restricted n^k-partitions (of any integer).

In one dimension ($k = 1$) the problem is trivial, $L_1(N, n) = \binom{N+n}{n}$. The problem for planes and higher dimensional solids was first studied by McMahon [2]. He generalized the concept of partitioning an integer into a linear array, and defined plane partitions and partitions "in solido," as two or more dimensional arrays of integers non-decreasing in each direction and summing up to a given integer m. He also considered partitions with restricted part magnitudes. McMahon was successful in obtaining generating functions for a wide variety of plane partitions (not necessarily rectangular). R. Stanley [3] gives a survey of many of the known results about plane partitions and some of the proofs involved. These proofs appeal to the theory of symmetric functions and the representation theory of symmetric groups. They are quite involved and apparently not trivially generalizable to higher dimensional lattices (except for some particularly simple 3-dimensional figures). Carlitz [1]

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uses a simple inductive proof to show that the number of $k\times r$ arrays of integers non-increasing in each direction for which the ith row is bounded above by n_i $(n_i \geqslant n_{i+1}\ \forall i)$ is given by the determinant of the $k\times k$ matrix:

$$\left|\binom{n_j+r}{r-i+j}\right|.$$

In case $k = r = n$ and $n_i = N\ \forall i$, the above reduces to:

$$\prod_{i=0}^{n-1}\prod_{j=1}^{n}\left(\frac{N+i+j}{i+j}\right) \tag{2}$$

which is the number of N-restricted n^2 partitions. The search for an exact count of solid N-restricted partitions has so far been unsuccessful. Even for our regular $n\times\ldots\times n$ lattice there seems to be no simple generalization of the above methods of attack in three or more dimensions. We are led, then, to finding upper and lower bounds on the desired count.

The upper bound is established by considering functions monotone in the first 2 coordinates and otherwise unconstrained. The lower bound is obtained by considering functions satisfying:

$$f(x_1, \ldots, x_k) \leqslant f(y_1, \ldots, y_k) \quad \text{if} \quad \sum_1^k x_i < \sum_1^k y_i.$$

It is shown that in the limit of large n and N, these bounds differ in the logarithm, by a factor of no more than $3\sqrt{k}$.

2. An upper bound on the number of N-restricted n^k-partitions

2.1. Consider the set F of functions f: $(0, n-1)^k \to (0, N)$. Then since:

$$\{f\in F \mid f \text{ monotone in all coordinates}\} \subseteq \{f\in F \mid f \text{ monotone in the first 2 coordinates}\}$$

we immediately obtain:

$$\log L_k(N, n) \leqslant n^{k-2}\log L_2(N, n), \quad \text{for} \quad k\geqslant 2. \tag{3}$$

Unfortunately (2) does not lend itself to easy manipulation in evaluating $\log L_2(N, n)$ for purposes of comparison. Thus it is convenient to formulate upper bounds on $\log L_2(N, n)$ in terms of functions that are easy to handle [1].

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[1] The asymptotic value of $\log L_2(N, n)$ can be obtained by approximating

$$\log L_2(N, n) = \sum_{i=0}^{n}\sum_{j=1}^{n}\log\left(\frac{N+i+j}{i+j}\right)$$

by the integral

$$\int_0^n\int_0^n \log\left(\frac{N+x+y}{x+y}\right)dx\,dy = n^2\left[\tfrac{1}{2}\left((g+2)^2\log(g+2) - 2(g+1)^2\log(g+1) + g^2\log g\right) - 2\log 2\right] \text{ where } g = N/n.$$

Using conventional techniques it can be shown that as n and N tend to ∞ the ratio of the sum to the integral tends to 1. However, even the latter formulation is somewhat cumbersome; the bounds derived above are simpler and suffice to establish the results of this paper.

