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On the representation of the integer by positive quadratic forms with square-free variables

by

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1. Introduction. Let

$$f = f(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} a_{ij}x_ix_j \quad (a_{ij} = a_{ji},\ 1 \leqslant i, j \leqslant k)$$

be a positive quadratic form with integral coefficients $a_{11}, \ldots, a_{kk}$, $2a_{12}, \ldots, 2a_{k-1,k}$ and determinant $D = \det(a_{ij}) \neq 0$. $R(f, n)$ denotes the number of representations of the positive integer n by the quadratic form f with square-free variables, i.e. the number of solutions of the equation

$$f(x_1, \ldots, x_k) = n \tag{1}$$

in square-free integers $x_1, \ldots, x_k$. Estermann [1] has obtained the asymptotic value of $R(f, n)$ for $k \geqslant 5$ and $f = x_1^2 + \ldots + x_k^2$; he has also considered the singular series (see also [3]). In [11] improvement has been obtained for the error term in the Estermann formula(1).

In the present paper we consider the asymptotic value of $R(f, n)$ in the case when f is an arbitrary positive quadratic form in $k \geqslant 4$ variables. We deduce the following

Theorem 1. *Let* $k \geqslant 4$, $\alpha = \dfrac{k-3}{4(k+1)}$, $\varepsilon > 0$ — *an arbitrary positive number. Then*

$$R(f, n) = \frac{\pi^{k/2}}{D^{1/2}\Gamma(k/2)} G(f, n)n^{k/2-1} + O(n^{k/2-1-\alpha+\varepsilon}) \tag{2}$$

where $G(f, n)$ *is the singular series:*

$$G(f, n) = \sum_{t_1, \ldots, t_k=1}^{\infty} \frac{\mu(t_1)\ldots\mu(t_k)}{t_1^2 \ldots t_k^2} H(f_{t_1,\ldots,t_k}, n);$$

(1) Unfortunatly, issues [5], [8] have been found to be mistaken (see [11]).

here $\mu(t)$ is Möbius' function,

$$f_{t_1,\ldots,t_k} = f(t_1^2 x_1, \ldots, t_k^2 x_k),$$

$$H(f_{t_1,\ldots,t_k}, n) = \sum_{q=1}^{\infty} q^{-k} \sum_{h(\bmod q)}' S(hf_{t_1,\ldots,t_k}, q)\, e\left(-\frac{nh}{q}\right),$$

$$e(z) = e^{2\pi iz},$$

$$S(hf_{t_1,\ldots,t_k}, q) = \sum_{x_1,\ldots,x_k=1}^{q} e\big(hf(t_1^2x_1, \ldots, t_k^2x_k)/q\big);$$

the constant implied in O depends only on f and ε.

In § 7 we obtain estimates for the singular series $G(f, n)$. We find that there is a finite set P_f of prime numbers p and integers N_p such that if the congruences

$$(3)\qquad f(x_1, \ldots, x_k) \equiv n (\bmod p^{N_p})$$

are soluble in integers $x_1, \ldots, x_k$ not divisible by p^2 for each $p \in P_f$, then

$$G(f, n) > G_\varepsilon^{(k)} n^{-\varepsilon}$$

for some $G_\varepsilon^{(k)} > 0$. Otherwise $G(f, n) = 0$.

From Theorems 1 and 2 it follows that for sufficiently large n the equation (1) is soluble in square-free integers $x_1, \ldots, x_k$ provided that congruences (3) are soluble.

The singular series $G(f, n)$ has been considered in [9] for $k \geqslant 5$. But one can apply arguments of that paper, strictly speaking only for diagonal forms $f = a_1x_1^2 + \ldots + a_kx_k^2$.

A combination of methods of this paper and [11] gives (for $k \geqslant 6$) the asymptotic formula for $R(f, n)$ with the error term $O(n^{\frac{k}{2}-\frac{5}{4}+\varepsilon})$.

A remark on notation. ε denotes a positive number as small as we please. The constant implied in O-notation will depend only on f and ε. For two vectors $\boldsymbol{a} = (a_1, \ldots, a_k)$, $\boldsymbol{b} = (b_1, \ldots, b_k)$ we define $\boldsymbol{ab} = (a_1b_1, \ldots, a_kb_k)$. Throughout this paper the vector $\boldsymbol{t} = (t_1, \ldots, t_k)$ will have square-free coordinates.

$$\mu(\boldsymbol{t}) := \mu(t_1) \ldots \mu(t_k),$$

$$\sum_{\boldsymbol{t} \leqslant a} := \sum_{1\leqslant t_1 \leqslant a} \cdots \sum_{1 \leqslant t_k \leqslant a}.$$

In the sum $\sum\limits_{h (\bmod q)}'$ the index h runs the reduced system of residues mod q.

2. Preliminary results. For any positive integers $t_1, \ldots, t_k$ we write

$$(4)\qquad f_{\boldsymbol{t}} = f_{\boldsymbol{t}}(\boldsymbol{x}) = f_{t_1,\ldots,t_k}(x_1, \ldots, x_k) = f(t_1^2x_1, \ldots, t_k^2x_k),\quad \boldsymbol{x} = (x_1, \ldots, x_k).$$

Let $N(f_{\boldsymbol{t}}, n)$ and $N^*(f_{\boldsymbol{t}}, n)$ denote the number of solutions of the equation

$$(5)\qquad f_{\boldsymbol{t}}(\boldsymbol{x}) = n$$

in integers and non-zero integers $x_1, \ldots, x_k$ respectively.

LEMMA 1. *There is a constant $c = c(f)$ such that for any solution $\boldsymbol{x} = (x_1, \ldots, x_k)$ of the equation* (1) *we have*

$$(6)\qquad |x_i| \leqslant cn^{1/2} \quad (i = 1, \ldots, k).$$

Proof. In the rational field f is equivalent to a diagonal form, say, $a_1y_1^2 + \ldots + a_ky_k^2$; $a_1 > 0, \ldots, a_k > 0$, and

$$(x_1, \ldots, x_k) = (y_1, \ldots, y_k)S$$

for some matrix $S = (S_{ij})_{i,j=1}^k$. If $x_1, \ldots, x_k$ is a solution of the equation (1), then

$$|y_i| \leqslant a_i^{-1} n^{1/2} \quad (i = 1, \ldots, k),$$

hence

$$|x_i| \leqslant k \cdot \max_{1\leqslant i,j\leqslant k} |S_{ij}| \cdot \max_{1\leqslant i\leqslant k} |y_i| \leqslant c(f) n^{1/2} = cn^{1/2} \quad (i = 1, \ldots, k).$$

COROLLARY. *The equation* (5) *does not have any non-zero integer solutions provided*

$$(7)\qquad \max_{1\leqslant i\leqslant k} |t_i| > c^{1/2} n^{1/4}.$$

Indeed, let $\boldsymbol{x}$ be a solution of the equation (5), then by (4) and Lemma 1

$$|t_i^2 x_i| \leqslant cn^{1/2} \quad (i = 1, \ldots, k).$$

This contradicts the above inequality.

LEMMA 2. *Let $x_3^0, \ldots, x_k^0$ be any fixed integers. Then there is a constant $\gamma = \gamma(f, \varepsilon)$ independent of $x_3^0, \ldots, x_k^0$ such that the number of solutions of the equation*

$$(8)\qquad f(x_1, x_2, x_3^0, \ldots, x_k^0) = n$$

in integers x_1, x_2 does not exceed γn^ε.

Proof. We have

$$f(x_1, \ldots, x_k) = d_1^{-1}y_1^2 + (d_1d_2)^{-1}y_2^2 + (d_1d_2)^{-1}\varphi(x_3, \ldots, x_k),$$

where

(9) $$y_1 = \sum_{i=1}^{k} a_{1i}x_i, \quad y_2 = \sum_{i=2}^{k} (a_{11}a_{2i} - a_{12}a_{1i})x_i;$$

(10) $$\varphi(x_3, \ldots, x_k) = \sum_{3\leqslant i,j\leqslant k} [(a_{11}a_{22} - a_{12}^2)(a_{11}a_{ij} - a_{1i}a_{1j}) - (a_{11}a_{2i} - a_{12}a_{1i})(a_{11}a_{2j} - a_{12}a_{1j})]x_ix_j$$

— the positive quadratic form with integer coefficients;

$$d_1 = a_{11} > 0, \quad d_2 = a_{11}a_{22} - a_{12}^2 > 0.$$

Now, let $x_3^0, \ldots, x_k^0$ be fixed. One can obtain different solutions of the equation (8) from (9) by using different solutions of the equation

(11) $$d_2y_1^2 + y_2^2 = d_1d_2n - \varphi(x_3^0, \ldots, x_k^0)$$

in integers y_1, y_2. It is known that the number of solutions of the equation (11) does not exceed

$$\gamma_{1,2}\big(d_1d_2n - \varphi(x_3^0, \ldots, x_k^0)\big)^\varepsilon \leqslant \gamma_{1,2}(d_1d_2)^\varepsilon n^\varepsilon \leqslant \gamma(f, \varepsilon)n^\varepsilon = \gamma n^\varepsilon$$

since φ is the positive quadratic form.

Lemma 3. *We have*

(12) $$R(f, n) = \sum_{t\leqslant c^{1/2}n^{1/4}} \mu(\boldsymbol{t})N^*(f_{\boldsymbol{t}}, n).$$

Proof. Since for $x \neq 0$

$$\mu^2(x) = \sum_{t^2|x} \mu(t) = \begin{cases} 1, & x \text{ is a square-free integer,} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$R(f, n) = \sum_{\substack{x\in Z^k \\ f(x)=n}} \mu^2(x) = \sum_{\substack{x\in(Z^*)^k \\ f(x)=n}} \prod_{j=1}^{k} \sum_{t_j^2|x_j} \mu(t_j) = \sum_{\substack{t<\infty \\ x\in(Z^*)^k \\ f_t(x)=n}} \mu(\boldsymbol{t}) = \sum_{t<\infty} \mu(\boldsymbol{t})N^*(f_{\boldsymbol{t}}, n).$$

If we have $t_j > c^{1/2}n^{1/4}$ for some j then $N^*(f_{\boldsymbol{t}}, n) = 0$ by corollary to Lemma 1. This completes the proof.

