Conspectus materiae tomi XXVIII, fasciculi 1

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange — Address of the Editorial Board and of the exchange — Die Adresse der Schriftleitung und des Austausches — Адрес редакции и книгообмена

ACTA ARITHMETICA
ul. Śniadeckich 8, 00-950 Warszawa

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires
The authors are requested to submit papers in two copies
Die Autoren sind gebeten um Zusendung von 2 Exemplaren jeder Arbeit
Рукописи статей редакция просит предлагать в двух экземплярах

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ACTA ARITHMETICA
XXVIII (1975)

Arithmetical functions with periodic zeros*

by

EMIL GROSSWALD (Haifa)

1. Introduction. In [4] D. H. Lehmer and S. Selberg studied the function

$$G(x) = \sum_{n\leqslant x} g(n), \qquad g(x) = \sum_{n\leqslant x} \frac{\mu(n)}{n} \qquad (\mu(n) = \text{Möbius' function})$$

and proved that for any constant K, $G(x)-K$ changes signs infinitely often. If we define a "zero" of $G(x)-2$ as an integer n such that $(G(n)-2)\times \times(G(n-1)-2) < 0$, then Lehmer and Selberg verified computationally that (with only two exceptions) the ratios of consecutive "zeros" are almost constant. Next, assuming the RH (Riemann hypothesis) they present a heuristic argument, which makes it plausible that at least on the RH said ratio should stay close to the constant $e^{\pi/\gamma_1} \simeq 1.2488968\ldots$ (here $\gamma_1 = 14.1\ldots$ is the ordinate of the smallest complex zero of Riemann's zeta function $\zeta(s)$). The average of the ratios found computationally for the first 56 "zeros" (up to $n = 2119483$) is indeed remarkably close to said value.

Their argument runs as follows. Assuming the RH and the simplicity of all complex zeros $\varrho_\nu = \frac{1}{2}+i\gamma_\nu$ of $\zeta(s)$ ($s = \sigma+it$, σ, t real), they prove the "explicit formula":

$$G(x-1)-2 = -2x^{1/2}\sum_{\nu=1}^{\infty}\frac{\cos(\gamma_\nu\log x-\alpha_\nu)}{(\frac{1}{4}+\gamma_\nu^2)R_\nu} + O(x^{-2}) \tag{1}$$

where $\zeta'(\varrho_\nu) = R_\nu e^{i\alpha_\nu}$. In general, the dominant term will be the first one, with $\gamma_1 = 14.1\ldots$, so that

$$G(x-1)-2 \simeq \frac{-2x^{1/2}}{(\gamma_1^2+\frac{1}{4})R_1}\cos(\gamma_1\log x-\alpha_1)$$

* This paper was written with partial support from the National Science Foundation, through the Grant NSF–GP–23170 (Amd. 2).

and $G(x-1)-2$ will change sign near those values x_n of x for which

$$\gamma_1 \log x_n - \alpha_1 = (2n+1)\frac{\pi}{2},$$

i.e., for

$$\log x_n = \{(n+\tfrac{1}{2})\pi+\alpha_1\}/\gamma_1 = \frac{\pi/2+\alpha_1}{\gamma_1} + n\frac{\pi}{\gamma_1} = \log x_0 + n(\pi/\gamma_1), \text{ say.}$$

Clearly, $x_n = x_0 e^{(\pi/\gamma_1)n}$ and the ratio of two consecutive values x_n/x_{n-1} is $e^{\pi/\gamma_1} \simeq 1.2488968\ldots$

Now, similar explicit formulae are known for many functions. Some of them, such as that for $M(x) = \sum_{n\leqslant x}\mu(n)$ are very similar in structure and the periodicy of the "zeros" (if it exists) should be even easier to verify. In fact, this periodicy is less regular and this is not surprising. Indeed, in the explicit formulas for $\mu(x)$ (see, e.g. [9], p. 318) the ordinates γ_ν occur at the first, rather than the second power, so that the first zero dominates far less the remaining part of the infinite series, than in (1).

The main purpose of this paper is to define sets of functions $G_k(x;f)$, depending on an integral valued parameter k and on an arithmetical function $f = f(n)$ and with properties similar to those of $G(x)$. We shall study in some detail only the case, $f(n) = \mu(n)$; the function $G_1(x;\mu)$ is essentially the $G(x)$ of Lehmer and Selberg (more precisely, $G_1(x;\mu) = G(x-1)$). We shall also consider briefly the choice $f(n) = \Lambda(n)$ ($= \log p$ if $n = p^m$; $= 0$ otherwise). It is clear that analogous sets of functions $G_k(x;f)$ can be defined for many other arithmetical functions $f(x)$. In all these cases we obtain explicit formulae of the general type

$$\tilde{G}_k(x;f) = G_k(x;f) - \Phi_k(x;f) - \varphi_k(x;f) = 2\sum_{\substack{\varrho=\beta+i\gamma\\ \gamma>0}} \frac{x^{k+\beta-1}}{\gamma^{k+1}}\frac{R}{\tau}\cos(\gamma\log x-\omega).$$

Here $\Phi_k(x;f)$ is (for fixed k) a fixed polynomial in x and $\log x$, while $\varphi_k(x;f) = O(x^{-1})$, is holomorphic in a neighborhood of $x=\infty$ and may be neglected for the purpose on hand. The quantities R, τ and ω depend (besides on f) mainly on the specific zero $\varrho = \beta+i\gamma$ and should properly be written as R_ϱ, τ_ϱ, ω_ϱ. However, to keep the notations simple, the subscript will usually be omitted here, as well as for β and γ themselves. R may also depend weakly (as a logarithm) on x, τ on k, and ω on both. Their exact definitions are given in Section 3 Theorem 4 and in Section 10.

By increasing k, we enhance the dominance of the first zero for "small" values of x. Nevertheless, it is clear that if the RH does not hold, then eventually the zeros ϱ with larger real parts will become dominant. The periodicity of successive zeros of $G_k(x;f)$ will then be governed by the ratio e^{π/γ_ϱ}. It is known [7] that $\gamma_\varrho > 10^6$; hence, $e^{\pi/\gamma_\varrho} \cong 1+\pi/\gamma_\varrho$ is

much closer to unity than the 1.2488... furnished by a dominant first zero. One concludes that if the RH does not hold, we may expect to find the following behavior of the ratios x_{n+1}/x_n, of successive zeros of $G_k(x;f)$, for appropriatedly selected k. At the beginning (for small x), this ratio will stay close to 1.2488... because the first zero $\varrho_1 = \frac{1}{2}+i(14.14\ldots)$ dominates in the expansion of $G_k(x;f)$. When x is sufficiently large, so that some zeros with larger β become important, the periodicity will become rather irregular. Finally, for still larger x, the term with largest β and smallest (for that β) γ, say, γ^* (if such a term exists) will dominate the behavior of $G_k(x;f)$. Now the periodicity of consecutive zeros will become again very accurate, but with the much shortened period $1+\pi/\gamma^*$. Obviously, one cannot hope to prove the RH by this approach. However, if one suspects the existence of a zero $\varrho = \beta+i\gamma$, $\beta > \frac{1}{2}$, it is conceivable that irregularities in the periodicity of the zeros of some $G_k(x;f)$ in certain ranges of x, followed by a considerable shortening of the period for still larger x will point strongly towards the existence of such a zero. This would presumably involve less numerical work than the actual determination of such a zero. Indeed at least for $f=\mu$, the values of the $G_k(x;f)$ can be computed rather easily by one of two methods, one direct, the other iterative (Theorems 5 and 6).

