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A partition problem of Frobenius, II

by

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As in [2], we consider the following question, often posed by Frobenius. What is the largest integer $M = M(a_1, a_2, \ldots, a_n)$ omitted by the linear form $\sum_{i=1}^{n} a_i x_i$, where $a_1, \ldots, a_n$ is an increasing sequence of fixed positive integers whose GCD is 1, and the x_i are n variable non-negative integers? Those integers which are taken on by this linear form will be called *attainable*. Much work has been done on this problem, and we refer the reader to [2] and [3] for an extensive bibliography.

In this paper we examine the situation when

$$(*)\qquad a_k \equiv k-1 \pmod{a_1}, \qquad 2 \leqslant k \leqslant n.$$

We describe an algorithm which yields the answer in all such cases, and then we give the explicit solution when $n \leqslant 5$. The following two lemmas will be our principal tools.

LEMMA 1. *Let* (*) *be satisfied. Then* $M \equiv -1 \pmod{a_1}$.

Proof. Let $N = N(a_1, a_2, \ldots, a_n)$ be the smallest attainable integer congruent to $-1 \pmod{a_1}$, so that $N - a_1$ is omitted. Consider the sequence of integers

$$(1)\quad \{N = a_2u_2 + \ldots + a_nu_n,\ a_2u_2 + \ldots + a_{n-1}(u_{n-1}+1) + a_n(u_n-1),$$
$$a_2u_2 + \ldots + a_{n-1}(u_{n-1}+2) + a_n(u_n-2),$$
$$\ldots,\ a_2u_2 + \ldots + a_{n-1}(u_{n-1}+u_n),$$
$$a_2u_2 + \ldots + a_{n-2}(u_{n-2}+1) + a_{n-1}(u_{n-1}+u_n-1),$$
$$\ldots,\ a_2(u_2 + \ldots + u_n),$$
$$a_2(u_2 + \ldots + u_n - 1), \ldots, a_2, 0\}.$$

Clearly (1) is a strictly decreasing sequence of attainable integers, and the kth integer in the sequence is congruent to $-k \pmod{a_1}$. Thus, for $2 \leqslant k \leqslant a_1$, we have an attainable integer smaller than N which is congruent to $-k \pmod{a_1}$. Therefore, $N-j$ is attainable for $0 \leqslant j < a_1$, and so $M = N - a_1$, which obviously yields the desired result.

LEMMA 2. *Let N be defined as in the proof of Lemma 1. Then*

$$\sum_{k=2}^{n} (k-1) u_k = a_1 - 1. \tag{2}$$

Proof. Since N is the only term in (1) which is congruent to $-1 \pmod{a_1}$, we see that (1) contains exactly a_1 terms. Also, by simply counting, we see that (1) contains

$$1 + u_n + (u_{n-1} + u_n) + \ldots + (u_2 + \ldots + u_n) = 1 + \sum_{k=2}^{n} (k-1) u_k$$

terms, and Lemma 2 is proven.

For $p \geqslant 1$ and $k \geqslant 3$, we let $\mathscr{R}(pa_k)$ denote any integer of the form $\sum_{j=2}^{k} a_j x_j$, where $x_j \geqslant 0$, $\sum_{j=2}^{k-1} x_j > 0$, and $\sum_{j=2}^{k} (j-1) x_j = p(k-1)$.

We let $R(pa_k)$ denote a particular $\mathscr{R}(pa_k)$.

We say that pa_k is *usable* if $pa_k < \mathscr{R}(pa_k)$ for all such $\mathscr{R}(pa_k)$, and we let α_k be the largest p such that pa_k is usable (where $\alpha_k = \infty$ is allowed). Our algorithm can now be described.