Lemma 4. *Let a be any positive number. Then*

$$\sum_{\substack{t\leqslant c^{1/2}n^{1/4} \\ \max_j t_j > n^a}} \mu(\boldsymbol{t})N^*(f_{\boldsymbol{t}}, n) \ll n^{k/2-1-a+\varepsilon}.$$

Proof. We have

$$\sum_{\substack{t\leqslant c^{1/2}n^{1/4} \\ \max_j t_j > n^a}} \mu(\boldsymbol{t})N^*(f_{\boldsymbol{t}}, n) \ll \sum_{j=1}^{k} \sum_{\substack{t\leqslant c^{1/2}n^{1/4} \\ t_j > n^a}} N^*(f_{\boldsymbol{t}}, n).$$

It is sufficient to estimate

$$\sum_{\substack{t\leqslant c^{1/2}n^{1/4} \\ t_k > n^a}} N^*(f_{\boldsymbol{t}}, n) = \sum_{t_k > n^a} \sum_{t_1,\ldots,t_{k-1}\leqslant c^{1/2}n^{1/4}} N^*(f_{\boldsymbol{t}}, n).$$

Now we fix some $t_k > n^a$. Let

$$(x_{i1}, x_{i2}, \ldots, x_{ik}) \quad (i = 1, \ldots, l)$$

be all the solutions of the equation

(13) $$f(x_1, \ldots, x_{k-1}, t_k^2x_k) = n$$

in non-zero integers $x_1, \ldots, x_{k-1}, x_k$. We write $x_{ij} = y_{ij}^2z_{ij}$ $(i = 1, \ldots, l;$ $j = 1, \ldots, k)$, where z_{ij} are square-free. For given t_k one can obtain all non-zero solutions of all equations of the type (5) with the fixed value of t_k from non-zero solutions of the equation (13), and from the solution $(x_{i1}, \ldots, x_{ik})$ we can obtain $\tau(y_{i1})\ldots\tau(y_{i,k-1})$ solutions of equations of type (5) with the fixed value of t_k (here $\tau(m)$ is the number of divisors of m). Thus for the fixed value of t_k

(14) $$\sum_{t_1,\ldots,t_{k-1}\leqslant c^{1/2}n^{1/4}} N^*(f_{t_1,\ldots,t_k}, n) = \sum_{i=1}^{l} \tau(y_{i1})\ldots\tau(y_{i,k-1}) \ll -n^\varepsilon l = n^\varepsilon N^*(f_{1,\ldots,1,t_k}, n).$$

For fixed values of $x_3, \ldots, x_k, t_k$ the number of solutions of the equation (13) does not exceed γn^ε (Lemma 2) and we may fix each of $x_3, \ldots$ $\ldots, x_{k-1}$ by $2cn^{1/2}$ manners (Lemma 1) and x_k by $2cn^{1/2}t_k^{-2}$ manners, hence

$$N^*(f_{1,\ldots,1,t_k}, n) \leqslant (2c)^{k-2}\gamma n^{k/2-1+\varepsilon}t_k^{-2} \ll t_k^{-2}n^{k/2-1+\varepsilon}.$$

Thus,

$$\sum_{t_k>n^a} \sum_{t_1,\ldots,t_{k-1}\leqslant c^{1/2}n^{1/4}} N^*(f_{t_1,\ldots,t_k}, n) \ll n^\varepsilon \sum_{t_k>n^a} N^*(f_{1,\ldots,1,t_k}, n) \ll n^{k/2-1+2\varepsilon} \sum_{t_k>n^a} t_k^{-2} \ll n^{k/2-1-a+2\varepsilon}.$$

The lemma is therefore proved.

Lemma 5. *Let $N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n)$ be the number of solutions of the equation*

$$f(t_1^2x_1, \ldots, t_r^2x_r, 0, \ldots, 0) = n$$

in integers $x_1, \ldots, x_r$. *Then*

$$\sum_{t_1,\ldots,t_k\leqslant n^a} N^*(f_{t_1;0,\ldots,0}, n) \ll n^{(k-1)a+\varepsilon}$$

and for $r = 2, \ldots, k-1$

$$\sum_{t_1,\ldots,t_k\leqslant n^a} N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n) \ll n^{(r-2)/2+(k-r)a+\varepsilon}.$$

Proof. We have for $1\leqslant r\leqslant k-1$

$$\sum_{t_1,\ldots,t_k\leqslant n^a} N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n) \leqslant n^{(k-r)a} \sum_{t_1,\ldots,t_r\leqslant n^a} N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n).$$

Let

$$(x_{i1}, \ldots, x_{ir}) \quad (i = 1, \ldots, l_r)$$

be all solutions of the equation

$$f(x_1, \ldots, x_r, 0, \ldots, 0) = n \tag{15}$$

in non-zero integers $x_1, \ldots, x_r$. Putting

$$x_{ij} = y_{ij}^2 z_{ij} \quad (i = 1, \ldots, l_r; j = 1, \ldots, r)$$

we have, as in the proof of Lemma 4, for $1\leqslant r\leqslant k-1$

$$\sum_{t_1,\ldots,t_r\leqslant n^a} N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n) \leqslant \sum_{i=1}^{l_r} \tau(y_{i1})\ldots\tau(y_{ir}) \ll n^\varepsilon l_r = n^\varepsilon N^*(f_{1,\ldots,1;0,\ldots,0}, n)$$

$$\ll \begin{cases} n^{2\varepsilon} & \text{for } r = 1, \\ n^{(r-2)/2+2\varepsilon} & \text{for } 2\leqslant r\leqslant k-1 \end{cases}$$

and the lemma is proved.

In particular, for $a = \dfrac{k-3}{4(k+1)}$ and $1\leqslant r\leqslant k-1$

$$\sum_{t_1,\ldots,t_k\leqslant n^a} N^*(f_{t_1,\ldots,t_r;0,\ldots,0}, n) \ll n^{k/2-1-a+\varepsilon}.$$

3. Estimations of exponential sums. Let A be the matrix of the quadratic form f. Minors of the matrix A which have the same diagonal, we shall call principal minors. Let $\mathcal{M}_f$ be the set of principal minors of the matrix A and

$$P_f = \{p\text{—prime}: p\mid m, m\in 2^{k+1}\mathcal{M}_f\}.$$

The matrix A has only non-vanishing principal minors (f is a positive quadratic form) and card $\mathcal{M}_f = 2^k-1$, therefore the set P_f is finite. In a particular case when f is a diagonal form the set P_f consists of all prime divisors of $2\det A$.

Lemma 6. *Let p be a prime, $p\notin P_f$, an integer $\nu\geqslant 1$, $\varepsilon_1, \ldots, \varepsilon_k$ be equal to 0 or 1. Then there exists a form φ, which is equivalent* mod p^ν *to the form f and satisfies the following two conditions:*

(1) *φ is a diagonal form;*

(2) *two forms* $f_{p^{\varepsilon_1},\ldots,p^{\varepsilon_k}}$ and $\varphi_{p^{\varepsilon_1},\ldots,p^{\varepsilon_k}}$ *of the type* (4) *are equivalent* mod p^ν.

Proof. Without loss of generality we may suppose that $\varepsilon_1 = \ldots = \varepsilon_r = 0$, $\varepsilon_{r+1} = \ldots = \varepsilon_k = 1$ for some $0\leqslant r\leqslant k$. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$$

where A_{11} is an $r\times r$ matrix. Then the quadratic form $f_{1,\ldots,1,p,\ldots,p}$ has the matrix

$$\begin{pmatrix} A_{11} & p^2A_{12} \\ p^2A_{21} & p^4A_{22}\end{pmatrix} = \begin{pmatrix} E_r & 0 \\ 0 & p^2E_{k-r}\end{pmatrix} A \begin{pmatrix} E_r & 0 \\ 0 & p^2E_{k-r}\end{pmatrix}$$

where E_t is the unit $t\times t$ matrix.

If $p\notin P_f$, then in $\boldsymbol{Z}(p^\nu)$ there is a triangular unimodular substitution $S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22}\end{pmatrix}$ which transforms the form f to a diagonal form $\varphi = a_1y_1^2 + \ldots + a_ky_k^2$. We may take the substitution S obtained by the Jacobi method of a reduction of the quadratic form to a diagonal form, since denominators of coefficients in the Jacobi formula, being some principal minors of A, are inversible mod p^ν. We have $p\nmid a_i$ $(i = 1, \ldots, k)$ and

$$S^TAS \equiv \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k\end{pmatrix} \pmod{p^\nu}.$$

Then the unimodular substitution

$$S_{1,\ldots,1,p,\ldots,p} = \begin{pmatrix} S_{11} & p^2S_{12} \\ 0 & S_{22}\end{pmatrix}, \quad \det S_{1,\ldots,1,p,\ldots,p} = \det S$$

transforms the form $f_{1,\ldots,1,p,\ldots,p}$ into the diagonal form

$$\varphi_{1,\ldots,1,p,\ldots,p}(y_1, \ldots, y_k) = a_1y_1^2 + \ldots + a_ry_r^2 + a_{r+1}p^4y_{r+1}^2 + \ldots + a_kp^4y_k^2. \tag{16}$$

The lemma is therefore proved.

For two vectors $\boldsymbol{l} = (l_1, \ldots, l_k)$ and $x = (x_1, \ldots, x_k)$ we put $(\boldsymbol{l}, x) = l_1x_1 + \ldots + l_kx_k$. Then

$$S(hf, q) = \sum_{x\in \boldsymbol{Z}^k(q)} e\big(hf(x)/q\big),$$

$$S(hf, \boldsymbol{l}, q) = \sum_{x\in \boldsymbol{Z}^k(q)} e\Big(\big(hf(x) + (\boldsymbol{l}, x)\big)/q\Big)$$

are homogeneous and non-homogeneous Gauss' sums respectively. We write

$$(a, q)_f = \prod_{p \in P_f} (a, p^{\nu_p(q)}) = \Big(a, \prod_{p \in P_f} p^{\nu_p(q)}\Big),$$

where $p^{\nu_p(q)} \| q$ (the greatest power of p, which divides q) and $(a, b) = \text{g.c.d.}(a, b)$.

Lemma 7. *Let $t_1, \ldots, t_k$ be square-free integers; $l_1, \ldots, l_k, u$ be integers; n, q be positive integers. Then*

$$(17)\qquad \sum_{h \,(\mathrm{mod}\, q)}{}' S(hf_{\boldsymbol{t}}, \boldsymbol{l}, q)\, e\big((-nh + uh^{-1(q)})/q\big) \ll q^{(k+1)/2+\varepsilon}(n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2}$$

where $h^{-1(q)}$ denotes h' such that $h'h \equiv 1 \pmod q$, and the constant implied in $\ll$ depends only on f and ε, and does not depend on $q, t_1, \ldots, t_k, l_1, \ldots, l_k, n, u$.