2.2. Upper bounds on $\log L_2(N, n)$. Let $g = N/n$.

ASSERTION 1.

$$\log L_2(N, n) \leqslant n^2\log\frac{e^{3/2}}{4}(1+g). \tag{4}$$

Proof.

$$\begin{aligned}\log L_2(N, n) &= \sum_{i=0}^{n-1}\sum_{j=1}^{n}\log\left(\frac{N+i+j}{i+j}\right)\\ &= \sum_{i=0}^{n-1}\sum_{j=1}^{n}\log(N+i+j) - \sum_{i=0}^{n-1}\sum_{j=1}^{n}\log(i+j).\end{aligned} \tag{5}$$

Now since the function $\log(x)$ is concave, $\log(a+b)+\log(a-b)\leqslant 2\log a$. Hence

$$\begin{aligned}&\sum_{i=0}^{n-1}\sum_{j=1}^{n}\log(N+i+j)\\ &= n\log(N+n) + \sum_{k=1}^{n-1}(n-k)\left[\log(N+n+k)+\log(N+n-k)\right] \leqslant n^2\log(N+n).\end{aligned}$$

Using similar arguments, it can be shown that

$$\int_i^{i+1}\int_{j-1}^{j}\log(x+y)\,dx\,dy \leqslant \log(i+j).$$

Substituting these in (5) we have:

$$\begin{aligned}\log L_2(N, n) &\leqslant n^2\log(N+n) - \int_0^n\int_0^n \log(x+y)\,dx\,dy\\ &= n^2\log(N+n) - n^2\log\frac{4n}{e^{3/2}} = n^2\log\frac{e^{3/2}}{4}(1+g).\end{aligned}$$

ASSERTION 2. *For large n,*

$$\log L_2(N, n) \leqslant 2Nn\log 2 = n^2(g\log 4). \tag{6}$$

Proof. From (5)

$$\log L_2(N,n) = \sum_{i=0}^{n-1}\sum_{j=1}^{n}[\log(N+i+j)-\log(i+j)]$$

$$\leqslant \sum_{i=0}^{n-1}\sum_{j=1}^{n} N\frac{d}{dx}\log x|_{i+j} \quad (\text{since } \log x \text{ is concave})$$

$$= \sum_{i=0}^{n-1}\sum_{j=1}^{n} N(\log e)\frac{1}{i+j}$$

$$= N\log e\Big(\sum_{i+j=1}^{n}(i+j)\frac{1}{i+j}+\sum_{i=1}^{n-1}\sum_{j=n+1}^{n+i}\frac{1}{j}\Big)$$

$$\leqslant N\log e\Big(n+\sum_{i=1}^{n-1}\int_{n}^{n+i}\frac{1}{x}\,dx\Big)$$

$$= N\log e\Big(n+\log_e\frac{(n+1)(n+2)\dots(2n-1)}{n^{n-1}}\Big)$$

$$= N\log e\Big(n+\log_e\frac{(2n)!}{2\cdot n!\,n^n}\Big).$$

Using Stirling's approximation:

$$\log_e\frac{(2n)!}{2\cdot n!\,n^n}\approx\log_e\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{2\cdot n^n e^{-n}\sqrt{2\pi n}\,n^n} \quad (\text{for large } n)$$

$$<\log_e\frac{2^{2n}}{e^n}=2n\log_e 2-n.$$

Hence for large n,

$$\log L_2(N,n)\leqslant N\log e[n+2n\log_e 2-n]=2nN\log 2,$$

q.e.d.

3. Lower bounds on the number of N-restricted n^k partitions

3.1. Partitions that are monotone along "diagonals" $S_j=\{x|\sum_1^k x_i=j\}$.

Consider partitioning the k-dimensional cube $(0,n-1)^k$, into $(n-1)k+1$ diagonal sections:

$$S_j=\Big\{x|\ \sum_1^k x_i=j\Big\},\quad 0\leqslant j\leqslant nk.$$

To find a lower bound we restrict our attention to those functions f that are non-decreasing along successive S_j's: for which $f(\boldsymbol{x})\leqslant f(\boldsymbol{y})$ whenever $\boldsymbol{x}\in S_i$, $\boldsymbol{y}\in S_j$ and $0\leqslant i<j\leqslant(n-1)k$.