First, let

$$y_k = \min\left(\alpha_k, \left[\!\left[\frac{a_1 - 1}{k-1}\right]\!\right]\right), \quad 2 \leqslant k \leqslant n.$$

Then, let

$$z_n = y_n, \quad \text{and} \quad z_j = \min\left(y_j, \left[\!\left[\frac{a_1 - 1 - \sum_{i=j}^{n-1} i z_{i+1}}{j-1}\right]\!\right]\right), \quad n-1 \geqslant j \geqslant 2.$$

By Lemma 2 and the above, we see that

$$N = \sum_{k=2}^{n} a_k z_k.$$

More simply, to arrive at N, and hence at $M = N - a_1$, we use, subject to (2), as many a_n as are usable, then as many a_{n-1}, etc. Since we need only check if pa_k is usable for $(k-1)p \leqslant a_1 - 1$, N will be computed by this scheme in a finite number of steps.

In fact, we now show that the number of steps can be greatly decreased, and actually can be made independent of a_1. This will require two additional lemmas.

LEMMA 3. (All sums in this lemma are from $j = 1$ to $j = m-1$.) *Let $m \geqslant 2$, let β_j be non-negative integers, $1 \leqslant j \leqslant m-1$, and let*

$$\sum j\beta_j \geqslant m^2. \tag{3}$$

Then there is a sequence of integers $\{\sigma_j\}$, $1 \leqslant j \leqslant m-1$, with

$$\sum j\sigma_j \equiv 0 \pmod{m} \quad \text{and} \quad \{0\} < \{\sigma_j\} < \{\beta_j\}$$

(where $\{s_j\} < \{t_j\}$ if $s_j \leqslant t_j$ for all j and $s_j < t_j$ for at least one j).

Proof[1]. By (3) we see that

$$\sum \beta_j > \sum \frac{j}{m} \beta_j \geqslant m,$$

so that

$$\sum \beta_j \geqslant m+1.$$

We now let r be the smallest index where $\beta_r > 0$. For $1 \leqslant i \leqslant \beta_r$, we define the sequence $\{\sigma_j^{(i)}\}$ by

$$\sigma_j^{(i)} = \begin{cases} i, & j = r, \\ 0, & j \neq r. \end{cases}$$

If s is the second smallest index where $\beta_s > 0$, we define the sequence $\{\sigma_j^{(i)}\}$, $\beta_r < i \leqslant \beta_r + \beta_s$, by

$$\sigma_j^{(i)} = \begin{cases} \beta_r, & j = r, \\ i - \beta_r, & j = s, \\ 0, & j \neq r, s. \end{cases}$$

Continuing in this way, we obtain a total of $\sum \beta_j - 1 = p \geqslant m$ sequences $\{\sigma_j^{(i)}\}$, with

$$0 < \{\sigma_j^{(1)}\} < \{\sigma_j^{(2)}\} < \ldots < \{\sigma_j^{(p)}\} < \{\beta_j\}.$$

Clearly, either

$$\sum j\sigma_j^{(i)} \equiv 0 \pmod{m} \quad \text{for some } i,\ 1 \leqslant i \leqslant p,$$

or

$$\sum j(\sigma_j^{(k)} - \sigma_j^{(i)}) \equiv 0 \pmod{m} \quad \text{for some } i, k,\ 1 \leqslant i < k \leqslant p.$$

In either case, we are done.

We remark that Lemma 3 would be false if m^2 in (3) was replaced by $(m-1)^2$. The choice $\beta_{m-1} = m-1$, all other $\beta_j = 0$, provides a counterexample.

[1] The author would like to thank M. Tomlinson for his suggestions concerning this proof.

LEMMA 4. *Let $k \geqslant 3$. If $(k-2)a_k$ is usable, then pa_k is usable for all $p \geqslant 1$.*

Proof. Case I: $1 \leqslant p \leqslant k-3$ (Case I does not occur if $k = 3$). If $\mathscr{R}(pa_k) \leqslant pa_k$ for some $\mathscr{R}$, then

$$\mathscr{R}(pa_k) + (k-2-p)a_k = \mathscr{R}\big((k-2)a_k\big) \leqslant (k-2)a_k,$$

contradicting the fact that $(k-2)a_k$ is usable.