If $(t_j^2, q)_f \nmid l_j$ for some j, $1 \leqslant j \leqslant k$, then

$$S(hf_{\boldsymbol{t}}, \boldsymbol{l}, q) = 0.$$

Proof. We have (see [10], p. 17) for $Q_p = q/p^{\nu_p(q)}$

$$(18)\qquad S(hf_{\boldsymbol{t}}, \boldsymbol{l}, q) = \prod_{p | q} S(hQ_p f_{\boldsymbol{t}}, \boldsymbol{l}, p^{\nu_p(q)}).$$

Consider each factor separately. We put $\nu_p(q) = \nu$ for brevity.

(1) $p \notin P_f$. Since $t_1, \ldots, t_k$ are square-free integers then there are numbers $\varepsilon_j = 0$ or 1 such that $p^{\varepsilon_j} \| t_j$, $t_j = p^{\varepsilon_j} t'_j$ $(j = 1, \ldots, k)$. We have

$$(19)\qquad f_{t_1, \ldots, t_k}(x_1, \ldots, x_k) = f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}(t_1'^2 x_1, \ldots, t_k'^2 x_k),$$
$$l'_j = l_j (t_j'^2)^{-1(p^\nu)}, \qquad \nu_p(l'_j) = \nu_p(l_j) \qquad (j = 1, \ldots, k).$$

If x runs over a complete residue system mod p^ν and $p \nmid t'$ then $t'^2 x$ also runs over a complete residue system mod p^ν. Therefore

$$(20)\qquad S(hQ_p f_{\boldsymbol{t}}, \boldsymbol{l}, p^\nu) = S(hQ_p f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}, \boldsymbol{l}', p^\nu).$$

By Lemma 6 f is equivalent mod p^ν to a diagonal form φ such that $f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$ is mod p^ν equivalent to the diagonal form $\varphi_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$. Without loss of generality we suppose that $\varepsilon_1 = \ldots = \varepsilon_r = 0$, $\varepsilon_{r+1} = \ldots = \varepsilon_k = 1$ for some r, $0 \leqslant r \leqslant k$. Then the substitution $S_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$, which transforms $f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$ to the form $\varphi_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$, is triangular and mod p^ν inversible. Let $S_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$ transform $\boldsymbol{l}'$ to $\boldsymbol{l}''$. We write

$$S_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}} = \begin{pmatrix} S_{11} & p^2 S_{12} \\ 0 & S_{22} \end{pmatrix}, \qquad \boldsymbol{l}' = (\boldsymbol{l}'_r, \boldsymbol{l}'_{k-r}), \qquad \boldsymbol{l}'' = (\boldsymbol{l}''_r, \boldsymbol{l}''_{k-r})$$

then

$$(21)\qquad (\boldsymbol{l}'_r, \boldsymbol{l}'_{k-r}) = (\boldsymbol{l}''_r, \boldsymbol{l}''_{k-r}) \begin{pmatrix} S_{11}^{-1(p^\nu)} & p^2 S_{11}^{-1(p^\nu)} S_{12} S_{22}^{-1(p^\nu)} \\ 0 & S_{22}^{-1(p^\nu)} \end{pmatrix}$$
$$= (\boldsymbol{l}''_r S_{11}^{-1(p^\nu)}, \boldsymbol{l}''_{k-r} S_{22}^{-1(p^\nu)} - p^2 \boldsymbol{l}''_r S_{11}^{-1(p^\nu)} S_{12} S_{22}^{-1(p^\nu)}).$$

We have by [10], p. 17, (16), (18) and (20)

$$(22)\qquad S(hQ_p f_{\boldsymbol{t}}, \boldsymbol{l}, p^\nu) = \prod_{j=1}^{k} S(hQ_p a_j p^{4\varepsilon_j}, l''_j, p^\nu).$$

It is known that if $(p^{4\varepsilon_j}, p^\nu) \nmid l''_j$ then $S(hQ_p a_j p^{4\varepsilon_j}, l''_j, p^\nu) = 0$. Therefore, and by (21), if for some j we have $(p^{2\varepsilon_j}, p^\nu) \nmid l_j$ then $S(hQ_p f_{\boldsymbol{t}}, \boldsymbol{l}, p^\nu) = 0$. Thus the second statement of the lemma is proved.

Putting $p^{\nu_j} = (p^{4\varepsilon_j}, p^\nu)$, $\nu'_j = \nu - \nu_j$, $p^{\nu_j} l'''_j = l''_j$ $(j = 1, \ldots, k)$, we have

$$S(hQ_p a_j p^{4\varepsilon_j}, l''_j, p^\nu) = p^{\nu_j} S(hQ_p a_j, l'''_j, p^{\nu'_j})$$

by [10], p. 17. Hence by [10], p. 20

$$S(hQ_p a_j, l'''_j, p^{\nu'_j}) = \left(\frac{hQ_p a_j}{p^{\nu'_j}}\right) i^{\left(\frac{p^{\nu'_j}-1}{2}\right)^2} p^{\frac12 \nu'_j} e\big(-(hQ_p a_j)^{-1(p^{\nu'_j})} c_j^2 / p^{\nu'_j}\big)$$
$$= \left(\frac{h}{p^{\nu'_j}}\right) p^{\frac12 \nu'_j} \xi_{j,p}\, e(-h^{-1(q)} \zeta_{j,p}/p)$$

where

$$c_j = \begin{cases} \frac12 l'''_j, & \text{if} \quad 2 | l'''_j, \\ \frac12 (l'''_j + p^{\nu'_j}), & \text{if} \quad 2 \nmid l'''_j, \end{cases}$$

$$\xi_{j,p} = \left(\frac{Q_p a_j}{p^{\nu'_j}}\right) i^{\left(\frac{p^{\nu'_j}-1}{2}\right)^2}$$

does not depend on h and $|\xi_{j,p}| = 1$,

$$\zeta_{j,p} = (Q_p a_j)^{-1(p^{\nu'_j})} c_j^2 q / p^{\nu'_j}$$

is integer. Hence by (22)

$$(23)\qquad S(hQ_p f_{\boldsymbol{t}}, \boldsymbol{l}, p^\nu) = \gamma_p \left(\frac{h}{p^{s_p}}\right) p^{\frac12 k\nu + \frac12 \sum_j \nu_j} \xi_p\, e(-h^{-1(q)} \zeta_p / q),$$

where

$$\xi_p = \prod_{j=1}^{k} \xi_{j,p}, \quad |\xi_p| = 1, \quad s_p = \sum_{j=1}^{k} \nu_j',$$

$$\zeta_p = \sum_{j=1}^{k} \zeta_{j,p}, \quad \gamma_p = 0 \text{ or } 1$$

are numbers which are independent of h.

(2) $p \in P_f \setminus \{2\}$. There is a diagonal form φ, which is mod p^ν equivalent to $f_{p^{s_1},\ldots,p^{s_k}}$ and

$$\varphi = \sum_{j=1}^{k} a_j p^{e_j} x_j^2$$

where

$$\sum_{j=1}^{k} e_j \leqslant \nu_p\,(\det f_{p^{s_1},\ldots,p^{s_k}}) \leqslant \nu_p(\det A) + 4k.$$

We have

$$(24) \qquad S(hQ_p f_t, \boldsymbol{l}, p^\nu) = \gamma_p \left(\frac{h}{p^{s_p}}\right) p^{\frac{1}{2}k\nu + \frac{1}{2}\sum_j e_j} \xi_p e(-h^{-1(q)} \zeta_p/q)$$

in the same way as we have used in the case (1).

(3) $p = 2$. In this case the form $f_{2^{s_1},\ldots,2^{s_k}}$ is mod 2^ν equivalent to a form $\varphi = \sum_{m=1}^{r} 2^{e_m}\varphi_m$, where variables of forms φ_m are not overlapping and φ_m have one of two forms,

$$(25) \qquad \varphi_m = \sum_{m_1=1}^{k_m} a_{mm_1} x_{mm_1}^2$$

or

$$(26) \qquad \varphi_m = \sum_{m_1=1}^{k_m/2} (2a'_{mm_1} x_{mm_1}^2 + 2a''_{mm_1} x_{mm_1} y_{mm_1} + 2a'''_{mm_1} y_{mm_1}^2)$$

and

$$-1 \leqslant e_1 < e_2 < \ldots < e_r < \nu, \quad d_m = \det\varphi_m, \quad 2 \nmid d_m.$$

We have

$$S(hQ_2 f_t, \boldsymbol{l}, 2^\nu) = S(hQ_2 f_{2^{s_1},\ldots,2^{s_k}}, \boldsymbol{l}', 2^\nu) = \prod_{m=1}^{r} S(hQ_2\varphi_m 2^{e_m}, \boldsymbol{l}''_m, 2^\nu) 2^{\nu(k - \sum_m k_m)}$$

where $\boldsymbol{l}''_m$ is the part of $\boldsymbol{l}''$ which have the same variables as φ_m. If 2^{e_m} does not divide some coordinate of $\boldsymbol{l}''_m$, then

$$S(hQ_2\varphi_m 2^{e_m}, \boldsymbol{l}''_m, 2^\nu) = S(hQ_2 f_t, \boldsymbol{l}, 2^\nu) = 0.$$

Otherwise, for $e_m \neq -1$

$$S(hQ_2\varphi_m 2^{e_m}, \boldsymbol{l}''_m, 2^\nu) = 2^{k_m e_m} S(hQ_2\varphi_m, \boldsymbol{l}'''_m, 2^{\nu - e_m}).$$