Next, generalizing the results of [4], which states that for every constant K, $G(x)-K$ changes signs infinitely often, it will be shown that for any $\varepsilon>0$ and $C>0$, arbitrarily large, and with $\theta = \sup\{\sigma | \zeta(\sigma+it) = 0\}$ the two inequalities

$$G_k(x;\mu) > Cx^{\theta+k-1-\varepsilon}, \quad G_k(x;\mu) < -Cx^{\theta+k-1-\varepsilon} \tag{2}$$

hold, each on a (naturally different) infinite set, say $x_1' < x_2' < \ldots < x_n' < \ldots$, and $x_1'' < x_2'' < \ldots < x_n'' < \ldots$ respectively, with $\lim_{n\to\infty} x_n' = \lim_{n\to\infty} x_n'' = \infty$. Such infinite increasing sets of positive numbers without finite limit point will be called (for want of a better name; see also [1] and [2]) simply X-sets. If a function satisfies two inequalities like (2) on X-sets we shall describe the situation simply by stating that each of the inequalities $G_k(x;\mu) \gtrless \pm Cx^{\theta+k-1-\varepsilon}$ holds on an X-set.

If there exists a zero $\varrho = \theta+i\gamma$, then we may take $\varepsilon = 0$ in (2), provided C is not too large (see Theorem 7). In particular, and without any unproven hypothesis, each of the inequalities $G_k(x;\mu) \gtrless \pm Cx^{k-1/2}$ holds on an X-sets. For $k=1$ this shows that the function $G(x)$ of [4] satisfies each of the inequalities $G(x) \gtrless \pm Cx^{1/2}$ on X-sets, a statement stronger than and implying that of [4].

We shall also consider briefly the case $f(n) = \Lambda(n)$. This choice of f has certain advantages, but also certain disadvantages in comparison with $f(n) = \mu(n)$.

The present paper has benefited greatly from critical remarks by H. Montgomery and I. Richards to whom I herewith express my gratitude.

2. Notations. Given any arithmetical function $g(m)$ the symbol $\sum_{m\leqslant x}' g(m)$ stands for $\sum_{1\leqslant m<x} g(m)$ if x is not an integer and for $\sum_{1\leqslant m<n} g(m)+\frac{1}{2}g(n)$, if $x=n$ is a positive integer. $[x]$ stands for the greatest integer function.

The symbol $\int_{(\sigma)} F(s)\,ds$ means $\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} F(s)\,ds$ whenever this limit exists.

For natural integer k we denote by $S_r^{(k)}$ (upper index usually omitted) the (absolute values of the) Stirling numbers of the first kind, i.e. the rth elementary symmetric function on the integers $1, 2, \ldots, k-1$. In particular,

$$S_1=\frac{1}{2!}k(k-1),\quad S_2=\frac{1}{4!}k(k-1)(k-2)(3k-1),\quad \ldots,\quad S_{k-1}=(k-1)!.$$

For convenience we also set $S_0=1$, for every k.

In what follows occur also certain constants defined by:

$$a_n=\sum_{r=1}^{2n+1}\frac{1}{r}-\sum_{r=1}^{k-1-2n}\frac{1}{r},$$

$$b_n=\frac{\Gamma'}{\Gamma}(2n+1)+\frac{\zeta'}{\zeta}(2n+1)-\log 2\pi=-\frac{1}{2}\frac{\zeta''}{\zeta'}(-2n),$$

$$c_n=a_n+b_n,\qquad d_n=a_n-b_n$$

and B_{2n}, the Bernoulli numbers in the even subscript notation.

If $\varrho=\beta+i\gamma$ is a complex zero of order r of $\zeta(s)$, then the coefficients $a_j=a_j(\varrho)$ are defined by

$$\zeta(s)=(s-\varrho)^r\sum_{j=0}^{\infty}a_j(s-\varrho)^j.$$

3. Main results

DEFINITION 1. For integral $r\geqslant -1$ and $\sigma>r+1$, we define $M_r(x)$ by

$$M_r(x)=\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s\zeta(s-r)}\,ds.$$

DEFINITION 2. For integral $k\geqslant 0$ the functions $G_k(x)=G_k(x;\ \mu)$ are defined by

$$G_k(x)=\frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}M_{r-1}(x)x^{k-r}.$$

THEOREM 1.

$$M_k(x)=\sum_{m\leqslant x}{}' m^k\mu(m).$$

THEOREM 2.

$$G_k(x)=\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^{s+k-1}}{(s-1)s(s+1)\ldots(s+k-1)\zeta(s)}\,ds\qquad(\sigma>1).$$

THEOREM 3. *$G_k(x)$ is a cardinal spline function. Specificially, $G_k(x)$ is a polynomial of exact degree k on any interval without square free integers. At the abscissae $x=n$, n square-free integer, $G_k(x)$ is differentiable if $k\geqslant 2$, and has $k-1$ continuous derivatives, while $G_k^{(k)}(x)$ has the discontinuity $\mu(n)/n$ at $x=n$.*

THEOREM 4. *Let $k'=[k/2]$ and $k''=[(k-1)/2]$ and set*

$$\Phi_k(x)=\Phi_k(x;\ \mu)=\frac{2}{k!}\Big\{kx^{k-1}-\sum_{n=1}^{k'}\binom{k}{2n}\frac{n}{B_{2n}}x^{k-2n}-$$
$$-\sum_{n=1}^{k''}(-1)^n\binom{k}{2n+1}\frac{(2\pi)^{2n}}{(2n)!}\frac{1}{\zeta(2n+1)}x^{k-2n-1}(\log x+c_n)\Big\}$$

and

$$\varphi_k(x)=2x^{k-1}\sum_{2n\geqslant k}(-1)^n\frac{(2n-k)!}{(2n)!(2n+1)!\zeta(2n+1)}\left(\frac{2\pi}{x}\right)^{2n};$$

then

$$G_k(x)=\Phi_k(x)+\varphi_k(x)+2\sum_{\gamma\geqslant 0}\frac{x^{k+\beta-1}}{\gamma^{k+1}}\frac{R}{\tau}\cos(\gamma\log x-\omega).\tag{3}$$

Here $R=R_\varrho(x)=\dfrac{(\log x)^{r-1}}{(r-1)!|a_0|}|1+\varphi_\varrho(x)|$, with $\varphi_\varrho(x)=\sum_{j=1}^{r-1}v_j(\log x)^{-j}$, where r is the multiplicity of the zero ϱ of $\zeta(s)$ and a_0 was defined in Section 2; also, the v_j's are numerical constants depending (for fixed k) only on ϱ. If $\lambda=\arg(1+\varphi_\varrho(x))$, $|\lambda|\leqslant\pi$, $\psi=\arg a_0$, and $\tau e^{i\delta}=\prod_{m=0}^{k}\left(1+\dfrac{\beta+m-1}{i\gamma}\right)$, $\tau>0$, δ real, then $\omega=\delta+\psi-\lambda+\pi(k+1)/2$.

