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On a conjecture of Norton

by

D. A. Burgess (Nottingham)

In memory of Yu. V. Linnik

Let χ denote a non-principal Dirichlet character to the modulus $n > 1$. In [1] Norton conjectured that for any positive integer h

$$\sum_{x=1}^{n}\Big|\sum_{y=1}^{h}\chi(x+y)\Big|^2 < nh. \tag{1}$$

He obtained the weaker upper bound $(9/8)nh$. The purpose of this paper is to prove (1).

Gallagher has proved that (1) holds if χ is a primitive character modulo n (see [1], Theorem 2.6). Thus we may assume that χ is not a primitive character. We prove (1) by induction on n.

If χ is a character modulo a proper divisor m of n then

$$\sum_{x=1}^{n}\Big|\sum_{y=1}^{h}\chi(x+y)\Big|^2 = \frac{n}{m}\sum_{x=1}^{m}\Big|\sum_{y=1}^{h}\chi(x+y)\Big|^2 < \frac{n}{m}mh = nh$$

by the inductive hypothesis. Thus we may suppose that

$$\chi = \chi_1\chi_2$$

where χ_1 is a primitive character modulo $n_1 > 1$, χ_2 is the principal character modulo $n_2 > 1$, $n = n_1n_2$, $(n_1, n_2) = 1$ and n_2 is square-free.

Let p denote a prime factor of n_2. Let χ_0 and χ_3 denote respectively the principal characters modulo p and n_2/p. Let

$$\psi = \chi_1\chi_3,$$

which is a non-principal character modulo $l = n/p$. Thus

$$\chi = \psi\chi_0.$$

Now we have

$$(2)\qquad \sum_{x=1}^{n}\Big|\sum_{y=1}^{h}\chi(x+y)\Big|^2 = \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{x=1}^{pl}\psi(x+y)\overline{\psi}(x+z)\chi_0(x+y)\overline{\chi}_0(x+z)$$

$$= \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{x=1}^{pl}\psi(x+y)\overline{\psi}(x+z) - \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\substack{x=1\\ p|(x+y)}}^{pl}\psi(x+y)\overline{\psi}(x+z) -$$

$$- \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\substack{x=1\\ p|(x+z)}}^{pl}\psi(x+y)\overline{\psi}(x+z) + \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\substack{x=1\\ p|(x+y)\\ p|(x+z)}}^{pl}\psi(x+y)\overline{\psi}(x+z)$$

$$= \Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4.$$

We evaluate these four sums.

First we have

$$\Sigma_1 = \sum_{x=1}^{pl}\Big|\sum_{y=1}^{h}\psi(x+y)\Big|^2 = p\sum_{x=1}^{l}\Big|\sum_{y=1}^{h}\psi(x+y)\Big|^2.$$

Next we see that

$$\Sigma_2 = \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\substack{x=1\\ p|(x+y)}}^{pl}\psi(x+y)\overline{\psi}(x+z) = \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\substack{t=1\\ p|t}}^{pl}\psi(t)\overline{\psi}(t+z-y)$$

$$= \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\lambda=1}^{l}\psi(p\lambda)\overline{\psi}(p\lambda+z-y) = \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\mu=1}^{l}\psi(\mu)\overline{\psi}(\mu+z-y)$$

since $(p, l) = 1$. Thus it follows that

$$\Sigma_2 = \sum_{y=1}^{h}\sum_{z=1}^{h}\sum_{\nu=1}^{l}\psi(\nu+y)\overline{\psi}(\nu+z) = \sum_{\nu=1}^{l}\Big|\sum_{y=1}^{h}\psi(\nu+y)\Big|^2.$$

It follows immediately that

$$\Sigma_3 = \overline{\Sigma}_2 = \sum_{\nu=1}^{l}\Big|\sum_{y=1}^{h}\psi(\nu+y)\Big|^2.$$

Thus we deduce from the inductive hypothesis that

$$(3)\qquad \Sigma_1 - \Sigma_2 - \Sigma_3 = (p-2)\sum_{x=1}^{l}\Big|\sum_{y=1}^{h}\psi(x+y)\Big|^2 < (p-2)lh.$$

It remains to estimate Σ_4. We have

$$\Sigma_4 = \sum_{x=1}^{pl}\Big|\sum_{\substack{y=1\\ p|(x+y)}}^{h}\psi(x+y)\Big|^2 = \sum_{u=1}^{l}\sum_{v=1}^{p}\Big|\sum_{\substack{y=1\\ p|(pu+lv+y)}}^{h}\psi(pu+lv+y)\Big|^2,$$

since $pu+lv$ runs through a complete set of residues modulo n. Since in addition ψ is a character modulo l it follows that

$$\Sigma_4 = \sum_{u=1}^{l}\sum_{v=1}^{p}\Big|\sum_{\substack{y=1\\ y\equiv -lv (\mathrm{mod}\, p)}}^{h}\psi(pu+y)\Big|^2 = \sum_{v=1}^{p}\sum_{u=1}^{l}\Big|\sum_{\substack{\lambda\\ 1\leqslant p\lambda-lv\leqslant h}}\psi(pu+p\lambda-lv)\Big|^2$$

$$= \sum_{v=1}^{p}\sum_{u=1}^{l}\Big|\sum_{\substack{\lambda\\ \frac{1+lv}{p}\leqslant\lambda\leqslant\frac{h+lv}{p}}}\psi(u+\lambda)\Big|^2 \leqslant \sum_{v=1}^{p} l \sum_{\substack{\lambda\\ \frac{1+lv}{p}\leqslant\lambda\leqslant\frac{h+lv}{p}}} 1$$

by the inductive hypothesis again. But the last expression

$$(4)\qquad = l\sum_{v=1}^{p}\sum_{\substack{\lambda\\ 1\leqslant p\lambda-lv\leqslant h}} 1 = l\sum_{v=1}^{p}\sum_{\substack{y=1\\ y\equiv -lv(\mathrm{mod}\, p)}}^{p} 1 = lh$$

since $(p, l) = 1$. (2), (3) and (4) together yield (1).

References

[1] K. K. Norton, *On character sums and power residues*, Trans. Amer. Math. Soc. 167 (1972), pp. 203–226.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
Nottingham, NG7 2RD
England

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