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On the difference between consecutive prime numbers

by

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Let p_n denote the nth prime number, and define

$$E = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n}.$$

The crude estimate $E \leqslant 1$ follows, as is easily seen, from the fact that $p_n \sim n\log n$ $(n\to\infty)$, which is equivalent to the prime number theorem. The long-standing conjecture that states that $E = 0$, which is obviously the case if there exist infinitely many pairs of primes p, q with a fixed non-zero difference, remains still unproved. The best result on the size of E that is known so far is due to G. Z. Pil'tjaĭ [2], who showed that

$$E \leqslant \tfrac{1}{4}(2\sqrt{2}-1) = 0.457106\ldots \tag{1}$$

improving a previous result of E. Bombieri and H. Davenport [1],

$$E \leqslant \tfrac{1}{8}(2+\sqrt{3}) = 0.466506\ldots$$

The purpose of the present article is to make a further improvement on these results. Indeed, we shall prove the following

Theorem.(1) *We have*

$$E \leqslant \frac{9-\sqrt{3}}{16} = 0.454246\ldots \tag{2}$$

An inspection of Pil'tjaĭ's paper [2] suggests a possibility of ameliorating the estimate (1) for E by an alternative choice of the various parameters therewith concerned. Our proof of (2) is thus a slight modification

(1) After the present paper had been submitted the writer learned from a kind letter of Prof. A. Schinzel that M. N. Huxley obtained, by improving Pil'tjaĭ's argument, the inequality $E \leqslant (4+\pi)/16 = 0.446349\ldots$, which supersedes (2).

of that of (1) given by Pil'tjaĭ, who has made an ingenious combination of some fundamental results due to Bombieri and Davenport [1] with the method of R. A. Rankin as described in [3]. However, we shall first reproduce without proofs some basic results that are indispensable to our argument.

1. We define for positive integers N and n

$$Z(N;2n) = \sum_{\substack{p,q\leqslant N\\ q-p=2n}} (\log p)\log q,$$

where the summation is extended over the prime numbers p, q satisfying the conditions indicated. Let k be any positive integer, and let

$$T(x) = t(0)+2\sum_{n=1}^{k} t(n)\cos 4\pi nx$$

be a trigonometrical polynomial with real coefficients $t(n)$, which is non-negative for all real values of x.

LEMMA 1. *If $k<(\log N)^C$ for some constant $C>0$, then for any fixed $\varepsilon>0$ we have*

$$\sum_{n=1}^{k} t(n)Z(N;2n) > 2N\sum_{n=1}^{k} t(n)H(n)-(\tfrac{1}{4}+\varepsilon)t(0)N\log N,$$

provided N is sufficiently large, where

$$H(n) = K\prod_{\substack{p\mid n\\ p>2}} \frac{p-1}{p-2}$$

with

$$K = \prod_{p>2}\left(1-(p-1)^{-2}\right).$$

LEMMA 2. *For any fixed $\delta>0$ we have*

$$Z(N;2n) < (8+\delta)H(n)N$$

for all n, provided N is large enough.

Lemmas 1 and 2 are due to Bombieri and Davenport [1], who proved these results by making essential use of Bombieri's mean value theorem of the remainder term in the prime number theorem for arithmetical progressions. In the proof of this mean value theorem a very important rôle is played by the large sieve, a fundamental and fruitful device invented by Yu. V. Linnik and developed by A. Rényi, E. Bombieri and others.

2. Let N be a sufficiently large positive integer, and put

$$M = \log N.$$

Let k, h and r be positive integers such that

$$k = [aM],\qquad h = [caM],$$
$$r = k-h = (1-c)aM+O(1),$$

where $a = 3(5-\sqrt{3})/32$ and c is any fixed number satisfying the condition

$$1 > c > c_0 = \frac{21+2\sqrt{3}}{33} > \frac{2}{3}.$$

Note that $c_0a = (9-\sqrt{3})/32$, and $x=(1-c_0)a$ is the lesser root of the quadratic equation $128x^2-48x+3=0$.

We now define the real numbers c_j $(0<j\leqslant k)$ by setting

$$c_j = \begin{cases} \alpha & \text{for}\quad 0<j\leqslant r,\\ 1 & \text{for}\quad r<j\leqslant h,\\ 1-\beta(j-h) & \text{for}\quad h<j\leqslant k,\end{cases}$$

where

$$\alpha = \frac{3\left(1-8(1-c)a\right)}{3-256(1-c)^2a^2},\qquad \beta = \frac{16a}{M}.$$

It is readily verified that $\alpha>0$, and that $1-\beta(j-h)\geqslant 0$ for $h<j\leqslant k$, provided N is large enough.