COROLLARY 1.

$$G_1(x)-2=-2\sum_{\gamma>0}\frac{x^\beta}{\gamma^2}\frac{R}{\tau}\cos(\gamma\log x-\psi-\delta+\lambda)+O(x^{-2}).$$

COROLLARY 2 (see [4]). *If we assume the RH and the simplicity of the zeros $\varrho=\frac{1}{2}+i\gamma$ of $\zeta(s)$, let $\alpha=\alpha_\varrho=\arg\zeta'(\varrho)$; then*

$$G_1(x)-2=-8x^{1/2}\sum_{\gamma>0}\frac{\cos(\gamma\log x-\alpha)}{(4\gamma^2+1)|\zeta'(\varrho)|}+O(x^{-2}).$$

Corollary 3. *If $\theta > \frac{1}{2}$ and if $\varrho^* = \theta + it_1$ is the zero of lowest ordinate with the abscissa θ, then, for sufficiently large x, the function $G_k(x) - \Phi_k(x)$ vanishes at a set of abscissae $\{x_n\}$, approximatively periodic and with*

$$\frac{x_{n+1}}{x_n} \cong e^{\pi/t_1} \cong 1 + \frac{\pi}{t_1}.$$

The values of the $G_k(x)$ may be computed with the help of the following two theorems.

Theorem 5.

$$G_k(x) = \frac{1}{k!}\sum_{m<x}\frac{\mu(m)}{m}(x-m)^k.$$

In order to state Theorem 6 we need the following definition.

Definition 3. Let

$$G^{(0)}(m_1) = \sum_{m=1}^{m_1}\frac{\mu(m)}{m}, \quad G^{(1)}(m_2) = \sum_{m_1=1}^{m_2} G^{(0)}(m_1),$$

and, in general if $G^{(r-1)}(m_r)$ is already defined, then $G^{(r)}(m_{r+1})$ is defined by

$$G^{(r)}(m_{r+1}) = \sum_{m_r=1}^{m_{r+1}} G^{(r-1)}(m_r).$$

Theorem 6. *The $G_k(n)$ may be computed recursively, if the $G_{k-r}(n)$ $(r \geqslant 1)$ are already known, by*

$$G_k(n) = G^{(k)}(n-1) - \frac{1}{k!}\sum_{r=1}^{k-1}(k-r)!\,S_r\cdot G_{k-r}(n),$$

with the $G^{(k)}(n-1)$ of Definition 3 and the S_r defined in Section 2.

The oscillatory properties of $G_k(x)$ are condensed in the following theorem.

Theorem 7. *Each of the following inequalities holds on an X-set, under the stated conditions:*

(a) $G_k(x) \lessgtr \pm Cx^{\theta+k-1-\varepsilon}$, *unconditionally, for $\varepsilon > 0$, arbitrarily small and $C > 0$, arbitrarily large;*

(b) $G_k(x) \lessgtr \pm Cx^{\theta+k-1}$, *assuming that there exists at least one zero $\varrho = \theta + i\gamma$ and that if ϱ is simple*

$$C < \max_{\varrho=\theta+i\gamma}\frac{1}{|(\varrho-1)\varrho(\varrho+1)\dots(\varrho+k-1)\zeta'(\varrho)|};$$

(c) $G_k(x) \lessgtr \pm Cx^{k-1/2}$, *with arbitrarily large C, assuming either that the RH does not hold or, if the RH does hold, that there exists at least one multiple zero $\varrho = \frac{1}{2} + i\gamma$;*

(d) $G_k(x) \lessgtr \pm Cx^{k-1/2}$, *holds unconditionally for*

$$C < \max_{\varrho}\frac{1}{|(\varrho-1)\varrho\dots(\varrho+k-1)\zeta'(\varrho)|}$$

(only the case of RH with all complex zeros simple is of interest here; otherwise (a), (b), (c) *yield stronger results).*

Corollary 4. *Under the conditions of* (d), *both inequalities $G_1(x) \lessgtr \pm Cx^{1/2}$ hold on X-sets for $C < \max_{\varrho}\frac{1}{|\varrho^2\zeta'(\varrho)|}$.*

4. Some lemmas. In the proofs of the theorems we shall need several lemmas of which some are known, but all are listed here, for ease of reference.

Lemma 1. *For integral, positive k and arbitrary, complex s not equal to an integer $\leqslant 1$,*

$$\sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}\frac{1}{s+r-1} = \frac{(k+1)!}{(s-1)s(s+1)\dots(s+k)}.$$

Lemma 2. *For integral, positive k and n,*

$$\sum_{m_1=m}^{n-1}\sum_{m_2=m_1}^{n-1}\dots\sum_{m_k=m_{k-1}}^{n-1}1 = \frac{1}{k!}(n-m)(n-m+1)\dots(n-m+k-1).$$

Lemma 3. *For integral, positive k and n,*

$$\sum_{m_1=m}^{n-1}\sum_{m_2=m_1}^{n-1}\dots\sum_{m_k=m_{k-1}}^{n-1}1 = \frac{1}{k!}\{(n-m)^k + (n-m)^{k-1}S_1 + \dots + (n-m)S_{k-1}\}.$$

Lemma 4. *For integral, positive k and n*

$$\sum_{m=1}^{n-1}\frac{\mu(m)}{m}\sum_{m_1=m}^{n-1}\dots\sum_{m_k=m_{k-1}}^{n-1}1 = \frac{1}{k!}\sum_{r=1}^{k} r!\,S_{k-r}\cdot G_r(n).$$

Lemma 5. *For positive, integral ν,*

$$\zeta'(-2\nu) = (-1)^\nu\cdot\tfrac{1}{2}(2\pi)^{-2\nu}(2\nu)!\,\zeta(2\nu+1),$$

$$\zeta''(-2\nu) = (-1)^\nu(2\pi)^{-2\nu}(2\nu)!\,\zeta(2\nu+1)\left\{\log 2\pi - \frac{\Gamma'}{\Gamma}(2\nu+1) - \frac{\zeta'}{\zeta}(2\nu+1)\right\}.$$

Lemma 6. *For integral k and n,*

$$G^{(k)}(n) = \sum_{m=1}^{n}\frac{\mu(m)}{m}\sum_{m_1=m}^{n}\dots\sum_{m_k=m_{k-1}}^{n}1.$$

LEMMA 7. *If* $F(s) = \int_1^\infty f(x)x^{-s}dx$ *converges for* $\sigma > a$ *but not for* $\sigma > a-\varepsilon$ $(\varepsilon > 0)$ *and represents a function* $F(s)$ *holomorphic in* $\sigma > a$, *but not in any larger half-plane, then, if* $F(s)$ *is holomorphic at* $s = a$, *it follows that* $f(x)$ *changes signs on an* X*-set.*

LEMMA 8. *If* $F(s)$ *is defined as in Lemma 7, is holomorphic for* $\sigma > a$, *but in no half-plane* $\sigma > a-\varepsilon$ $(\varepsilon > 0)$ *and* $s = a$ *is a singular point of* $F(s)$, *so that* $\lim_{\sigma\to a} F(\sigma) = +\infty$, *then one can still assert that* $f(x)$ *changes signs on an* X*-set, provided that there exists a* $t \neq 0$ *such that*

$$\overline{\lim_{\sigma\to a}} \left| \frac{F(\sigma+it)}{F(\sigma)} \right| = a > 1.$$

LEMMA 9 (H. Montgomery [5]). *For any given* $\varepsilon > 0$ *there exists a* $T_0 = T_0(\varepsilon)$ *such that for* $T \geqslant T_0$ *the following holds: Between* T *and* $2T$ *there exists a* t, *for which* $|\zeta(\sigma\pm it)|^{-1} < Ct^\varepsilon$ *for* $0 \leqslant \sigma \leqslant 1$, *and with an absolute constant* $C > 0$.

LEMMA 10 (A. Selberg [8]). *There exists an absolute constant* $A > 0$ *and an infinite set of real numbers* T_ν, $\lim_{\nu\to\infty} T_\nu = +\infty$, *such that for any positive integer* m, *the inequality*

$$\left| \frac{\zeta'}{\zeta}(s) \right| < A\log^2 m$$

holds on the contour $\sigma > -2m-1$, $t = \pm T_\nu$; $\sigma = -2m-1$, $|t| < T_\nu$.