Case II: $p \geqslant k-1$. It is clearly sufficient to show that if $(p-1)a_k$ is usable, then pa_k is usable. Thus, assume that $(p-1)a_k$ is usable and $\mathscr{R}(pa_k) \leqslant pa_k$ for some $\mathscr{R}$, say

$$R(pa_k) = \sum_{j=2}^{k} a_j v_j.$$

If $v_k > 0$, then

$$\mathscr{R}\big((p-1)a_k\big) = \sum_{j=2}^{k-1} a_j v_j + a_k(v_k-1) \leqslant (p-1)a_k,$$

which contradicts the usability of $(p-1)a_k$. Thus, $v_k = 0$, so that

$$R(pa_k) = \sum_{j=2}^{k-1} a_j v_j,$$

with

$$\sum_{j=2}^{k-1} (j-1)v_j = p(k-1) \geqslant (k-1)^2.$$

By Lemma 3, there is a sequence $\{w_j\}$, $2 \leqslant j \leqslant k-1$, such that for some r, $0 < r < p$,

$$\sum_{j=2}^{k-1} (j-1)w_j = r(k-1), \quad \text{and} \quad \{0\} < \{w_j\} < \{v_j\}.$$

Then, applying Case I, we have

$$R(pa_k) = \sum_{j=2}^{k-1} a_j w_j + \sum_{j=2}^{k-1} a_j(v_j - w_j) = \mathscr{R}(ra_k) + \mathscr{R}\big((p-r)a_k\big)$$
$$> ra_k + (p-r)a_k = pa_k,$$

contradicting the definition of R, and completing the proof.

We observe that, by the remark following Lemma 3, Lemma 4 would be false if $(k-2)$ in the hypothesis was replaced by any smaller integer.

Thus, we see that in checking whether pa_k is usable, we need only consider $p \leqslant k-2$. We describe the resulting algorithm.

"Usability" Chart for $n = 5$

If							Use
$a_3 < 2a_2$	$a_4 < a_2 + a_3$	$2a_4 < 3a_3$	$a_5 < 2a_3$	$a_5 < a_2 + a_4$	$2a_5 < 2a_4 + a_3$	$3a_5 < 4a_4$	a_5, a_4, a_3, a_2
						$3a_5 \geqslant 4a_4$	$\leqslant 2a_5, a_4, a_3, a_2$
					$2a_5 \geqslant 2a_4 + a_3$		$\leqslant 1a_5, a_4, a_3, a_2$
				$a_5 \geqslant a_2 + a_4$			a_4, a_3, a_2
			$a_5 \geqslant 2a_3$				a_4, a_3, a_2
		$2a_4 \geqslant 3a_3$	$a_5 < 2a_3$	$a_5 < a_2 + a_4$			$a_5, \leqslant 1a_4, a_3, a_2$
				$a_5 \geqslant a_2 + a_4$			$\leqslant 1a_4, a_3, a_2$
			$a_5 \geqslant 2a_3$				$\leqslant 1a_4, a_3, a_2$
	$a_4 \geqslant a_2 + a_3$	$a_5 < 2a_3$					a_5, a_3, a_2
		$a_5 \geqslant 2a_3$					a_3, a_2
$a_3 \geqslant 2a_2$	$a_4 < 3a_2$	$a_5 < 4a_2$	$a_5 < a_2 + a_4$	$3a_5 < 4a_4$			a_5, a_4, a_2
				$3a_5 \geqslant 4a_4$			$\leqslant 2a_5, a_4, a_2$
			$a_5 \geqslant a_2 + a_4$				a_4, a_2
		$a_5 \geqslant 4a_2$					a_4, a_2
	$a_4 \geqslant 3a_2$	$a_5 < 4a_2$					a_5, a_2
		$a_5 \geqslant 4a_2$					a_2

First, check the usability, in the order given, of

$$a_3,\ a_4,\ 2a_4,\ a_5,\ 2a_5,\ 3a_5,\ a_6,\ \ldots,\ a_n,\ 2a_n,\ \ldots,\ (n-2)a_n$$