For example in three dimensions:

$$S_0=\{(0,0,0)\},\quad S_1=\{(1,0,0),(0,1,0),(0,0,1)\},$$

$$S_2=\{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\},\ \text{etc.},$$

and we consider functions for which

$$f(0,0,0)\leqslant f(1,0,0)\ \text{and}\dots\text{and}\ f(0,0,1)$$
$$\leqslant f(2,0,0,)\ \text{and}\dots\text{and}\ f(0,1,1)\leqslant \text{etc.}$$

DEFINITION. Let $\leqslant$ be the partial order defined on subsets (Q_m) of non-negative integers given by

$$Q_{m_1}\leqslant Q_{m_2}\Leftrightarrow\forall x\in Q_{m_1},\ \forall y\in Q_{m_2},\ x\leqslant y.$$

THEOREM 1. *Given any sequence of $(n-1)k+1$ finite subsets Q_i of the non-negative integers, such that*

$$\{0\}\leqslant Q_0\leqslant Q_1\leqslant\dots\leqslant Q_{(n-1)k}\leqslant\{N\} \tag{7}$$

the product $\prod_{j=0}^{k(n-1)}|Q_j|^{|S_j|}$ is a lower bound on the number of N-restricted n^k-partitions, where

$$S_j=\Big\{x|\sum_1^k x_i=j\Big\}.$$

Proof. The product counts the total number of functions $f\colon(0,n-1)^k\to(0,N)$ for which $f(S_j)\subseteq Q_j$, with Q_j's satisfying (7).

Hence all we have to show is that such functions are monotone. So let f be such a function and let $\boldsymbol{x}<\boldsymbol{y}$ then

$$j_{\boldsymbol{x}}=\sum_1^k x_i<\sum_1^k y_i=j_{\boldsymbol{y}}.$$

This implies, from (7) that

$$Q_{j_{\boldsymbol{x}}}\leqslant Q_{j_{\boldsymbol{y}}}$$

and so,

$$f(\boldsymbol{x})(\epsilon Q_{j_{\boldsymbol{x}}})\leqslant f(\boldsymbol{y})(\epsilon Q_{j_{\boldsymbol{y}}}).$$

Hence $\boldsymbol{x}\leqslant\boldsymbol{y}\Rightarrow f(\boldsymbol{x})\leqslant f(\boldsymbol{y})$, q.e.d.

COROLLARY 1. *Let $\Pi_N=\{P_0,P_1,\dots,P_{(n-1)k}\}$ be a partition of an integer $M\leqslant N$ into $(n-1)k+1$ non-negative parts that are not necessarily ordered.*

Then:

$$L_k(N, n) \geqslant \prod_{0}^{k(n-1)} (1+P_j)^{|S_j|}.$$

The proof follows immediately from the theorem by letting

$$q_j = \sum_{0}^{j-1} P_i \quad \text{for} \quad 1 \leqslant j \leqslant (n-1)k+1 \quad (q_0 = 0)$$

and

$$Q_j = \{q_j, q_j+1, q_j+2, \ldots, q_{j+1}\}.$$

EXAMPLE. Consider 12 restricted 2^3-partitions and let

$$\Pi_N = (2, 4, 4, 2)$$

be a partition of $N = 12$, into $(n-1)k+1 = 4$ parts as specified in the corollary. Then,

P_j	Q_j	$\lvert Q_j\rvert$	S_j	$\lvert S_j\rvert$
2	$Q_0 = 0, 1, 2$	3	$S_0 = (0, 0, 0)$	1
4	$Q_1 = 2, 3, 4, 5, 6$	5	$S_1 = (1, 0, 0), (0, 1, 0), (0, 0, 1)$	3
4	$Q_2 = 6, 7, 8, 9, 10$	5	$S_2 = (1, 1, 0), (1, 0, 1), (0, 1, 1)$	3
2	$Q_3 = 10, 11, 12$	3	$S_3 = (1, 1, 1)$	1

It should now be clear that function values on each S_j can be chosen independently of each other from the elements of the corresponding Q_j, without interfering with monotonicity constraints (see Fig. 1).