5. Proofs of the lemmas

Lemma 1 is the classical decomposition into partial fractions (see also [6], Prob. 5, p. 29).

Lemma 2 seems well known and may be proved by double induction on k and $n-m$ (no exact reference comes to mind, but see the related formula in [6], p. 183).

Lemma 3 is an immediate corollary of Lemma 2.

Lemma 4 is used only in the proof of Theorem 6; its proof requires Theorem 5 and is postponed to Section 8.

Proof of Lemma 5. One differentiates the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$

and sets $s = 1+2\nu$. In all terms but one the cosine vanishes and one obtains the first statement. If one differentiates the equation twice, collects terms and sets $s = 1+2\nu$, one obtains the second statement.

Proof of Lemma 6. By definition, $G^{(0)}(m_1) = \sum_{m=1}^{m_1} \mu(m)/m$, and

$$G^{(1)}(m_2) = \sum_{m_1=1}^{m_2} G^{(0)}(m_1) = \sum_{m_1=1}^{m_2} \sum_{m=1}^{m_1} \frac{\mu(m)}{m} = \sum_{m=1}^{m_2} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_2} 1$$

so that the lemma holds for $k = 0$ and $k = 1$. To complete the proof by induction, we assume the lemma already proved for $k = r-1$, i.e. that

$$G^{(r-1)}(m_r) = \sum_{m=1}^{m_r} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_r} \cdots \sum_{m_{r-1}=m_{r-2}}^{m_r} 1.$$

Then

$$\begin{aligned} G^{(r)}(m_{r+1}) &= \sum_{m_r=1}^{m_{r+1}} G^{(r-1)}(m_r) = \sum_{m_r=1}^{m_{r+1}} \sum_{m=1}^{m_r} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_r} \cdots \sum_{m_{r-1}=m_{r-2}}^{m_r} 1 \\ &= \sum_{m=1}^{m_{r+1}} \frac{\mu(m)}{m} \sum_{m_r=m}^{m_{r+1}} \sum_{m_1=m}^{m_r} \cdots \sum_{m_{r-1}=m_{r-2}}^{m_r} 1 \\ &= \sum_{m=1}^{m_{r+1}} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_{r+1}} \sum_{m_r=m_1}^{m_{r+1}} \sum_{m_2=m_1}^{m_r} \cdots \sum_{m_{r-1}=m_{r-2}}^{m_r} 1 \\ &= \sum_{m=1}^{m_{r+1}} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_{r+1}} \sum_{m_2=m_1}^{m_{r+1}} \cdots \sum_{m_{r-2}=m_{r-3}}^{m_{r+1}} \sum_{m_r=m_{r-2}}^{m_{r+1}} \sum_{m_{r-1}=m_{r-2}}^{m_r} 1 \\ &= \sum_{m=1}^{m_{r+1}} \frac{\mu(m)}{m} \sum_{m_1=m}^{m_{r+1}} \sum_{m_2=m_1}^{m_{r+1}} \cdots \sum_{m_{r-2}=m_{r-3}}^{m_{r+1}} \sum_{m_{r-1}=m_{r-2}}^{m_{r+1}} \sum_{m_r=m_{r-1}}^{m_{r+1}} 1 \end{aligned}$$

and the formula holds for $k = r$, hence, for all k.

Lemma 7 is a classical theorem of Landau (see [3]).

Lemma 8 is Theorem C in [2], or Theorem 1 in [1].

Lemma 9 is due to H. Montgomery [5].

Lemma 10 is due to A. Selberg (see [8]).

6. Proofs of Theorems 1, 2 and 3

6.1. Proof of Theorem 1. For $\sigma > r+1$,

$$\frac{1}{\zeta(s-r)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{s-r}},$$

so that

$$M_r(x) = \frac{1}{2\pi i}\int_{(\sigma)} \frac{x^s}{s}\left\{\sum_{m=1}^{\infty}\frac{\mu(m)}{m^{s-r}}\right\} ds$$
$$= \sum_{m=1}^{\infty}\mu(m)m^r\cdot\frac{1}{2\pi i}\int_{(\sigma)}\frac{(x/m)^s}{s}\,ds = \sum_{m\leqslant x} m^r\mu(m),$$

the interchange of integration and summation being easily justified for $\sigma\geqslant r+1+\varepsilon$ $(\varepsilon>0)$. This proves Theorem 1.

6.2. Proof of Theorem 2. By Definitions 1 and 2, and with $\sigma_r > r$,

$$G_k(x) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}M_{r-1}(x)x^{k-r}$$
$$= \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}x^{k-r}\cdot\frac{1}{2\pi i}\int_{(\sigma_r)}\frac{x^s}{s\zeta(s-r+1)}\,ds$$
$$= \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}x^{k-r}\cdot\frac{1}{2\pi i}\int_{(\sigma_r')}\frac{x^{s+r-1}}{(s+r-1)\zeta(s)}\,ds$$
$$(\sigma_r' = \sigma_r - r + 1 > 1)$$
$$= \frac{1}{2\pi i}\int_{(\sigma')}\frac{x^{s+k-1}}{\zeta(s)}\cdot\frac{1}{k!}\left\{\sum_{r=0}^{k}(-1)^r\frac{\binom{k}{r}}{s+r-1}\right\}ds \qquad (\sigma' = \sigma_r' > 1).$$

By Lemma 1, the inner sum equals $\dfrac{k!}{(s-1)s(s+1)\ldots(s+k-1)}$, so

$$G_k(x) = \frac{1}{2\pi i}\int_{(\sigma')}\frac{x^{s+k-1}}{\zeta(s)(s-1)s\ldots(s+k-1)}\,ds$$

as claimed.

6.3. Proof of Theorem 3. This follows immediately from Definition 2. On the intervals between squarefree integers, all $M_k(x)$ are constants so that $G_k(x)$ is a polynomial in x, with highest coefficient $M_{r-1}(z+0)$ and where z is the largest squarefree integer less than x. By differentiation on these intervals,

$$G_k^{(m)}(x) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}(k-r)(k-r-1)\ldots(k-r-m+1)M_{r-1}(x)x^{k-r-m}$$
$$= \frac{1}{k!}\sum_{r=0}^{k-m}(-1)^r\frac{k!}{r!(k-r-m)!}M_{r-1}(x)x^{k-r-m}$$
$$= \frac{1}{(k-m)!}\sum_{r=0}^{k-m}(-1)^r\binom{k-m}{r}M_{r-1}(x)x^{(k-m)-r} = G_{k-m}(x).$$

Hence, the assertions concerning the continuity will be proved, if we verify the continuity of $G_1(x)$ at all integers. Now

$$\lim_{x\to n^-}G_1(x) = \lim_{x\to n^-}\{M_{-1}(x)x - M_0(x)\} = \lim_{x\to n}\left\{x\sum_{m\leqslant n-1}\frac{\mu(m)}{m} - \sum_{m\leqslant n-1}\mu(m)\right\}$$
$$= n\sum_{m\leqslant n-1}\frac{\mu(m)}{m} - \sum_{m\leqslant n-1}\mu(m)$$
$$= n\sum_{m\leqslant n-1}\frac{\mu(m)}{m} + n\frac{\mu(n)}{n} - \left(\sum_{m\leqslant n-1}\mu(m)+\mu(n)\right)$$
$$= n\sum_{m\leqslant n}\frac{\mu(m)}{m} - \sum_{m\leqslant n}\mu(m) = \lim_{x\to n^+}\left\{x\sum_{m\leqslant n}\frac{\mu(m)}{m} - \sum_{m\leqslant n}\mu(m)\right\}$$
$$= \lim_{x\to n^+}\{xM_{-1}(x) - M_0(x)\} = \lim_{x\to n^+}G_1(x)$$

and $G_1(x)$ is continuous at all integers $x = n$.