If 2 divides each coordinate of $\boldsymbol{l}'''_m$, then see [10], p. 29

$$S(hQ_2\varphi_m, \boldsymbol{l}'''_m, 2^{\nu-e_m}) = S(hQ_2\varphi_m, 2^{\nu-e_m}) e(-h^{-1(q)} \zeta_2^{(m)}/q),$$

$$(27) \qquad S(hQ_2\varphi_m, 2^{\nu-e_m})$$
$$= \gamma(\nu - e_m)(-1)_*^{\frac{h-1}{2}(-1)^{\frac{1}{2}k_m(k_m+1)}\frac{1}{2}(d_m-1)} (-1)^{\frac{h^2-1}{8}k_m(\nu-e_m)} i^{\left(\frac{h-1}{2}\right)^2 k_m^2} \xi_2^{(m)} 2^{\frac{1}{2}k_m(\nu+1)}$$

where $\gamma(\nu - e_m) = 0$, if $\nu = e_m + 1$ and φ_m is of the type (25), otherwise $\gamma(\nu - e_m) = 1$; $*$ denotes omission of the multiplier for φ_m which is of the type (26); $\xi_2^{(m)}$ is independent of h and $|\xi_2^{(m)}| = 1$. If $2 \nmid \boldsymbol{l}'''_m$ and φ_m is of the type (26) then $S(hQ_2\varphi_m, \boldsymbol{l}'''_m, 2^{\nu-e_m}) = 0$. If $2 \nmid \boldsymbol{l}'''_m$ and φ_m is of the type (25) then

$$S(hQ_2\varphi_m, \boldsymbol{l}'''_m, 2^{\nu-e_m}) = \prod_{m_1=1}^{k_m} S(hQ_2 a_{mm_1}, l'''_{mm_1}, 2^{\nu-e_m}).$$

We have

$$S(a, l, 2^\nu) = \begin{cases} S(a, 2^\nu) e\left(-\left(\frac{l}{2}\right)^2 a^{-1(2^\nu)}/2^\nu\right) & \text{if } 2 \mid l, \\ 0 & \text{if } 2 \nmid l,\ \nu \geqslant 2, \\ S(a, 2^\nu) & \text{if } 2 \nmid l,\ \nu = 1, \end{cases}$$

$$S(a, 2^\nu) = \gamma(\nu)(-1)^{\nu\left(\frac{a^2-1}{8}\right)} (1 + i^a) 2^{\nu/2},$$

where $\gamma(1) = 0$ and $\gamma(\nu) = 1$ for $\nu \geqslant 2$. Hence formulae (27) hold in this case, as in the case when $e_m = -1$. Thus formulae (27) are true in all cases.

(4) Let $L \mid q$, $(j, L) = 1$ and every prime p which divides Q divides q. Putting

$$K_Q(n, u, j, L, q) = \sum_{\substack{h (\operatorname{mod} q) \\ h \equiv j (\operatorname{mod} L)}}' \left(\frac{h}{Q}\right) e\left(\frac{-nh + uh^{-1(q)}}{q}\right)$$

we have by [10], p. 51

$$|K_Q(n, u, j, L, q)| \leqslant \varkappa_\varepsilon q^{\frac{1}{2}+\varepsilon} (n, q)^{1/2}$$

where $\varkappa_\varepsilon > 0$ depends only on ε.

(5) Putting $2^\nu \| q$ and

$$\sigma = \sum_{h (\operatorname{mod} q)}' S(hf_t, \boldsymbol{l}, q) e\left(\frac{-nh + uh^{-1(q)}}{q}\right)$$

we have four cases:

(a) $\nu = 0$. Then by (18), (23) and (24)

$$\sigma = \prod_{\substack{p\in P_f \\ p|q}} \{\xi_p p^{\frac{k}{2}\nu_p(q)+\frac{1}{2}\sum_j e_j(p)}\} \prod_{\substack{p\notin P_f \\ p|q}} \Big\{\prod_{j=1}^{k} (p^{4\nu_p(t_j)}, q)^{1/2} \xi_p p^{\frac{k}{2}\nu_p(q)}\Big\} \times$$

$$\times \sum_{h (\operatorname{mod} q)}{}' \left(\frac{h}{\prod_{p|q} p^{s_p}}\right) e\left(\frac{-nh+u_1 h^{-1(q)}}{q}\right).$$

Here we put

$$u_1 = u - \sum_{p|q} \zeta_p, \qquad Q = \prod_{p|q} p^{s_p}.$$

Hence,

$$|\sigma| \leqslant |\det A|^{1/2} \prod_{p\in P_f} \{p^{2k}\} q^{k/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2} |K_Q(n, u, 1, 1, q)|$$

$$\ll q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2}.$$

Henceforth we shall write $\varphi(e_m)$ instead of φ_m and $k(e_m)$ instead of k_m. Everywhere we have: $\varphi(-1)$ is of the type (26), $k(-1)$ is even, $\varphi(\nu-1)$ is a diagonal form. If $\varphi(\nu-1) \neq 0$ then $\sigma = 0$, therefore we may suppose that $\varphi(\nu-1) = 0$.

(b) $\nu = 1$, $e_m = -1$ or 0. In the first case

$$S(hQ_2 f_{2^{\varepsilon_1},\ldots,2^{\varepsilon_k}}, \boldsymbol{l}, 2) = \xi_2' 2^{k+\frac{1}{2}e_1} e\left(-\frac{h^{-1(q)}\zeta_2}{q}\right)$$

where ξ_2' does not depend on h. We omit the second case because $\nu-1 = 0$.

$$|\sigma| \leqslant 2^{k/2} c_f q^{k/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2} |K_s(n, u_1, 1, 1, q)| \ll q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)^{1/2}$$

where

$$c_f = |\det A|^{1/2} \prod_{p\in P_f} p^{2k}.$$

(c) $\nu = 2$, $e_m = -1, 0$, or 1. We have

$$\sigma = \sum_{h (\operatorname{mod} q)}{}' = \sum_{\substack{h (\operatorname{mod} q) \\ h\equiv 1 (\operatorname{mod} 4)}}{}' + \sum_{\substack{h (\operatorname{mod} q) \\ h\equiv 3 (\operatorname{mod} 4)}}{}' .$$

If $h \equiv h_1 (\operatorname{mod} 4)$ then

$$\frac{h-1}{2} \equiv \frac{h_1-1}{2} (\operatorname{mod} 2), \qquad \left(\frac{h-1}{2}\right)^2 \equiv \left(\frac{h_1-1}{2}\right)^2 (\operatorname{mod} 4).$$

In formula (27) $\sum_m k_m(2-e_m)$ is the even number, therefore

$$|\sigma| \leqslant 2^{k/2} c_f q^{k/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2} \sum_{h_1=1,3} |K_Q(n, u_1, h_1, 4, q)|$$

$$\ll q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2}.$$

(d) $\nu \geqslant 3$. If $h \equiv h_1 (\operatorname{mod} 8)$ then

$$\frac{h-1}{2} \equiv \frac{h_1-1}{2} (\operatorname{mod} 2), \qquad \frac{h^2-1}{8} \equiv \frac{h_1^2-1}{8} (\operatorname{mod} 2),$$

$$\left(\frac{h-1}{2}\right)^2 \equiv \left(\frac{h_1-1}{2}\right)^2 (\operatorname{mod} 4)$$

and

$$\sigma = \sum_{h (\operatorname{mod} q)}{}' = \sum_{h_1=1,3,5,7} \sum_{\substack{h (\operatorname{mod} q) \\ h\equiv h_1 (\operatorname{mod} 8)}}{}'$$

hence

$$|\sigma| \leqslant 2^{k/2} c_f q^{k/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2} \sum_{h_1=1,3,5,7} |K_Q(n, u_1, h_1, 8, q)|$$

$$\ll q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2}.$$

The lemma is therefore proved.

4. The main term of the asymptotic formula for $R(f, n)$. We need to find an asymptotic value of $\sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N(f_{\boldsymbol{t}}, n)$. For $\boldsymbol{t} \leqslant n^{\alpha}$ and for a complex number w, $|w| < 1$, we put

$$F_{\boldsymbol{t}}(w) = \sum_{x\in Z^k} w^{f_{\boldsymbol{t}}(x)} = \sum_{n=0}^{\infty} N(f_{\boldsymbol{t}}, n) w^n.$$

Then

$$N(f_{\boldsymbol{t}}, n) = \frac{1}{2\pi i} \int_{\Gamma} F_{\boldsymbol{t}}(w) w^{-n-1} dw,$$

where

$$\Gamma = \{w\colon |w| = e^{-1/n}\}.$$

Putting

$$n_0 = [n^{1/2}], \qquad I_{n_0} = \left[-\frac{1}{1+n_0}, 1-\frac{1}{1+n_0}\right]$$

we have

$$N(f_{\boldsymbol{t}}, n) = \int_{I_{n_0}} F_{\boldsymbol{t}}(e^{-1/n+2\pi iu}) e^{1-2\pi inu} du.$$

We make a Farey dissection of the order n_0 of the interval I_{n_0} and we put

$$\gamma_{h,q} = \left(-\frac{1}{qq_2}, \frac{1}{qq_1}\right)$$

where q_1 and q_2 are denominators of adjacent Farey fractions to h/q. Then

$$N(f_{\boldsymbol{t}}, n) = \sum_{q \leqslant n_0} \sideset{}{'}\sum_{h(\operatorname{mod} q)} e\left(-\frac{nh}{q}\right) \int_{\gamma_{h,q}} F_{\boldsymbol{t}}(e^{-\frac{1}{n}+2\pi i\left(\frac{h}{q}+\theta\right)}) e^{1-2\pi in\theta} d\theta.$$

Putting

$$v = \frac{1}{n} - 2\pi i\theta, \quad w = e^{-v+2\pi i\frac{h}{q}}$$

we have

$$F_{\boldsymbol{t}}(w) = \sum_{\boldsymbol{x}\in\mathbf{Z}^k} e^{-\left(v-2\pi i\frac{h}{q}\right)f_{\boldsymbol{t}}(\boldsymbol{x})} = \sum_{\boldsymbol{r}\in\mathbf{Z}^k(q)} \sum_{\boldsymbol{y}\in\mathbf{Z}^k} e^{-\left(v-2\pi i\frac{h}{q}\right)f_{\boldsymbol{t}}(q\boldsymbol{y}+\boldsymbol{r})}$$
$$= \sum_{\boldsymbol{r}\in\mathbf{Z}^k(q)} e\left(\frac{h}{q} f_{\boldsymbol{t}}(\boldsymbol{r})\right) \sum_{\boldsymbol{y}\in\mathbf{Z}^k} e^{-vq^2 f_{\boldsymbol{t}}\left(\boldsymbol{y}+\frac{\boldsymbol{r}}{q}\right)}.$$

LEMMA 8. *Let $f^{-1}(\boldsymbol{x})$ be the quadratic form with a matrix which is inverse to the matrix of the quadratic form f; $\delta, \xi_1, \ldots, \xi_k$ are complex numbers and $\operatorname{Re}\delta > 0$. Then*

$$\sum_{\boldsymbol{y}\in\mathbf{Z}^k} e^{-\delta\pi f(\boldsymbol{y}+\boldsymbol{\xi})} = \frac{1}{\delta^{k/2} D^{1/2}} \sum_{\boldsymbol{l}\in\mathbf{Z}^k} e^{-\frac{\pi}{\delta} f^{-1}(\boldsymbol{l}) + 2\pi i(\boldsymbol{\xi}, \boldsymbol{l})}.$$

Proof, see, for example [10], p. 76.