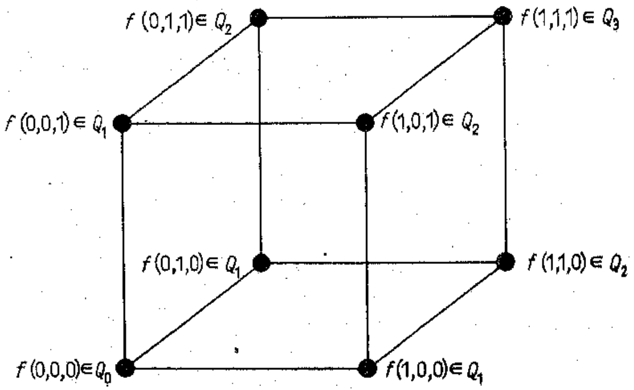

Fig. 1

The number of such functions is

$$3^1 \times 5^3 \times 5^3 \times 3^1 = \prod_{1}^{4} (1+P_j)^{|S_j|}.$$

3.2. To obtain the highest possible bounds from this scheme one should now try to maximize the product in Corollary 1, over all partitions of integers $\leqslant N$. Efforts to do so are hampered by digital and positivity constraints, and lack of analytic information about the sizes of the sets S_j. Something close to a maximum, however, can be obtained by a choice of the parts P_j to approximate the "Lagrange Multiplier" result:

$$1+P_j = \lambda |S_j| \tag{8}$$

as follows:

Let the sets $S_j = \{x \mid \sum_{1}^{k} x_i = j\}$ be ordered according to size:

$$|S_{j_1}| \geqslant |S_{j_2}| \geqslant \ldots \geqslant |S_{j_{(n-1)k+1}}|.$$

Consider the largest m among the sets S_j, and let $S^m = \{S_{j_1}, S_{j_2}, \ldots, S_{j_m}\}$.

COROLLARY 2a. *Let*

$$M_m = \sum_{i=1}^{m} |S_{j_i}| = \sum_{\text{largest } m} |S_j|, \quad 1 \leqslant m \leqslant (n-1)k+1,$$

then

$$L_k(N, n) \geqslant \left(\frac{N}{m}\right)^{M_m}.$$

Proof. Approximating the result (8) for elements of S^m, in Corollary 1 let Π_N be defined as:

$$P_j = \begin{cases} \left[\dfrac{N}{M_m}|S_j|\right], & S_j \in S^m, \\ 0, & \text{else} \end{cases} \tag{9}$$

(where $[x]$ = greatest integer $\leqslant x$).

Then since

$$\sum P_j \leqslant \sum_{i=1}^{m} \frac{N}{M_m} |S_{j_i}| = N,$$

the corollary applies and therefore:

$$L_k(N, n) \geqslant \prod_{i=1}^{m} \left(1+\left[\frac{N}{M_m}|S_{j_i}|\right]\right)^{|S_{j_i}|} \geqslant \prod_{i=1}^{m} \left(\frac{N}{M_m}|S_{j_i}|\right)^{|S_{j_i}|}$$

$$= \left(\frac{N}{M_m}\right)^{\sum_{i=1}^{m}|S_{j_i}|} \prod_{i=1}^{m} |S_{j_i}|^{|S_{j_i}|}.$$

Now

$$\log\prod_{i=1}^{m}|S_{j_i}|^{|S_{j_i}|} = \sum_{i=1}^{m}|S_{j_i}|\log|S_{j_i}|.$$