Finally, for all natural integers k, $G_k^{(k)}(x) = G_0(x) = M_{-1}(x) = \sum_{m\leqslant x}'\mu(m)/m$, so that $G_k^{(k)}(n^+) - G_k^{(k)}(n^-) = \mu(n)/n$ as claimed.

7. Proof of Theorem 4. We use the representation of $G_k(x)$ given by Theorem 2 and estimate the integral by Cauchy's Theorem on residues. We apply it to the rectangle of vertices $\sigma_1\pm iT$ $(\sigma_1 = 1+1/\log x)$, $-U\pm iT$, and where $U\to+\infty$ through odd integers and $T\to\infty$ through a sequence of values $T = t_\nu$, $\lim_{\nu\to\infty}t_\nu = \infty$ and with $|\zeta(\sigma\pm it_\nu)|^{-1} < Ct_\nu^\varepsilon$ for $0\leqslant\sigma\leqslant 1$. The existence of such a sequence is guaranteed by Lemma 9.

We must:

(i) show that the integrals $\int_{\sigma_1+iT}^{-U+iT}$, $\int_{-U-iT}^{-U+iT}$ and $\int_{-U-iT}^{\sigma_1-iT}$ can be made arbitrarily small in absolute value, when U and T increase in previously stated manner;

(ii) compute the sum of the residues and take the limit of that sum, as $U\to\infty$, $T\to\infty$.

Then if we denote the integral along the rectangular contour by $\int_{\mathscr{C}}$, we shall be able to conclude that

$$G_k(x) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_1-iT}^{\sigma_1+iT}\frac{x^{s+k-1}}{(s-1)s(s+1)\ldots(s+k-1)\zeta(s)}\,ds \tag{4}$$
$$= \lim_{T\to\infty}\left\{\lim_{U\to\infty}\frac{1}{2\pi i}\left(\int_{\mathscr{C}} - \int_{\sigma_1+iT}^{-U+iT} - \int_{-U+iT}^{-U-iT} - \int_{-U-iT}^{\sigma_1-iT}\right)\right\}.$$

(i) The estimation of the last three integrals while somewhat lengthy is routine. If $U = 1-2N$ (N natural integer, $N\to\infty$), then the functional equation of the zeta function and the Stirling formula for the Γ-function (for T arbitrarily large, but constant, and $0<N\to\infty$) yield that

$$|\zeta(1-2N+it)| = \left(\frac{N}{\pi e}\right)^{2N-1/2} e^{-1/2}\left(e^{\pi t/2}+e^{-\pi t/2}\right)\left(1+o(1)\right).$$

From this follows that

$$\left|\int_{-U+iT}^{-U-iT} \frac{x^{s+k-1}}{(s-1)s\dots(s+k-1)\zeta(s)}\,ds\right| \leqslant \frac{x^{-2N+k}}{(2N+1)^{k+1}}\, C\left(\frac{\pi e}{N}\right)^{2N-1/2} e^{1/2}\int_0^T e^{-\pi t/2}\,dt$$
$$< C_1\left(\frac{\pi e}{Nx}\right)^{2N-1/2}\left(\frac{x}{2N+1}\right)^{k+1} x^{-3/2}\to 0$$

for $N\to\infty$.

Also,

$$\left|\int_{\sigma_1+iT}^{-U+iT} \frac{x^{s+k-1}}{(s-1)s\dots(s+k-1)\zeta(s)}\,ds\right| < \int_{-U}^{\sigma_1}\frac{x^{\sigma+k-1}}{T^{k+1}|\zeta(\sigma+iT)|}\,d\sigma$$
$$\leqslant \int_{-\infty}^{\sigma_1}\frac{x^{\sigma+k-1}}{T^{k+1}|\zeta(\sigma+iT)|}\,d\sigma = \frac{x^{k-1}}{T^{k+1}}\left\{\int_{-\infty}^{0}+\int_0^1+\int_1^{1+1/\log x}\right\}\frac{x^{\sigma}}{|\zeta(\sigma+iT)|}\,d\sigma = I,$$

say, and clearly, also $\left|\int_{-U-iT}^{\sigma_1-iT}\right| < I$.

On $1<\sigma<1+1/\log x$,

$$\frac{1}{|\zeta(\sigma+iT)|} = O(\log T),$$

so that

$$\int_1^{1+1/\log x}\frac{x^\sigma}{|\zeta(\sigma+iT)|}\,d\sigma = O\left(x\frac{\log T}{\log x}\right).$$

Next,

$$\int_{-\infty}^{0}\frac{x^\sigma}{|\zeta(\sigma+iT)|}\,d\sigma = \int_0^\infty\frac{x^{-\sigma}}{|\zeta(-\sigma+iT)|}\,d\sigma$$

and, by the functional equation,

$$|\zeta(-\sigma+iT)| = 2^{-\sigma}\pi^{-\sigma-1}\left|\sin\frac{\pi}{2}(-\sigma+iT)\right|\,|\Gamma(1+\sigma-iT)|\,|\zeta(1+\sigma-iT)|.$$

Here

$$\left|\sin\frac{\pi}{2}(-\sigma+iT)\right| = \tfrac{1}{2}e^{\pi T/2}\left(1+o(1)\right)$$

and

$$\frac{1}{|\zeta(1+\sigma-iT)|} = \begin{cases} O(\log T) & \text{for}\quad 0<\sigma<T-1,\\ O(1) & \text{for}\quad \sigma>T-1.\end{cases}$$

By Stirling's formula for the Γ-function

$$\Gamma(1+\sigma-iT) > \begin{cases} C_1 e^{-\pi T/2}T^{\sigma+1/2} & \text{for}\quad \sigma<T-1,\\ C_2\left(\dfrac{\sigma+1}{e^{(\pi+2)/4}}\right)^{\sigma+1/2} & \text{for}\quad \sigma>T-1.\end{cases}$$

An easy estimate now leads to

$$\int_{-\infty}^{0}\frac{x^\sigma}{|\zeta(\sigma+iT)|}\,d\sigma = O(T^{-1/2}),$$

so that

$$\frac{x^{k-1}}{T^{k+1}}\left\{\int_{-\infty}^{0}+\int_1^{1+1/\log x}\dots\right\} = \frac{x^{k-1}}{T^{k+1}}\left\{O\left(x\frac{\log T}{\log x}\right)+O(T^{-1/2})\right\}\to 0$$

for $T\to\infty$.

It is only in order to cross the critical strip that we have to take $T = t_\nu$ and use Lemma 9. With this choice we obtain

$$\int_0^1\frac{x^\sigma}{|\zeta(\sigma+iT)|}\,d\sigma = O\left(\frac{x}{\log x}T^{\varepsilon}\right)$$

so that

$$I = O\left(\frac{x^k}{\log x}T^{\varepsilon-k-1}\right)$$

and vanishes for $T\to\infty$.