(certainly, if ra_k is not usable for some $r < k-2$, then it is unnecessary to check pa_k for $r+1 \leqslant p \leqslant k-2$). Then use, subject to (2), as many a_n as are usable, then as many a_{n-1}, etc., until N is arrived at (i.e., until (2) is satisfied). As an example, the preceding chart describes all "usability possibilities" for $n = 5$. An a_k in the last column indicates that, in that particular row, all a_k are usable, while the notation "$\leqslant ra_k$" indicates that at most ra_k are usable. The absence of a particular a_k in the last column indicates that, in that case, a_k is not usable.

By using the above chart, the explicit values of M for $n \leqslant 5$ can obviously be written down immediately. Thus, for $n = 4$, we have

$$M(a_1, a_2, a_3, a_4) = \begin{cases} \left[\dfrac{a_1-1}{3}\right] a_4 + \left[\dfrac{a_1-1-3\left[\dfrac{a_1-1}{3}\right]}{2}\right] a_3 + \\ \quad + \left(a_1-1-3\left[\dfrac{a_1-1}{3}\right]-2\left[\dfrac{a_1-1-3\left[\dfrac{a_1-1}{3}\right]}{2}\right]\right) a_2 - a_1 \\ \qquad \text{if } a_3 < 2a_2,\ a_4 < a_2+a_3,\ 2a_4 < 3a_3; \\ a_4 + \left[\dfrac{a_1-4}{2}\right] a_3 + \left(a_1-4-2\left[\dfrac{a_1-4}{2}\right]\right) a_2 - a_1 \\ \qquad \text{if } a_3 < 2a_2,\ a_4 < a_2+a_3,\ 2a_4 \geqslant 3a_3; \\ \left[\dfrac{a_1-1}{2}\right] a_3 + \left(a_1-1-2\left[\dfrac{a_1-1}{2}\right]\right) a_2 - a_1 \\ \qquad \text{if } a_3 < 2a_2,\ a_4 \geqslant a_2+a_3; \\ \left[\dfrac{a_1-1}{3}\right] a_4 + \left(a_1-1-3\left[\dfrac{a_1-1}{3}\right]\right) a_2 - a_1 \\ \qquad \text{if } a_3 \geqslant 2a_2,\ a_4 < 3a_2; \\ (a_1-1)a_2 \quad \text{if } a_3 \geqslant 2a_2,\ a_4 \geqslant 3a_2. \end{cases}$$

For particular examples when $n = 5$, we have

$$M(211, 634, 1057, 1691, 2114) = 1057\cdot 105 - 211 = 110774;$$

$$M(729, 2917, 2918, 4377, 4378) = 4378\cdot 182 - 729 = 796067;$$

$$M(1019, 6115, 7135, 12231, 13251) = 13251\cdot 254 + 7135 - 1019 = \\ = 3371870.$$

Clearly, a chart such as the above can be constructed for any n, although for $n \geqslant 8$ or so the large number of subcases makes this impractical.

In conclusion, we observe that when the a_i's are consecutive integers, Lemmas 1 and 2 immediately yield the result

$$M(a_1, a_1+1, \ldots, a_1+n-1) = \frac{a_1-j}{n-1} a_1 + \left[\frac{n-3+j}{n-1}\right] a_1 - 1,$$

$$\text{where} \quad a_1 \equiv j \pmod{n-1},\ 1 \leqslant j \leqslant n-1.$$

This agrees with Brauer's result [1], which was obtained by a considerably more complex method.

References

[1] Alfred Brauer, *On a problem of partitions*, Amer. J. Math. 64 (1942), pp. 299–312.
[2] J. S. Byrnes, *On a partition problem of Frobenius*, J. of Combinatorial Theory, Series A 17 (1974), pp. 162–166.
[3] P. Erdös and R. L. Graham, *On a linear diophantine problem of Frobenius*, Acta Arith. 21 (1972), pp. 399–408.

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