Since the function $f(x) = x\log x$ is convex,

$$\frac{1}{m}\sum_{i=1}^{m}|S_{j_i}|\log|S_{j_i}| \geqslant \frac{M_m}{m}\log\frac{M_m}{m},$$

$$\therefore \quad \sum_{i=1}^{m}|S_{j_i}|\log|S_{j_i}| \geqslant M_m\log\frac{M_m}{m}$$

and

$$\prod_{i=1}^{m}|S_{j_i}|^{|S_{j_i}|} \geqslant \left(\frac{M_m}{m}\right)^{M_m},$$

$$\therefore \quad L_k(N, n) \geqslant \left(\frac{N}{M_m}\right)^{M_m}\left(\frac{M_m}{m}\right)^{M_m} = \left(\frac{N}{m}\right)^{M_m}.$$

COROLLARY 2b. *For* $N \leqslant (n-1)k+1$

$$L_k(N, n) \geqslant 2^{M_N} \quad \textit{where} \quad M_N = \sum_{i=1}^{N}|S_{j_i}|.$$

For small values of N the choice (9) of the parts P_j may yield an exceeding number of zeros. A better choice is:

$$P_j = \begin{cases} 1, & S_j \in S^N, \\ 0, & S_j \notin S^N. \end{cases}$$

This is a valid choice for all $N \leqslant (n-1)k+1$. Substituting into Corollary 1:

$$L_k(N, n) \geqslant \prod_j (P_j+1)^{|S_j|} = \prod_{j=1}^{N} 2^{|S_{j_i}|} = 2^{M_N},$$

q.e.d.

Theorem 1 and its ensuing corrolaries constitute the basic results of this paper. What remains to be done is an investigation of how close these bounds are to the actual values of $\log L_k(N, n)$. However, before we can do that, we need an estimate of $M_m = \sum_{i=1,\ldots,m}|S_{j_i}| = \sum_{\text{largest } m}|S_j|$.

3.3. To obtain analytic information about $M_m = \sum_{\text{largest } m}|S_j|$, we appeal to the methods of probability theory.

In particular, think of each coordinate x_i, as a random variable taking on discrete values $0, 1, 2, \ldots, n-1$ with equal probability $1/n$ independently of all the other coordinates.

Then

$$S_j = \text{number of points whose coordinates add up to } j$$
$$= n^k \times \text{Prob. [sum of } x_i = j].$$

Formally let x_i be k independent identically distributed random variables, each with density

$$p_{x_i} = \frac{1}{n}\sum_{l=0}^{n-1}\mu_0(x-l).$$

Let s be the random variable defined by their sum: $s = \sum_1^k x_i$. Then

(10) $$|S_j| = n^k p_s(j) \quad \text{where} \quad p_s(j) = \text{Prob. } (s = j).$$

Notice the following facts about p_s:

(a) Since each p_{x_i} has mean $\frac{n-1}{2}$ and variance $\frac{n^2-1}{12}$ $\left(\approx \frac{n^2}{12}\text{ for large } n\right)$, s has mean $\mu_s = \left(\frac{n-1}{2}\right)k$ and variance $\sigma_s^2 \approx \frac{kn^2}{12}$.

(b) p_s is symmetric unimodal about a peak which occurs on or near the mean μ_s.

We are now in a position to estimate M_m using the Chebyshev Inequality.

THEOREM 2. *Let* $M_m = \sum_{\text{largest } m}|S_j|$, *then for large* m,

$$M_m \geqslant \begin{cases} \left(\frac{2m}{3\sqrt{k}n}\right)n^k, & m \leqslant n\sqrt{k}, \\ \left(1 - \frac{kn^2}{3m^2}\right)n^k, & m \geqslant n\sqrt{k}. \end{cases}$$

Proof. $M_m = n^k \sum_{\text{largest } m} p_s(j)$ from (10). As a result of (b) above, the latter is equal to,

(11) $$n^k\,\text{Prob.}\left[-\frac{m-1}{2} \leqslant s-\mu_s \leqslant \frac{m}{2}\right] \approx n^k\,\text{Prob.}\left[|s-\mu_s| \leqslant \frac{m}{2}\right]$$

for large m.