We have shown that

$$G_k(x) = \sum_{-\infty<\sigma<\sigma_1}\text{Residues}.$$

It has to be observed, however, that for $k=0$, the sum that appears in (3), and which corresponds to the sum of the residues of the complex zero of $\zeta(s)$, may not converge in the usual sense (and certainly does not converge uniformly — otherwise $G_0(x)$ would be continuous, which it is not), but only by grouping the terms as dictated by

$$\lim_{\nu\to\infty}\sum_{|\gamma|<t_\nu}\Big(\sum_{-\infty<\sigma<\sigma_1}\text{Residues}\Big).$$

For $k \geqslant 1$, the convergence may well be absolute and uniform in x, because $\sum |\gamma|^{-m}$ converges for $m \geqslant 2$ and $G_k(x)$ is continuous, but we have no information about the way in which $R = R_\varrho$ varies and neither the absolute convergence of the series, nor its uniform (in x) convergence are proved.

(ii) The residues. The integrand in (4) is holomorphic at $s = 1$ and actually for $\sigma > \theta$. It has the following poles, with corresponding residues:

(a) $s = 0$; simple pole with residue $\dfrac{2}{(k-1)!}x^{k-1}$.

(b) $s = 1-2\nu$, $1 \leqslant 2\nu-1 \leqslant k-1$; simple poles with residues

$$\{\zeta(1-2\nu)(-2\nu)(1-2\nu)\dots(-1)\cdot 1\cdot 2\cdot\ldots\cdot(k-2\nu)\}^{-1}x^{k-2\nu}$$
$$= \left\{-\frac{B_{2\nu}}{2\nu}(2\nu)!(k-2\nu)!\right\}^{-1}x^{k-2\nu} = -\frac{2\nu}{B_{2\nu}}\frac{1}{k!}\binom{k}{2\nu}x^{k-2\nu}.$$

(c) $s = -2\nu$, $0 < 2\nu \leqslant k-1$; double poles with residues

$$-\frac{1}{k!}\binom{k}{2\nu+1}\frac{x^{k-2\nu-1}}{\zeta'(-2\nu)}\left(\log x + a_\nu - \frac{1}{2}\frac{\zeta''}{\zeta'}(-2\nu)\right),$$

with a_ν as defined in Section 2.

This computation is somewhat lengthy, but routine, and will not be reproduced here. One may now use Lemma 5, in order to replace $\zeta'(-2\nu)$ and $\zeta''(-2\nu)$ by more convenient functions.

(d) $s = -2n$, $2n \geqslant k$, simple poles with residues

$$\frac{(2n-k)!}{(2n+1)!}\frac{x^{k-2n-1}}{\zeta'(-2n)};$$

here $\zeta'(-2n)$ is to be replaced by its value from Lemma 5.

(e) $s = \beta + i\gamma$, the complex zeros of $\zeta(s)$; the order of these zeros is unknown, but at least the first 3500000 are known (see [7]) to be simple and with $\beta = \frac{1}{2}$. To find the residue at an r-fold zero ϱ, we consider the expansion of the integrand J around $s = \varrho$, set $y = s-\varrho$ and obtain:

$$J = \frac{x^{s+k-1}}{(s-1)s(s+1)\dots(s+k-1)\zeta(s)}$$
$$= \frac{x^{\varrho+k-1}e^{y\log x}}{(y+\varrho-1)(y+\varrho)\dots(y+k+\varrho-1)y^r}\left(\sum_{j=0}^{\infty}a_j y^j\right)^{-1}$$
$$= \frac{x^{\varrho+k-1}}{a_0 y^r}\sum_{j=0}^{\infty}\frac{(y\log x)^j}{j!}\left\{\prod_{m=0}^{k}(i\gamma+\beta-1+m)\cdot\prod_{m=0}^{k}\left(1+\frac{y}{\varrho-1+m}\right)\times\right.$$
$$\left.\times\left(1+\sum_{j=1}^{\infty}\frac{a_j}{a_0}y^j\right)\right\}^{-1}$$

$$= \frac{x^{\varrho+k-1}}{a_0 y^r (i\gamma)^{k+1}\tau e^{i\delta}}\left[1 + y\left(\log x - \sum_{m=0}^{k}\frac{1}{\varrho+m-1} - \frac{a_1}{a_0}\right) + \ldots + \right.$$
$$+ y^{r-1}\left\{\frac{(\log x)^{r-1}}{(r-1)!} - \frac{(\log x)^{r-2}}{(r-2)!}\left(\sum_{m=0}^{k}\frac{1}{\varrho+m-1} + \frac{a_1}{a_0}\right) + \ldots + \right.$$
$$+ \frac{\log x}{1!}\left((-1)^{r-2}\sum_{m=0}^{k}\frac{1}{(\varrho+m-1)^{r-2}} - \frac{a_{r-2}}{a_0} + \ldots\right) +$$
$$+ \left((-1)^{r-1}\sum_{\substack{\nu_1+\nu_2+\ldots+\nu_{k+1}=r-1\\ \nu_j\geqslant 0}}\frac{1}{(\varrho-1)^{\nu_1}\varrho^{\nu_2}\dots(\varrho+k-1)^{\nu_{k+1}}} + \right.$$
$$\left.\left.\left.+\text{similar sums involving also the } (a_j/a_0)\text{'s}\right)\right\} + \ldots\right].$$

Hence, if $\psi = \arg a_0$, the coefficient of y^{r-1} is of the form

$$\frac{(\log x)^{r-1}}{(r-1)!}\big(1+\varphi_\varrho(x)\big),$$

with $\varphi_\varrho(x) = \sum_{j=1}^{r-1} v_j(\log x)^{-1} = O(1/\log x)$ and with coefficients v_j that depend (for fixed k) only on ϱ (which determines r as well as all the a_j's). It follows that the residue of the pole $s = \varrho$ of J equals

$$\frac{x^{k+\varrho-1}(\log x)^{r-1}|1+\varphi_\varrho(x)|}{\gamma^{k+1}\tau|a_0|(r-1)!}\exp\left\{i\left(-\frac{\pi}{2}(k+1)-\delta-\psi+\lambda\right)\right\}$$

with $\lambda = \arg\big(1+\varphi_\varrho(x)\big)$. If we pair together the residues corresponding to complex conjugate zeros, their sum is

$$2\frac{x^{\beta+k-1}(\log x)^{r-1}|1+\varphi_\varrho(x)|}{\gamma^{k+1}\tau|a_0|(r-1)!}\cos(\gamma\log x-\omega)$$

with $\omega = \delta+\psi-\lambda+\pi(k+1)/2$. It is clear that, for given k, $\delta\to 0$ as $\gamma\to\infty$ and $\lambda\to 0$ for $x\to\infty$, while ψ depends only on ϱ and $\pi(k+1)/2$ is constant. If we set also

$$R = R_\varrho(x) = \frac{(\log x)^{r-1}}{(r-1)!}\left|\frac{1+\varphi_\varrho(x)}{a_0}\right|,$$

the sum of all residues corresponding to the complex zeros ϱ with $0 < \gamma < t_\nu$ equals

$$2\sum_{0<\gamma<t_\nu}\frac{x^{\beta+k-1}}{\gamma^{k+1}}\frac{R}{\tau}\cos(\gamma\log x-\omega).$$

By adding up the residues and by using the definitions of $\Phi_k(x)$ and of $\varphi_k(x)$, we obtain (3), hence Theorem 4. Corollary 1 follows if we set $k=1$ in (3). If we also assume that all complex zeros are simple then $\varphi_\varrho(x)=\lambda=0$ and $a_0=\zeta'(\varrho)$; assuming also (with $k=1$) the RH, then

$$\tau e^{i\delta}=\left(1-\frac{1}{2i\gamma}\right)\left(1+\frac{1}{2i\gamma}\right)=1+\frac{1}{4\gamma^2},$$

and Corollary 2 is proved (observe an error of sign in (7) of [4]). Corollary 3 follows from Theorem 4 if we observe that $\Phi_k(x)+\varphi_k(x)=O(x^{k-1})$ and that (at least for sufficiently large k),

$$G_k(x)\sim\pm\frac{x^{k+\theta-1}}{t_1^{k+1}}\,\frac{R_{\varrho^*}}{\tau_{\varrho^*}},$$

as $x\to\infty$ through values for which the cosine equals ±1; hence, $G_k(x)$, and more accurately still $\tilde G_k(x)$, vanishes only close to values where $\cos(t_1\log x-\omega^*)=0$, so that with $x_0=e^{(\omega^*+\pi/2)/t_1}$, $x_n=x_0e^{n\pi/t_1}$.