Consequently, by Lemma 8

$$\sum_{\boldsymbol{y}\in\mathbf{Z}^k} e^{-vq^2 f_{\boldsymbol{t}}\left(\boldsymbol{y}+\frac{\boldsymbol{r}}{q}\right)} = \frac{\pi^{k/2} D^{-1/2} q^{-k}}{v^{k/2} \prod_{j=1}^{k} t_j^2} \sum_{\boldsymbol{l}\in\mathbf{Z}^k} e^{-\frac{\pi^2}{vq^2} f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l})} e\left(\left(\frac{\boldsymbol{r}}{q}, \boldsymbol{l}\right)\right),$$

hence, and by the definition of Gauss' sum,

$$F_{\boldsymbol{t}}(w) = \frac{\pi^{k/2} D^{-1/2} q^{-k}}{v^{k/2} \prod_{j=1}^{k} t_j^2} \sum_{\boldsymbol{l}\in\mathbf{Z}^k} e^{-\frac{\pi^2}{vq^2} f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l})} S(hf_{\boldsymbol{t}}, \boldsymbol{l}, q).$$

Further, by Lemma 7, if for some j $(1 \leqslant j \leqslant k)$ we have $b_j = (t_j^2, q)_f \nmid l_j$ then $S(hf_{\boldsymbol{t}}, \boldsymbol{l}, q) = 0$. Therefore

$$F_{\boldsymbol{t}}(w) = \frac{\pi^{k/2} D^{-1/2} q^{-k}}{v^{k/2} \prod_{j=1}^{k} t_j^2} \sum_{\boldsymbol{l}\in\mathbf{Z}^k} e^{-\frac{\pi^2}{vq^2} f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l}\boldsymbol{b})} S(hf_{\boldsymbol{t}}, \boldsymbol{l}\boldsymbol{b}, q).$$

If we put

$$\gamma_q^+ = \left(-\frac{1}{qn^{1/2}}, \frac{1}{qn^{1/2}}\right), \quad \gamma_q^- = \left(-\frac{1}{2qn^{1/2}}, \frac{1}{2qn^{1/2}}\right)$$

then it is known that

$$\gamma_q^- \subset \gamma_{h,q} \subset \gamma_q^+$$

then for

$$g(h, q, \theta) = \begin{cases} 1, & \text{if} \quad h/q + \theta \in \gamma_{h,q}, \\ 0 & \text{otherwise}, \end{cases}$$

$$A_{\boldsymbol{t}}(\theta, \boldsymbol{l}\boldsymbol{b}, q) = \sideset{}{'}\sum_{h(\operatorname{mod} q)} S(hf_{\boldsymbol{t}}, \boldsymbol{l}\boldsymbol{b}, q) e\left(-\frac{nh}{q}\right) g(h, q, \theta)$$

we have

$$N(f_{\boldsymbol{t}}, n) = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} \int_{\gamma_q^+} \sum_{\boldsymbol{l}\in\mathbf{Z}^k} e^{-\frac{\pi^2}{vq^2} f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l}\boldsymbol{b})} A_{\boldsymbol{t}}(\theta, \boldsymbol{l}\boldsymbol{b}, q) v^{-k/2} e^{nv} d\theta. \tag{28}$$

LEMMA 9. *For any $q \leqslant n_0$ and any θ there are numbers $c_1, \ldots, c_q$ such that*

$$\sum_{r=1}^{q} |c_r| \ll q^{\varepsilon}, \tag{29}$$

and for any h, $(h, q) = 1$

$$g(h, q, \theta) = \sum_{r=1}^{q} c_r e\left(\frac{rh^{-1(q)}}{q}\right).$$

Proof, see [2], p. 435.

COROLLARY. *It is uniformly in $\boldsymbol{l}\boldsymbol{b}$ and θ*

$$A_{\boldsymbol{t}}(\theta, \boldsymbol{l}\boldsymbol{b}, q) \ll q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \prod_{j=1}^{k} (t_j^4, q)_f^{1/2}. \tag{30}$$

Indeed, by Lemma 9

$$A_{\boldsymbol{t}}(\theta, \boldsymbol{lb}, q) = \sum_{r=1}^{q} c_r \sum_{h \,(\mathrm{mod}\, q)}' S(hf_{\boldsymbol{t}}, \boldsymbol{lb}, q)\, e\left(\frac{-nh + rh^{-1(q)}}{q}\right),$$

hence by (17) and (29) we obtain (30).

We put

$$N(f_{\boldsymbol{t}}, n) = N_{\boldsymbol{t}}^{(0)} + N_{\boldsymbol{t}}^{(1)},$$

where

$$N_{\boldsymbol{t}}^{(0)} = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} \int_{\gamma_q^+} A_{\boldsymbol{t}}(\theta, \boldsymbol{0}, q)\, v^{-k/2} e^{nv}\, d\theta, \tag{31}$$

$$N_{\boldsymbol{t}}^{(1)} = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} \int_{\gamma_q^+} \sum_{\boldsymbol{l} \in \boldsymbol{Z}^k \setminus \{\boldsymbol{0}\}} e^{-\frac{\pi^2}{vq^2} f^{-1}(\boldsymbol{t}^{-2} \boldsymbol{lb})} A_{\boldsymbol{t}}(\theta, \boldsymbol{lb}, q)\, v^{-k/2} e^{nv}\, d\theta. \tag{32}$$

LEMMA 10. *Let $k \geqslant 4$. Then*

$$\sum_{\boldsymbol{t} \leqslant n^{\alpha}} \mu(\boldsymbol{t}) N_{\boldsymbol{t}}^{(0)} = \frac{\pi^{k/2}}{D^{1/2}\Gamma(k/2)} G(f, n)\, n^{k/2-1} + O(n^{k/2-1-\alpha+\varepsilon}). \tag{33}$$

Proof. For $\theta \in \gamma_q^-$

$$A_{\boldsymbol{t}}(\theta, \boldsymbol{0}, q) = \sum_{h \,(\mathrm{mod}\, q)}' S(hf_{\boldsymbol{t}}, q)\, e\left(-\frac{nh}{q}\right) = A_{\boldsymbol{t}}(q).$$

Therefore, we may put

$$N_{\boldsymbol{t}}^{(0)} = M_{\boldsymbol{t}}^{(0)} + M_{\boldsymbol{t}}^{(1)} - M_{\boldsymbol{t}}^{(2)},$$

where

$$M_{\boldsymbol{t}}^{(0)} = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} A_{\boldsymbol{t}}(q) \int_{\boldsymbol{R}} v^{-k/2} e^{nv}\, d\theta,$$

$$M_{\boldsymbol{t}}^{(1)} = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} \int_{\gamma_q^+ \setminus \gamma_q^-} A_{\boldsymbol{t}}(\theta, \boldsymbol{0}, q)\, v^{-k/2} e^{nv}\, d\theta,$$

$$M_{\boldsymbol{t}}^{(2)} = \frac{\pi^{k/2} D^{-1/2}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} A_{\boldsymbol{t}}(q) \int_{\boldsymbol{R} \setminus \gamma_q^-} v^{-k/2} e^{nv}\, d\theta.$$

By (30)

$$\sum_{\boldsymbol{t} \leqslant n^{\alpha}} \mu(\boldsymbol{t})\, M_{\boldsymbol{t}}^{(1)} \ll \sum_{q \leqslant n_0} q^{-k} \sum_{\boldsymbol{t} \leqslant n^{\alpha}} \prod_{j=1}^{k} \frac{(t_j^4, q)_f^{1/2}}{t_j^2} q^{\frac{k+1}{2}+\varepsilon} (n, q)^{1/2} \int_{1/(2qn^{1/2})}^{1/(qn^{1/2})} \theta^{-k/2}\, d\theta$$

$$\ll n^{\frac{k}{4}-\frac{1}{2}} \sum_{q \leqslant n_0} q^{-1/2+\varepsilon} (n, q)^{1/2} \left(\sum_{t \leqslant n^{\alpha}} \frac{(t^4, q)_f^{1/2}}{t^2}\right)^{k}.$$

We have

$$\sum_{t \leqslant n^{\alpha}} \frac{(t^4, q)_f^{1/2}}{t^2} \leqslant \sum_{t \leqslant n^{\alpha}} \frac{(t, q)^2}{t^2} \leqslant \sum_{\delta | q} \sum_{t_1 \leqslant n^{\alpha}/\delta} \frac{\delta^2}{\delta^2 t_1^2} \ll \sum_{\delta | q} 1 \ll q^{\varepsilon}. \tag{34}$$

On the other hand

$$\sum_{q \leqslant n_0} q^{-1/2+\varepsilon} (n, q)^{1/2} \ll \sum_{\delta | n} \delta^{\varepsilon} \sum_{q_1 \leqslant n_0/\delta} q_1^{-1/2+\varepsilon} \ll n^{\varepsilon} n_0^{1/2} \ll n^{1/4+\varepsilon}. \tag{35}$$

Hence

$$\sum_{\boldsymbol{t} \leqslant n^{\alpha}} \mu(\boldsymbol{t})\, M_{\boldsymbol{t}}^{(1)} \ll n^{k/4-1/4+\varepsilon} \ll n^{k/2-1-\alpha+\varepsilon}.$$

Similarly

$$\sum_{\boldsymbol{t} \leqslant n^{\alpha}} \mu(\boldsymbol{t})\, M_{\boldsymbol{t}}^{(2)} \ll n^{k/4-1/4+\varepsilon} \ll n^{k/2-1-\alpha+\varepsilon}.$$

By Hankel's formula for the Γ-function

$$\int_{\boldsymbol{R}} e^{nv} v^{-k/2}\, d\theta = n^{k/2-1} \left\{\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} u^{-k/2} e^{u}\, du\right\} = \frac{n^{k/2-1}}{\Gamma(k/2)}.$$