Now from the Chebyshev Inequality:

(12) $$\text{Prob. }[|s-\mu_s| \leqslant x] \geqslant 1 - \frac{\sigma_s^2}{x^2} \approx 1 - \frac{kn^2}{12x^2}.$$

This is an accurate estimate for values of x somewhat larger than one standard deviation, but is of little use otherwise. To obtain a more accurate estimate for small values of x, notice that since $p_s(x)$ is symmetric unimodal

$$\text{Prob.}\ [|s-\mu_s| \leqslant x] \geqslant \frac{x}{y}\,\text{Prob.}\ [|s-\mu_s| \leqslant y] \quad \text{for} \quad y \geqslant x$$

$$\geqslant \frac{x}{y}\left(1-\frac{kn^2}{12y^2}\right) \quad \text{from (12)}.$$

Now as a function of y the right-hand side reaches a maximum at $\bar{y} = \frac{\sqrt{k}n}{2}$ independently of x, so that substituting $\bar{y}$ in the above, for $x \leqslant \frac{1}{2}\sqrt{k}n$

$$\text{Prob.}\ [|s-\mu_s| \leqslant x] \geqslant \frac{4x}{3\sqrt{k}n}. \tag{13}$$

The theorem is then established by substitution from (12) and (13) into (11).

4. Comparison of asymptotic bounds

4.1. Introduction. To present the final results, it is necessary to consider four separate regions.

For each region:

(i) Using Corollary 2 together with Theorem 2 an asymptotic lower bound is established on $\log L_k(N, n)$:

$$l_k(N, n) \leqslant \log L_k(N, n).$$

(ii) Using Assertions 1 and 2 an asymptotic upper bound is obtained on $n^{k-2}\log L_2(N, n)$:

$$n^{k-2}\log L_2(N, n) \leqslant u_k(N, n).$$

(From (3) $\log L_k(N, n) \leqslant n^{k-2}\log L_2(N, n)$.)

(iii) It is shown that the asymptotic ratio of the bounds, $\frac{u_k(N, n)}{l_k(N, n)}$ is $\leqslant 3\sqrt{k}$. Then since

$$\frac{n^{k-2}\log L_k(N, n)}{l_k(N, n)} \leqslant \frac{u_k(N, n)}{l_k(N, n)}$$

it follows that for large n and N:

$$\frac{1}{3\sqrt{k}}\,n^{k-2}\log L_2(N, n) \leqslant \log L_k(N, n) \leqslant n^{k-2}\log L_2(N, n).$$

4.2. Lower bounds. As an immediate consequence of Corollary 2b ($\log L_k(N, n) \geqslant M_N \log 2$) we obtain:

(a) for $N \leqslant n\sqrt{k}$

$$\log L_k(N, n) \geqslant M_N\log 2 \geqslant \frac{2g}{3\sqrt{k}}\,n^k \log 2, \quad \text{where} \quad g = \frac{N}{n}.$$

(The last inequality follows from Theorem 2.)

(b) Similarly for $n\sqrt{k} \leqslant N \leqslant (n-1)k+1$

$$\log L_k(N, n) \geqslant M_N \log 2 \geqslant n^k\left(1-\frac{kn^2}{3N^2}\right)\log 2 \geqslant \frac{2}{3}\,n^k\log 2.$$

From Corollary 2a we have:

$$\log L_k(N, n) \geqslant M_m \log(N/m) \quad \text{for any } m \leqslant (N-1)k+1.$$

Using this:

(c) for $(n-1)k+1 \leqslant N \leqslant n^2$ choose $m = [n\sqrt{k}] \approx n\sqrt{k}$, then

$$\log L_k(N, n) \geqslant M_{n\sqrt{k}}\log(N/n\sqrt{k}) \geqslant n^k\left(1-\frac{kn^2}{3n^2k}\right)\log(N/n\sqrt{k})$$

$$= \frac{2n^k}{3}\log(g/\sqrt{k}).$$

(The last inequality follows from Theorem 2.)