8. Proofs of Theorems 5, 6 and of Lemma 4

8.1. Proof of Theorem 5. By Definition 2 and Theorem 1

$$\begin{aligned}G_k(x)&=\frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}M_{r-1}(x)x^{k-r}=\frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}x^{k-r}{\sum_{m\leqslant x}}' m^{r-1}\mu(m)\\&=\frac{1}{k!}{\sum_{m\leqslant x}}'\frac{\mu(m)}{m}\sum_{r=0}^{k}(-1)^r\binom{k}{r}x^{k-r}m^r=\frac{1}{k!}{\sum_{m\leqslant x}}'\frac{\mu(m)}{m}(x-m)^k\\&=\frac{1}{k!}\sum_{m<x}\frac{\mu(m)}{m}(x-m)^k\end{aligned}$$

as claimed.

Remark. This theorem puts well into evidence the continuity of $G_k(x)$ and of its first $k-1$ derivatives at all integers. Indeed, if x is an integer, then for $m=x$, the last term in the sum vanishes, so that $\sum_{m<x}$, ${\sum}'_{m\leqslant x}$, $\sum_{m\leqslant x}$ stand all for the same sum.

8.2. Proof of Lemma 4. By Lemma 3

$$\begin{aligned}\sum_{m=1}^{n-1}\frac{\mu(m)}{m}\sum_{m_1=m}^{n-1}\ldots\sum_{m_k=m_{k-1}}^{n-1}1&=\sum_{m=1}^{n-1}\frac{\mu(m)}{m}\cdot\frac{1}{k!}\sum_{r=1}^{k}(n-m)^rS_{k-r}\\&=\frac{1}{k!}\sum_{r=1}^{k}S_{k-r}\sum_{m=1}^{n-1}\frac{\mu(m)}{m}(n-m)^r.\end{aligned}$$

By Theorem 5, the coefficient of S_{k-r} equals $r!G_r(n)$ and the lemma is proved.

8.3. Proof of Theorem 6. From Lemma 4 it follows that

$$\sum_{m=1}^{n-1}\frac{\mu(m)}{m}\sum_{m_1=m}^{n-1}\ldots\sum_{m_k=m_{k-1}}^{n-1}1=\frac{1}{k!}\sum_{r=0}^{k-1}(k-r)!S_r\cdot G_{k-r}(n).$$

By Lemma 6, the first member equals $G^{(k)}(n-1)$, so that

$$G^{(k)}(n-1)=G_k(n)+\frac{1}{k!}\sum_{r=1}^{k-1}(k-r)!S_r\cdot G_{k-r}(n)$$

and Theorem 6 is proved.

9. Proof of Theorem 7.

By Theorem 2,

$$x^{1-k}G_k(x)=\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{(s-1)s(s+1)\ldots(s+k-1)\zeta(s)}\,ds\qquad(\sigma>1).$$

By Mellin's inversion theorem, it follows that

$$\frac{1}{(s-1)s(s+1)\ldots(s+k-1)\zeta(s)}=\int_1^\infty G_k(x)x^{1-k}x^{-1-s}\,dx=\int_1^\infty G_k(x)x^{-s-k}\,dx.$$

Consequently, for any real constants C and a,

$$\int_1^\infty\big(G_k(x)-Cx^a\big)x^{-s-k}\,dx=\frac{1}{(s-1)s\ldots(s+k-1)\zeta(s)}-\frac{C}{s+k-a-1}=F(s),$$

say. For $\sigma>a+1-k>1$, $F(s)$ is holomorphic and the integral converges. One verifies that if $a\neq k$, then $F(s)$ is holomorphic also at $s=1$ and has poles at $s=0,-1,-2,\ldots,-k+1$; at $s=a+1-k$; and at the complex zeros ϱ of $\zeta(s)$. If $a<\theta+k-1$, then $F(s)$ is holomorphic for $\sigma>\theta$, but not holomorphic in any half-plane $\sigma\geqslant\theta-\varepsilon$ $(\varepsilon>0)$. However, $F(s)$ is holomorphic at $s=\theta$, so that Lemma 7 (Landau's Theorem) is applicable and it follows that, for $a<\theta+k-1$, $G_k(x)-Cx^a$ changes signs infinitely often, regardless of the size, or sign of C. In particular, for arbitrarily large, positive C, $G_k(x)-Cx^a>0$ on a set $x_1',x_2',\ldots,x_n',\ldots,x_n'\to\infty$; and also with $C_1=-C$, $G_k(x)-C_1x^a<0$ holds for $x_1'',x_2'',\ldots,x_n'',\ldots,x_n''\to\infty$; in other words $G_k(x)>Cx^a$ and $G_k(x)<-Cx^a$ both hold on X-sets. This proves (a).

If $a=\theta+k-1$ and $\varrho=\theta+it$ is a zero of $\zeta(s)$, then, by Lemma 8, the same conclusion holds, provided that

$$\varlimsup_{\sigma\to\theta^+}\left|\frac{F(\sigma+it)}{F(\sigma)}\right|=a>1. \tag{5}$$

If ϱ is a multiple zero of $\zeta(s)$, then (5) holds trivially. If ϱ is a simple zero, then

$$F(\sigma) \cong \frac{C}{\sigma-\theta} \quad \text{and} \quad F(\sigma+it) \cong \frac{1}{(\varrho-1)\varrho\ldots(\varrho+k-1)\zeta'(\varrho)(\sigma-\theta)},$$

so that

$$\lim_{\sigma\to\theta^+}\left|\frac{F(\sigma+it)}{F(\sigma)}\right| = \frac{1}{C}\,\frac{1}{|(\varrho-1)\varrho\ldots(\varrho+k-1)\zeta'(\varrho)|}$$

and (5) becomes

$$C < \frac{1}{|(\varrho-1)\varrho\ldots(\varrho+k-1)\zeta'(\varrho)|};$$

this proves (b).

If the RH does not hold, then $\theta > \frac{1}{2}$ and (c) (and more) follows from (a) for sufficiently small $\varepsilon > 0$. If the RH holds, but there exist multiple zeros $\varrho = \frac{1}{2}+it$, then (5) holds for arbitrarily large $C > 0$. Finally, if the RH holds and all complex zeros of $\zeta(s)$ are simple, one obtains (d) from (b) with $\theta = \frac{1}{2}$. Corollary 4 follows immediately from Theorem 7 for $k=1$, and the remark that $|\varrho-1| = |\frac{1}{2}+it-1| = |\varrho|$.

10. The case $f(n) = \Lambda(n)$**.** This section is due to a remark of Professor Ian Richards.

The case $f = \Lambda$ is easier to handle than $f = \mu$, because now all complex zeros of $\zeta(s)$ lead to poles of the first order.