Hence

$$M_{\boldsymbol{t}}^{(0)} = \frac{\pi^{k/2} D^{-1/2}}{\Gamma(k/2)} \frac{n^{k/2-1}}{\prod_{j=1}^{k} t_j^2} \sum_{q \leqslant n_0} q^{-k} A_{\boldsymbol{t}}(q) = M_{\boldsymbol{t}}^{(3)} - M_{\boldsymbol{t}}^{(4)},$$

where

$$M_{\boldsymbol{t}}^{(3)} = \frac{\pi^{k/2} D^{-1/2}}{\Gamma(k/2)} \frac{n^{k/2-1}}{\prod_{j=1}^{k} t_j^2} \sum_{q=1}^{\infty} q^{-k} A_{\boldsymbol{t}}(q),$$

$$M_{\boldsymbol{t}}^{(4)} = \frac{\pi^{k/2} D^{-1/2}}{\Gamma(k/2)} \frac{n^{k/2-1}}{\prod_{j=1}^{k} t_j^2} \sum_{q > n_0} q^{-k} A_{\boldsymbol{t}}(q).$$

By (30) and (34)

$$\sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(4)} \ll \sum_{q>n_0} q^{-k}\sum_{\boldsymbol{t}\leqslant n^a}\prod_{j=1}^{k}\frac{(t_j^4, q)_f^{1/2}}{t_j^2}\, q^{k/2+1/2+\varepsilon}(n, q)^{1/2} n^{k/2-1}$$

$$\ll n^{k/2-1}\sum_{q>n_0} q^{-k/2+1/2+(k+1)\varepsilon}(n, q)^{1/2}$$

where

$$\sum_{q>n_0} q^{k/2+1/2+\varepsilon}(n, q)^{1/2} \leqslant \sum_{\delta|n}\delta^{-k/2+1+\varepsilon}\sum_{q_1>n_0\delta^{-1}} q_1^{-k/2+1/2+\varepsilon} \ll n_0^{-k/2+3/2+\varepsilon} n^{\varepsilon}$$

$$\ll n^{-k/4+3/4+2\varepsilon}.$$

Hence,

$$\sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(4)} \ll n^{k/4-1/4+\varepsilon} \ll n^{k/2-1-a+\varepsilon}.$$

Furthermore,

$$\sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(3)} = \sum_{\boldsymbol{t}<\infty}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(3)} - \sum_{\substack{\boldsymbol{t}<\infty\\ \max_j t_j>n^a}}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(3)}.$$

We have

$$\Big|\sum_{\substack{\boldsymbol{t}<\infty\\ \max_j t_j>n^a}}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(3)}\Big| \leqslant \sum_{j=1}^{k}\sum_{\substack{\boldsymbol{t}<\infty\\ t_j>n^a}}|M_{\boldsymbol{t}}^{(3)}|.$$

For a fixed value of j we have by (30) and (34)

$$\sum_{\substack{\boldsymbol{t}<\infty\\ t_j>n^a}}|M_{\boldsymbol{t}}^{(3)}| \leqslant n^{k/2-1}\sum_{q=1}^{\infty} q^{-k/2+1/2+\varepsilon}(n, q)^{1/2}\sum_{t>n^a}\frac{(t^4, q)^{1/2}}{t^2}.$$

Let $\delta = (t^4, q)$ and δ^* be the least positive integer among integers δ_1 such that $(\delta\delta_1)^{1/4}$ is an integer. Then for some integer t_1 we have $t_1^4\delta\delta^* = t^4$. Therefore

$$\sum_{t>n^a}\frac{(t^4, q)^{1/2}}{t^2} \leqslant \sum_{\delta|q}\sum_{t_1>\frac{n^a}{(\delta\delta^*)^{1/4}}}\frac{\delta^{1/2}}{\delta^{1/2}\delta^{*1/2}t_1^2} = \sum_{\delta|q}\frac{1}{\delta^{*1/2}}\sum_{t_1>\frac{n^a}{(\delta\delta^*)^{1/4}}}\frac{1}{t_1^2}$$

$$\ll \sum_{\delta|q}\Big(\frac{\delta}{\delta^*}\Big)^{1/4} n^{-a} \ll q^{1/4+\varepsilon}n^{-a}.$$

Since for $k\geqslant 4$ it is $k/2-3/4-\varepsilon>1$ then

$$\sum_{q=1}^{\infty}\frac{(n, q)^{1/2}}{q^{k/2-3/4-\varepsilon}} \leqslant \sum_{\delta|n}\delta^{-\left(\frac{k}{2}-\frac{5}{4}-\varepsilon\right)}\sum_{q_1=1}^{\infty} q_1^{-\frac{k}{2}+\frac{3}{4}+\varepsilon} \ll n^{\varepsilon}.$$

Hence

$$\sum_{\substack{\boldsymbol{t}<\infty\\ \max_j t_j<n^a}}\mu(\boldsymbol{t})M_{\boldsymbol{t}}^{(3)} \ll n^{\frac{k}{2}-1-a+\varepsilon}.$$

The truth of the lemma now follows from the identity

$$\sum_{\boldsymbol{t}<\infty}\mu(\boldsymbol{t})\sum_{q=1}^{\infty} q^{-k}A_{\boldsymbol{t}}(q) = G(f, n).$$

5. An evaluation of $\sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})N_{\boldsymbol{t}}^{(1)}$. The following result is known.

LEMMA 11. *There is a positive constant $\varkappa = \varkappa(f)$ such that for any real numbers $x_1, \ldots, x_k$*

$$f^{-1}(x_1, \ldots, x_k) \geqslant \varkappa(x_1^2+\ldots+x_k^2).$$

LEMMA 12. *We have*

$$\sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})N_{\boldsymbol{t}}^{(1)} \ll n^{\frac{k}{2}-1-a+\varepsilon}.$$

Proof. Putting

$$\eta = 1+4\pi^2n^2\theta^2, \quad d_j = (t_j^4, q)_f^{1/2} \quad (j = 1, \ldots, k), \quad A = \frac{\varkappa\pi^2 n}{\eta q^2}$$

we have

$$(36)\qquad \sum_{\boldsymbol{t}\leqslant n^a}\mu(\boldsymbol{t})N_{\boldsymbol{t}}^{(1)}$$

$$\ll n^{k/2}\sum_{q\leqslant n_0} q^{-\frac{k}{2}+\frac{1}{2}+\varepsilon}(n, q)^{1/2}\int_0^{1/(qn^{1/2})}\sum_{\boldsymbol{t}\leqslant n^a}\prod_{j=1}^{k}\frac{d_j}{t_j^2}\sum_{\boldsymbol{l}\in\mathbf{Z}^k\setminus\{0\}} e^{-\frac{\pi^2 n}{\eta q^2}f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l}\boldsymbol{b})}\eta^{-k/4}d\theta$$

and by Lemma 11

$$\sum_{\boldsymbol{l}\in\mathbf{Z}^k\setminus\{0\}} e^{-\frac{\pi^2 n}{\eta q^2}f^{-1}(\boldsymbol{t}^{-2}\boldsymbol{l}\boldsymbol{b})} \leqslant \sum_{\boldsymbol{l}\in\mathbf{Z}^k\setminus\{0\}} e^{-A\sum_{j=1}^{k} t_j^{-4}b_j^2 l_j^2}.$$

Let

$$\sigma(l) = \sum_{t\leqslant n^\alpha} \frac{d}{t^2} e^{-Ab^2l^2t^{-4}}$$

where

$$d = (t^4, q)_f^{1/2}, \quad b = (t^2, q)_f, \quad d \leqslant b$$

then

$$(37) \qquad \sum_{t\leqslant n^\alpha} \prod_{j=1}^{k} \frac{d_j}{t_j^2} \sum_{l\in Z^k\setminus\{0\}} e^{-\frac{\pi^2 n}{\eta q^2} f^{-1}(t^{-2}lb)} \ll \Big\{\sum_{l=1}^{\infty} \sigma(l)\Big\}\Big\{\sum_{l=0}^{\infty}\sigma(l)\Big\}^{k-1}.$$

By (34) and by the inequality $d\leqslant b$

$$\sum_{l=0}^{\infty}\sigma(l) = \sum_{t\leqslant n^\alpha}\frac{d}{t^2}\sum_{l=0}^{\infty} e^{-Ab^2l^2t^{-4}} = \sum_{t\leqslant n^\alpha}\frac{d}{t^2} + \sum_{t\leqslant n^\alpha}\frac{d}{t^2}\sum_{l=1}^{\infty} e^{-Ab^2l^2t^{-4}}$$

$$\ll q^\varepsilon + \sum_{t\leqslant n^\alpha}\frac{d}{t^2}\int_0^\infty e^{-Ab^2l^2t^{-4}}dl \ll q^\varepsilon + \sum_{t\leqslant n^\alpha}\frac{d}{t^2}\frac{t^2}{b}A^{-1/2} \ll q^\varepsilon + A^{-1/2}n^\alpha.$$

Hence

$$\Big\{\sum_{l=0}^{\infty}\sigma(l)\Big\}^{k-1} \ll q^\varepsilon + A^{-\frac{k-1}{2}} n^{(k-1)\alpha}.$$

Similarly

$$\sum_{l=1}^{\infty}\sigma(l) = \sum_{t\leqslant n^\alpha}\frac{d}{t^2}\sum_{l=1}^{\infty} e^{-Ab^2l^2t^{-4}} = \sum_{t\leqslant n^\alpha}\frac{d}{t^2}e^{-Ab^2t^{-4}}\sum_{l=1}^{\infty} e^{-Ab^2t^{-4}(l^2-1)}$$

$$\leqslant \sum_{t\leqslant n^\alpha}\frac{d}{t^2}e^{-Ab^2t^{-4}}\sum_{l=0}^{\infty}e^{-Ab^2l^2t^{-4}} \ll \sum_{t\leqslant n^\alpha}\frac{d}{t^2}e^{-Ab^2t^{-4}}\Big(1+A^{-1/2}\frac{t^2}{b}\Big)$$

$$= \sum_{t\leqslant n^\alpha}\frac{d}{t^2}e^{-Ab^2t^{-4}} + A^{-1/2}\sum_{t\leqslant n^\alpha}e^{-Ab^2t^{-4}}.$$