(d) Similarly for $n^2 \leqslant N$: Letting $m = (n-1)k+1$,

$$\log L_k(N, n) \geqslant n^k \log\frac{N}{(n-1)k+1} > n^k \log\frac{g}{k}.$$

4.3. Upper bounds.

(a) For $N \leqslant n\sqrt{k}$, using (3) together with (6), we have

$$\log L_k(N, n) \leqslant n^k g \log 4 \quad \text{where} \quad g = N/n.$$

For the remaining regions we use (3) together with (4):

(b) For $n\sqrt{k} \leqslant N \leqslant (n-1)k+1$

$$\log L_k(N, n) \leqslant \log L_k(nk, n) \leqslant n^2 \log\frac{e^{3/2}}{4}(1+k) \leqslant n^2\log 2k \quad \text{for} \quad k \geqslant 3.$$

(c) and (d). For $N \geqslant (n-1)k+1$

$$\log L_k(N, n) \leqslant n^2\log\frac{e^{3/2}}{4}(1+g) \leqslant n^2 \log 2g.$$

(The last inequality is true for $g = N/n \geqslant k \gtrapprox 3$.)

4.4. Asymptotic ratios. Using the results of the previous two paragraphs we can now calculate the asymptotic ratio of the derived bounds. The results are best summarized in a table:

Table

Bounds on the normalized logarithm of the count, $L_k(N, n)$, as a function of the normalized maximum $g = (N/n)$

Region	Asymptotic lower bound on $(\log L_k(N, n))/(n^k)$	Asymptotic upper bound on $(\log L_k(N, n))/(n^k)$	Asymptotic ratio is $\leqslant$
$N \leqslant n\sqrt{k}$	$\frac{g\log 4}{3\sqrt{k}}$	$g\log 4$	$3\sqrt{k}$
$n\sqrt{k} \leqslant N \leqslant (n-1)k+1$	$\frac{2}{3}\log 2$	$\log(2k)$	$\frac{3}{2}\log_2(2k)$
$(n-1)k+1 \leqslant N \leqslant n^2$	$\frac{2}{3}\log(g/\sqrt{k})$	$\log(2g)$	5
$n^2 \leqslant N$	$\log(g/k)$	$\log(2g)$	1

Remarks. (a) For $N \leqslant n\sqrt{k}$ the ratio is obviously $3\sqrt{k}$.

(b) For $n\sqrt{k} \leqslant N \leqslant (n-1)k+1$ the ratio is $\frac{3}{2}\log_2 2k$.

To prove that the latter is $\leqslant 3\sqrt{k}$ $(k \geqslant 3)$ it is sufficient to prove $\frac{1}{2}\log_2 2k \leqslant \sqrt{k}$ or $\frac{1}{2}+\log_2\sqrt{k} \leqslant \sqrt{k}$ which can be readily checked for $k \geqslant 3$.

(c) For $(n-1)k+1 \leqslant N \leqslant n^2$ the ratio is $\frac{3}{2}\frac{\log 2g}{\log\frac{g}{\sqrt{k}}}$, which is a decreasing function of g. Since in this region $g = \frac{N}{n} \gtrapprox k \geqslant 3$ for large n,

$$\frac{3}{2}\frac{\log 2g}{\log\frac{g}{\sqrt{k}}} \leqslant \frac{3}{2}\frac{\log 6}{\frac{1}{2}\log 3} < 5 < 3\sqrt{k} \quad \text{for} \quad k \geqslant 3.$$

(d) For $N \geqslant n^2$ the ratio is

$$\frac{\log 2g}{\log\frac{g}{k}} = \frac{\log g+\log 2}{\log g-\log k}.$$

Since $g > N/n \geqslant n$, asymptotically the ratio reaches unity.

The final result:

$$\frac{1}{3\sqrt{k}}n^{k-2}\log L_2(N, n) \leqslant \log L_k(N, n) \leqslant n^{k-2}\log L_2(N, n)$$

follows immediately from the discussion in Section 4.1.

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