We now set for integral $r \geqslant -1$ and $\sigma > r+1$,

$$L_r(x) = -\frac{1}{2\pi i}\int_{(\sigma)}\frac{\zeta'}{\zeta}(\sigma-r)\frac{x^s}{s}\,ds = {\sum_{n\leqslant x}}'\Lambda(n)n^r$$

and define

$$G_k(x;\,\Lambda) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}L_{r-1}(x)x^{k-r}.$$

Clearly, as before,

$$G_k(x;\,\Lambda) = \frac{1}{k!}\sum_{n\leqslant x}\frac{\Lambda(n)}{n}(x-n)^k.$$

It follows, as in the proof of Theorem 3, that $G_k(x;\Lambda)$ is a spline function, namely a polynomial of exact degree k in the intervals between prime powers, while at $x = p^m$, $G_k(x;\Lambda)$ and its first $k-1$ derivatives, are continuous, but

$$G_k^{(k)}(p^m+0;\,\Lambda) - G_k^{(k)}(p^m-0;\,\Lambda) = p^{-m}\log p.$$

From the integral representation of $L_r(x)$ it follows that

$$(6)\qquad G_k(x;\,\Lambda) = -\frac{1}{2\pi i}\int_{(\sigma)}\frac{\zeta'}{\zeta}(s)\frac{x^{s+k-1}}{(s-1)s\ldots(s+k-1)}\,ds \qquad (\sigma > 1).$$

We can use (6) in order to obtain an explicit formula for $G_k(x;\Lambda)$, the crossing of the critical strip being now justified by Lemma 10.

The details of the computations may be suppressed; indeed, they are analogous to those for $G_k(x;\mu)$ of Section 7, only simpler (the poles at $s=\varrho$ are now all simple).

With k', k'', τ and δ defined as in Theorem 4 and with $\omega = \delta+(k+1)\pi/2$, the result reads as follows:

Theorem 8. *Set*

$$k!\,\Phi_k(x;\Lambda) = x^k\log x\left(1-\left(\gamma+\sum_{m=1}^{k}m^{-1}\right)\log^{-1}x\right) + k(\log 2\pi)x^{k-1} -$$
$$-\sum_{n=1}^{k}\binom{k}{2n}\frac{\zeta'}{\zeta}(1-2n)x^{k-2n} + \sum_{n=1}^{k''}\binom{k}{2n+1}x^{k-2n-1}(\log x + d_n),$$
$$k!\,\varphi_k(x;\Lambda) = (-1)^k\sum_{n=1}^{k'}\frac{(2n-k)!}{(2n+1)!}x^{-(2n+1-k)};$$

then

$$G_k(x;\Lambda) = \Phi_k(x;\Lambda)+\varphi_k(x;\Lambda) - 2\sum_{\gamma>0}\frac{x^{k+\beta-1}}{\tau\gamma^{k+1}}\cos(\gamma\log x-\omega),$$

each term of the sum being counted according to the multiplicity of the respective zero $\varrho = \beta+i\gamma$.

The discussion of Section 1 applies, of course, as well to

$$\tilde G_k(x;\Lambda) = G_k(x;\Lambda) - \Phi_k(x;\Lambda) - \varphi_k(x;\Lambda)$$
$$= -2x^{1/2}\sum_{\gamma>0}\frac{x^{\beta-1/2}}{\tau\gamma^{k+1}}\cos(\gamma\log x-\omega)$$

as to

$$\tilde G_k(x;\mu) = G_k(x;\mu) - \Phi_k(x;\mu) - \varphi_k(x;\mu)$$
$$= 2x^{1/2}\sum_{\gamma>0}\frac{x^{\beta-1/2}}{\gamma^{k+1}}\frac{R}{\tau}\cos(\gamma\log x-\omega).$$

The proof of an exact analog of Theorem 7 does not go through, because $s=1$, the singularity of largest real part of the integrand in (6) is itself real. In fact, it follows from (6) (by proceeding as in the proof of the prime number theorem) that $G_k(x;\Lambda) \cong \dfrac{x^k\log x}{k!}$ and increases monotonically for sufficiently large x. The correct oscillation theorem is

THEOREM 9. *Set*

$$H(x) = G_k(x; \Lambda) - x^k \log x + x^k\left(\gamma + 1 + \frac{1}{2} + \ldots + \frac{1}{k}\right);$$

then both inequalities

$$H(x) \gtrless \pm C x^{k+\theta-1-\varepsilon}$$

hold on X-sets, for arbitrarily large $C > 0$ and arbitrarily small $\varepsilon > 0$ (or with $\varepsilon = 0$ for $C > 0$ sufficiently small, if there exists a zero $\varrho = \theta + i\gamma$).

The proof follows from the fact that

$$H(x) = -\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{\zeta'}{\zeta}(s) + \frac{e^{-\gamma(s-1)}}{s-1}\right)\frac{x^{s+k-1}}{(s-1)s\ldots(s+k-1)}\,ds$$

and its details may be suppressed.

The proofs are simpler for $f = \Lambda$, then for $f = \mu$, and at first glance one may also hope to be able to compute the $G_k(x; \Lambda)$'s easier than the $G_k(x; \mu)$'s, because it is easier to check whether $n = p^m$, or $n \neq p^m$, than to factor n completely into primes. In fact, in order to compute $G_k(x; \Lambda)$ one has to add logarithms, instead of making sums of the form $1+1+ +\ldots+1$, followed by sums of $\pm 1/m$ (see Lemmas 2, 3, 4 and 6), as needed for $G_k(x; \mu)$. Also, $G_k(x; \mu)$ is asymptotically equal to the oscillating sum, which is the relevant element of the explicit formula, while for $f = \Lambda$ only the more complicated function $H(x)$ is asymptotically equal to that sum. Finally, $\Phi_k(x; \Lambda)$ is much more difficult to compute than $\Phi_k(x; \mu)$, especially for large k. Actually, already for $k = 2$, $\Phi_2(x; \mu) = 2x - 6$, while

$$\Phi_2(x; \Lambda) = \frac{x^2}{2}(\log x - \gamma - 3/2) + (x + 1/2)\log 2\pi + \tfrac{1}{2}\Gamma'(2) + 3\pi^{-2}\zeta'(2).$$

For these reasons no analogues to Theorems 5 and 6 will be given.

Bibliography

[1] E. Grosswald, *On some generalizations of theorems by Landau and Pólya*, Israel J. Math. 3 (1965), pp. 211–220.

[2] — *Oscillation theorems of arithmetical functions*, Transactions of the AMS, 126 (1967), pp. 1–28.

[3] E. Landau, *Über einen Satz von Tschebyschef*, Math. Ann. 61 (1905), pp. 527–550.

[4] D. H. Lehmer and S. Selberg, *A sum involving the function of Möbius*, Acta Arith. 6 (1960), pp. 111–114.

[5] H. L. Montgomery, Unpublished paper.

[6] J. Riordan, *Combinatorial Identities*, New York–London–Sidney 1968.

[7] J. B. Rosser, J. M. Yohe and L. Schoenfeld, *Rigorous computation of the zeros of the Riemann zeta-function*, Information Processing 68 (Proc. IFIP Congress, Edinburgh 1968), vol. 1; Mathematics, Software, pp. 70–76, Amsterdam 1969.

[8] A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Archiv for Mathematik og Naturvidenskab B.v. 48, No. 5 (1946).

[9] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford 1951.

THE TECHNION
Haifa, Israel
TEMPLE UNIVERSITY
Philadelphia, USA

Received on 23. 10. 1973 (481)