Consequently, by (36) and (37),

$$\sum_{t\leqslant n^\alpha}\mu(t)N_t^{(1)} \ll n^{k/4}\sum_{q\leqslant n_0} q^{-\frac{k}{2}+\frac{1}{2}+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}\Bigg\{\int_0^{1/(qn^{1/2})}\Big(\frac{d}{t^2}q^\varepsilon + A^{-1/2}q^\varepsilon +$$

$$+ A^{-k/2}n^{(k-1)\alpha} + \frac{d}{t^2}A^{-\frac{k-1}{2}}n^{(k-1)\alpha}\Big)e^{-Ab^2t^{-4}}\eta^{-k/4}d\theta\Bigg\} = J_1+J_2+J_3+J_4.$$

It is known that for any positive numbers S and A

$$A^Se^{-A} \leqslant S^Se^{-S}.$$

Then, by (35),

$$J_1 \ll n^{k/4}\sum_{q\leqslant n_0} q^{1/2+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}\frac{d}{t^2}\frac{t^k}{b^{k/2}}\int_0^{1/(qn^{1/2})}\Big(\frac{Ab^2}{t^4}\Big)^{k/4}e^{-Ab^2/t^4}d\theta$$

$$\ll n^{k/4-1/2}\sum_{q\leqslant n_0} q^{-1/2+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}t^{k-2} \ll n^{k/4-1/4+\alpha(k-1)+\varepsilon} \ll n^{k/2-1-\alpha+\varepsilon},$$

$$J_2 \ll n^{k/4}\sum_{q\leqslant n_0} q^{1/2+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}t^{k-2}b^{-k/2+1}\int_0^{1/(qn^{1/2})}\Big(\frac{Ab^2}{t^4}\Big)e^{-Ab^2/t^4}d\theta$$

$$\ll n^{k/4-1/4+\alpha(k-1)+\varepsilon} \ll n^{k/2-1-\alpha+\varepsilon}.$$

We have for $0<\theta<\dfrac{1}{qn^{1/2}}$ and $q\leqslant n^{1/2}$

$$A = \frac{\varkappa\pi^2 n}{(1+4\pi^2n^2\theta^2)q^2} \geqslant \frac{\varkappa\pi^2 n}{q^2+4\pi^2 n} \geqslant \frac{\varkappa\pi^2}{1+4\pi^2}.$$

Hence

$$J_3 \ll n^{k/4}\sum_{q\leqslant n_0} q^{1/2+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}n^{(k-1)\alpha}\int_0^{1/(qn^{1/2})}A^{-k/4}e^{-Ab^2/t^4}d\theta$$

$$\ll n^{k/4-1/4+\alpha k+\varepsilon} \ll n^{k/2-1-\alpha+\varepsilon},$$

$$J_4 \ll n^{k/4+(k-1)\alpha}\sum_{q\leqslant n_0} q^{1/2+\varepsilon}(n,q)^{1/2}\sum_{t\leqslant n^\alpha}\frac{d}{t^2}\int_0^{1/(qn^{1/2})}A^{-(k/4-1/2)}e^{-Ab^2/t^4}d\theta \ll n^{k/2-1-\alpha+\varepsilon}.$$

The lemma is therefore proved.

6. A proof of Theorem 1. By Lemmas 1 and 2

$$R(f,n) = \sum_{t\leqslant c^{1/2}n^{1/4}}\mu(t)N^*(f_t,n) + \sum_{\substack{t\leqslant c^{1/2}n^{1/4}\\ \max_j t_j>n^\alpha}}\mu(t)N^*(f_t,n)$$

and by Lemma 3 the second term is $\ll n^{k/2-1-\alpha+\varepsilon}$. The first term we shall represent in the form

$$\sum_{t\leqslant n^\alpha}\mu(t)N^*(f_t,n) = \sum_{t\leqslant n^\alpha}\mu(t)N(f_t,n) - \sum_{j_1,\ldots,j_l}\sum_{t\leqslant n^\alpha}\mu(t)\sum_{\substack{x,f(x)=n\\ x_{j_1},\ldots,x_{j_l}=0\\ \text{others}\neq 0}}1$$

where the sum $\sum_{j_1,\ldots,j_l}$ is the sum on all sets $(j_1,\ldots,j_l)\subset(1,\ldots,k)$ $(l=1,\ldots$ $\ldots,k-1)$. By Lemma 4 each term of the sum (there are 2^k-2 such sums) is

$$\ll n^{k/2-1-\alpha+\varepsilon}.$$

Thus

$$R(f,n) = \sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N(f_{\boldsymbol{t}}, n) + O(n^{k/2-1-\alpha+\varepsilon})$$
$$= \sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N_{\boldsymbol{t}}^{(0)} + \sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N_{\boldsymbol{t}}^{(1)} + O(n^{k/2-1-\alpha+\varepsilon})$$

and by Lemma 12

$$\sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N_{\boldsymbol{t}}^{(1)} \ll n^{k/2-1-\alpha+\varepsilon}.$$

By Lemma 10 we have

$$\sum_{t\leqslant n^{\alpha}} \mu(\boldsymbol{t}) N_{\boldsymbol{t}}^{(0)} = \frac{\pi^{k/2} D^{-1/2}}{\Gamma(k/2)} G(f,n) + O(n^{k/2-1-\alpha+\varepsilon})$$

which implies the result stated.

7. The singular series. We have

$$G(f,n) = \sum_{t<\infty} \frac{\mu(\boldsymbol{t})}{\prod_{j=1}^{k} t_j^2} \sum_{q=1}^{\infty} q^{-k} A_{\boldsymbol{t}}(q)$$

where

$$A_{\boldsymbol{t}}(q) = \sum_{h\,(\mathrm{mod}\, q)}' S(hf_{\boldsymbol{t}}, q)\, e\left(-\frac{nh}{q}\right).$$

This series is absolutely convergent on $t_1, \ldots, t_k, q$ for $k \geqslant 4$ by (30) and (34).

LEMMA 13. *Let* $\boldsymbol{d} = (d_1, \ldots, d_k)$. *Then*

$$G(f,n) = \left(\frac{6}{\pi^2}\right)^k \sum_{q=1}^{\infty} q^{-k} \prod_{p|q} (1-p^{-2})^{-k} \sum_{d_1|q,\ldots,d_k|q} \frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k} d_j^2} A_{\boldsymbol{d}}(q). \tag{38}$$

Proof, see [9], p. 46. Taking $d_j|q, (t_j, q) = 1, (j = 1, \ldots, k)$ we have

$$f_{\boldsymbol{td}}(\boldsymbol{x}) = f_{\boldsymbol{d}}(\boldsymbol{t}^2\boldsymbol{x}), \quad (t_j^2, q) = 1 \quad (j = 1, \ldots, k)$$

and $t_j^2 x_j$ runs over a complete residue system mod q when x_j does. Therefore

$$S(hf_{\boldsymbol{td}}, q) = S(hf_{\boldsymbol{d}}, q), \quad A_{\boldsymbol{td}}(q) = A_{\boldsymbol{d}}(q).$$

Hence

$$G(f,n) = \sum_{q=1}^{\infty} q^{-k} \sum_{d_1|q,\ldots,d_k|q} \sum_{\substack{\boldsymbol{t} \\ (t_j,q)=1 \\ j=1,\ldots,k}} \frac{\mu(\boldsymbol{t})}{\prod_{j=1}^{k} t_j^2} A_{\boldsymbol{t}}(q)$$
$$= \sum_{q=1}^{\infty} q^{-k} \sum_{d_1|q,\ldots,d_k|q} \sum_{\substack{\boldsymbol{t} \\ (t_j,q)=d_j \\ j=1,\ldots,k}} \frac{\mu(\boldsymbol{td})}{\prod_{j=1}^{k} (t_j d_j)^2} A_{\boldsymbol{td}}(q)$$
$$= \sum_{q=1}^{\infty} q^{-k} \sum_{d_1|q,\ldots,d_k|q} \frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k} d_j^2} A_{\boldsymbol{d}}(q) \left\{ \sum_{(t,q)=1} \frac{\mu(t)}{t^2} \right\}^k$$

and the formula (38) follows from the identity

$$\sum_{(t,q)=1} \frac{\mu(t)}{t^2} = \frac{6}{\pi^2} \prod_{p|q} (1-p^{-2})^{-1}.$$

We put

$$T(hf, q) = \sum_{d_1|q,\ldots,d_k|q} \frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k} d_j^2} S(hf_{\boldsymbol{d}}, q)$$

and

$$B(q) = \sum_{h\,(\mathrm{mod}\, q)}' T(hf, q)\, e\left(-\frac{nh}{q}\right).$$

LEMMA 14. *For* $k \geqslant 4$

$$G(f,n) = \left(\frac{6}{\pi^2}\right)^k \prod_{p} G_p(f,n)$$

where

$$G_p(f,n) = 1 + (1-p^{-2})^{-k} \sum_{\nu=1}^{\infty} p^{-k\nu} B(p^{\nu}) \tag{39}$$

and the product is taken over all primes p.

Proof, see [9], p. 47. Let $(q_1, q_2) = 1$, $d_j|q_1$, $\delta_j|q_2$ $(j = 1, \ldots, k)$. Then by the identity

$$S(hf_{\boldsymbol{d\delta}}, q_1q_2) = S(hq_1f_{\boldsymbol{\delta}}, q_2)\, S(hq_2f_{\boldsymbol{d}}, q_1)$$

we have

$$T(hf, q_1q_2) = \sum_{d_1|q_1,\ldots,d_k|q_1} \frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k} d_j^2} \sum_{\delta_1|q_2,\ldots,\delta_k|q_2} \frac{\mu(\boldsymbol{\delta})}{\prod_{j=1}^{k} \delta_j^2} S(hf_{\boldsymbol{d\delta}}, q_1q_2)$$
$$= T(hq_1f_1, q_2)\, T(hq_2f, q_1).$$

Furthermore, since $h_1q_2+h_2q_1$, runs over a reduced residue system mod q_1q_2 when h_1 runs over a reduced residue system mod q_1, and h_2 runs over a reduced residue system mod q_2, we have

$$B(q_1q_2) = \sum_{h=h_1q_2+h_2q_1 \pmod{q_1q_2}}' T\big((h_1q_2+h_2q_1)f,\, q_1q_2\big)\, e\left(-\frac{(h_1q_2+h_2q_1)n}{q_1q_2}\right)$$

$$= \sum_{h_1 \pmod{q_1}}' \sum_{h_2 \pmod{q_2}}' T\big((h_1q_1q_2+h_2q_1^2)f,\, q_2\big)\, T\big((h_1q_2^2+h_2q_1q_2)f,\, q_1\big) \times$$

$$\times\, e\left(-\frac{nh_1}{q}\right) e\left(-\frac{nh_2}{q}\right) = B(q_1)B(q_2)$$

and

$$\prod_{p|q_1q_2}(1-p^{-2})^{-k} = \prod_{p|q_1}(1-p^{-2})^{-k}\prod_{p|q_2}(1-p^{-2})^{-k}.$$

Hence

$$\sum_{q=1}^{\infty} q^{-k}\prod_{p|q}(1-p^{-2})^{-k}B(q) = \prod_{p} G_p(f,n).$$

Lemma 15. *Let p be a prime, $p^w \| n$. Then*

$$B(p^\nu) = 0 \quad \textit{if} \quad \begin{cases}\nu > w+1, & p>2,\\ \nu > w+3, & p=2.\end{cases} \tag{40}$$

Proof. We have

$$B(p^\nu) = \sum_{d_1|p,\ldots,d_k|p} \frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k} d_j^2} A_{\boldsymbol{d}}(p^\nu)$$

and the result follows from that (see [10], p. 61, 68) for

$$A_{\boldsymbol{d}}(p^\nu) = 0 \quad \text{if} \quad \begin{cases}\nu > w+1, & p>2,\\ \nu > w+3, & p=2.\end{cases}$$

For another proof, see [9], p. 51.