Put

$$T(x) = \Big|\sum_{j=1}^{k} c_j e^{4\pi ijx}\Big|^2 = t(0)+2\sum_{n=1}^{k} t(n)\cos 4\pi nx.$$

Then we have

$$t(n) = \sum_{j=1}^{k-n} c_jc_{j+n}\quad (0\leqslant n<k),\qquad t(k)=0.$$

It is not difficult to see that the coefficient $t(n)$ of $T(x)$, as a function of n, $0\leqslant n<k$, can be represented in each of the intervals $0\leqslant n<r$, $r\leqslant n<h-r$, $h-r\leqslant n<h$, $h\leqslant n<k$, as a sum of several monomials of degree at most 3 in n, and the absolute value of each of the monomials therein involved has for $0\leqslant n<k$ the order of magnitude $O(M)$, the O-constant being, of course, numerical. We have, in particular,

$$t(0) = \alpha^2r+h-\beta r(r+1)+\tfrac{1}{6}\beta^2r(r+1)(2r+1),$$

$$2\sum_{n=1}^{k} t(n) = \left(\alpha r+h-\tfrac{1}{2}\beta r(r+1)\right)^2-t(0)$$

and

$$\sum_{n=h}^{k} t(n) = \tfrac{1}{2}\alpha r(r+1)-\tfrac{1}{6}\alpha\beta r(r+1)(2r+1),$$

whence we get after some simplifications

$$2\sum_{n=1}^{k} t(n) = \big(A(c)\big)^2 M^2 + O(M)$$

and

$$32\sum_{n=h}^{k} t(n) + t(0)M = A(c)M^2 + O(M)$$

with

$$A(c) = \frac{3a-48(1-c)^2a^2-64(1-c)^2(7c-3)a^3}{3-256(1-c)^2a^2}.$$

Also, we have

$$\sum_{n=l}^{k} t(n)H(n) = \sum_{n=l}^{k} t(n) + O\big(M(\log M)^2\big)$$

for $l = 1$ and h. This is an immediate consequence of the following lemma which has been proved by Rankin [3].

LEMMA 3. *If s is a non-negative real number, then for arbitrary integers a, b with $0 \leqslant a < b \leqslant M$ we have*

$$\sum_{n=a}^{b} n^s H(n) = \sum_{n=a}^{b} n^s + O\big(M^s(\log M)^2\big),$$

where the O-constant may depend only on s.

3. Let the integer N be sufficiently large. We define as in Pil'tjaĭ [2]

$$W(N;\,2n) = Z(N;\,2n) - Z(NM^{-2};\,2n),$$

where we have set $M = \log N$, as before.

It follows from Lemmas 1 and 2 that one has for any fixed $\varepsilon > 0$

$$\sum_{n=1}^{k} t(n)\, W(N;2n) > 2N\sum_{n=1}^{k} t(n)H(n) - (\tfrac{1}{4}+2\varepsilon)t(0)NM,$$

since

$$\sum_{n=1}^{k} t(n)Z(NM^{-2};2n) = O(N).$$

Assume now that $W(N;2n) = 0$ for all integers n lying in the interval $1 \leqslant n < h$. Then, by Lemma 2 we have for any fixed $\delta > 0$

$$\sum_{n=1}^{k} t(n)\, W(N;2n) \leqslant \sum_{n=h}^{k} t(n)Z(N;2n) < (8+\delta)N\sum_{n=h}^{k} t(n)H(n),$$

and we arrive at the inequality

$$(8+\delta)\sum_{n=h}^{k} t(n)H(n) + (\tfrac{1}{4}+2\varepsilon)t(0)M > 2\sum_{n=1}^{k} t(n)H(n)$$

or

$$32\sum_{n=h}^{k} t(n) + t(0)M > 8\sum_{n=1}^{k} t(n) - 3\varepsilon M^2,$$

on appealing to the results obtained in § 2.

It follows from this that

$$A(c) > 4\big(A(c)\big)^2 - 4\varepsilon$$

or

$$1 > 4A(c) - \varepsilon',$$

where ε' is a positive real number that can be made as small as we please, if we let N tend to infinity. However, this is obviously impossible, since we have taken $1 > c > c_0$ and this implies that

$$4A(c) > 4A(c_0) = 1;$$

in fact, it will suffice to observe that the quantity

$$(1-c)^2(28ac-12a-1)$$

is, as a function of c, monotonically decreasing in the interval

$$c_1 = \frac{44+\sqrt{3}}{63} < c < 1, \quad \text{where} \quad c_1 < c_0.$$

We thus have proved that, if N is sufficiently large, then we must always have

$$W(N;2n) > 0$$

for at least one n with $1 \leqslant n < h = [ca\log N]$, which implies that $E \leqslant 2ca$. Since c may be taken arbitrarily close to c_0, this concludes the proof of our assertion (2).

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