Lemma 16. *Let p be a prime, $p^{w_p(f)}$ be the greatest degree of p, which divides the determinant $2D$ of the form f. Then*

$$B(p^\nu) = 0 \quad \textit{if} \quad \begin{cases}\nu > w_p(f)+3, & p>2,\\ \nu > w_p(f)+5, & p=2.\end{cases} \tag{41}$$

Proof. It is sufficient to prove the same result for $T(hf, p^\nu)$. Indeed, we have

$$T(hf, p^\nu) = \sum_{p^2\nmid x_1,\ldots,p^2\nmid x_k} e\left(\frac{hf(x_1,\ldots,x_k)}{p^\nu}\right) = \sum_{\substack{b_1,\ldots,b_k \pmod{p^2}\\ b_j\not\equiv 0 \pmod{p^2}\ (j=1,\ldots,k)}} S_{p^2;b_1,\ldots,b_k}(hf, p^\nu),$$

where

$$S_{p^2;b_1,\ldots,b_k}(hf, p^\nu) = \sum_{\substack{x,\,x_j\equiv b_j \pmod{p^2}\\ (j=1,\ldots,k)}} e\left(\frac{hf(x_1,\ldots,x_k)}{p^\nu}\right).$$

It is known, see [10], p. 35, that

$$S_{p^2;b_1,\ldots,b_k}(hf, p^\nu) = 0 \quad \text{for} \quad t \geqslant \tau$$

where for an odd prime p the number τ is defined in the following manner. Let f be equivalent to $\varphi = p^{e_1}a_1x_1^2+\ldots+p^{e_k}a_kx_k^2$ and $(b_1,\ldots,b_k)$ transforms to $(b_1',\ldots,b_k')$ by the same substitution. If $p^{v_j}\| b_j'$, then

$$\tau = \min_j(2+v_j+e_j) \leqslant 2+w_p(f)+\min_j v_j \leqslant 3+w_p(f).$$

Similarly, for $p=2$ we have numbers v_j, $\min v_j \leqslant 1$ and e_j, $e_j \leqslant w_2(f)$ and three subsets of indicies J_1, J_2, J_3 such that

$$\tau = \min\{\min_{j\in J_1}(3+v_j+e_j),\, \min_{j\in J_2}(4+v_j+e_j),\, \min_{j\in J_3}(3+v_j+e_j)\}$$
$$\leqslant 4+w_2(f)+\min v_j \leqslant 5+w_2(f).$$

The lemma is therefore proved.

Lemma 17. *Let*

$$N = N_p = \begin{cases}\min\{5+w_2(f),\, w_2+3\}, & p=2,\\ \max\{\min\{w_p+1,\, w_p(f)+3\},\, 2\}, & p>2\end{cases} \tag{42}$$

and $\varrho(f, p^N, n)$ be the number of solutions of the congruence

$$f(x_1,\ldots,x_k) \equiv n \pmod{p^N} \tag{43}$$

in integers $x_1,\ldots,x_k$ not divisible by p^2. Then

$$G_p(f,n) = p^{-(k-1)N}(1-p^{-2})^{-k}\varrho(f,p^N,n). \tag{44}$$

Proof. For every prime p by (39), (40) and (41) we have

$$G_p(f,n) = 1+(1-p^{-2})^{-k}\sum_{\nu=1}^{\infty}p^{-k\nu}B(p^\nu) = 1+(1-p^{-2})^{-k}\sum_{\nu=1}^{N}p^{-k\nu}B(p^\nu)$$

$$= (1-p^{-2})^{-k}\sum_{\nu=0}^{N}p^{-k\nu}\sum_{d_1|p,\ldots,d_k|p}\frac{\mu(\boldsymbol{d})}{\prod_{j=1}^{k}d_j^2}A_{\boldsymbol{d}}(p^\nu)$$

$$= (1-p^{-2})^{-k}\sum_{e_1,\ldots,e_k=0,1}\left(-\frac{1}{p^2}\right)^{\sum_j e_j}\sum_{\nu=0}^{N}p^{-k\nu}A_{\boldsymbol{d}}(p^\nu).$$

It is known (see [10], p. 70) that

$$\sum_{\nu=0}^{N} p^{-k\nu} A_d(p^\nu) = \frac{\sigma(f_d, p^N, n)}{p^{(k-1)N}} \tag{45}$$

where $\sigma(f_d, p^N, n)$ is the number of solutions of the congruence

$$f_d(x_1, \ldots, x_k) \equiv n \pmod{p^N} \tag{46}$$

in integers $x_1, \ldots, x_k$ and $\boldsymbol{d} = (p^{\varepsilon_1}, \ldots, p^{\varepsilon_k})$.

Hence

$$G_p(f, n) = p^{-(k-1)N}(1-p^{-2})^{-k} \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0,1} \left(-\frac{1}{p^2}\right)^{\sum_j \varepsilon_j} \sigma(f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}, p^N, n).$$

It is easy to see that each solution of the congruence (46) corresponds to one solution of the congruence (45) and each solution of the congruence (43) corresponds to $\prod_{j=1}^{k} p^{2\varepsilon_j}$ solutions of the congruence (46). Therefore denoting that $\varrho_{\varepsilon_1, \ldots, \varepsilon_k}(f, p^N, n)$ is the number of solutions of the congruence (43) in integers $x_1, \ldots, x_k$, $p^{2\varepsilon_1} | x_1, \ldots, p^{2\varepsilon_k} | x_k$ we have

$$\sum_{\varepsilon_1, \ldots, \varepsilon_k = 0,1} \left(-\frac{1}{p^2}\right)^{\sum_j \varepsilon_j} \sigma(f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}, p^N, n) = \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0,1} (-1)^{\sum_j \varepsilon_j} \varrho_{\varepsilon_1, \ldots, \varepsilon_k}(f, p^N, n) = \varrho(f, p^N, n)$$

where the latter equality results according to the including-excluding principle.

For another proof, see [9], p. 55.

Corollary. *$G(f, n)$ is a real positive number or zero, since by Lemma 17 $G_p(f, n) \geqslant 0$ for every prime p.*

Lemma 18. *There is a constant $c_\varepsilon^{(k)}$ such that*

$$\prod_{p \nmid P_f} G_p(f, n) \geqslant \begin{cases} c_\varepsilon^{(4)} n^{-\varepsilon} & \text{if } k = 4, \\ c_\varepsilon^{(k)} & \text{if } k \geqslant 5. \end{cases}$$

Proof. Let $p \nmid P_f$ and $p^w \| n$. For $\nu \geqslant 1$ we put

$$B(p^\nu, w) = \sum_{h \,(\mathrm{mod}\, p^\nu)}{}' T(hf, p^\nu) e\left(-\frac{nh}{p^\nu}\right).$$

To evaluate $G_p(f, n)$ we consider four cases.

(a) $w = 0$. Then, by (39) and (40)

$$G_p(f, n) = 1 + (1-p^{-2})^{-k} p^{-k} B(p, 0).$$

By Lemma 6 for every set $(\varepsilon_1, \ldots, \varepsilon_k)$ the form f is equivalent mod p to a diagonal form

$$\varphi^{(\varepsilon_1, \ldots, \varepsilon_k)} = \sum_{j=1}^{k} a_j^{(\varepsilon_1, \ldots, \varepsilon_k)} y_j^2$$

such that $f_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$ is equivalent mod p to the form $\varphi^{(\varepsilon_1, \ldots, \varepsilon_k)}_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}$. Hence

$$S(hf_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}, p) = \prod_{j=1}^{k} \left(\frac{h}{p}\right)^{1-\varepsilon_j} \left(\frac{a_j^{(\varepsilon_1, \ldots, \varepsilon_k)}}{p}\right)^{1-\varepsilon_j} i^{\left(\frac{p-1}{2}\right)^2 (1-\varepsilon_j)} p^{\frac{1+\varepsilon_j}{2}}$$

$$= \left(\frac{h}{p}\right)^{k - \sum_j \varepsilon_j} \left(\frac{\prod_{j=1}^{k} a_j^{(\varepsilon_1, \ldots, \varepsilon_k)(1-\varepsilon_j)}}{p}\right) i^{(k-\sum_j \varepsilon_j)\left(\frac{p-1}{2}\right)^2} p^{\frac{k}{2} + \frac{1}{2}\sum_j \varepsilon_j}.$$

We have

$$T(hf, p) = \sum_{k - \sum_j \varepsilon_j \equiv 0 \,(\mathrm{mod}\, 2)} + \left(\frac{h}{p}\right) \sum_{k - \sum_j \varepsilon_j \equiv 1 \,(\mathrm{mod}\, 2)} = \Sigma_1 + \left(\frac{h}{p}\right) \Sigma_2,$$

$$B(p, 0) = \left\{ \sum_{h \,(\mathrm{mod}\, p)}{}' e\left(-\frac{nh}{p}\right) \right\} \Sigma_1 + \left\{ \sum_{h \,(\mathrm{mod}\, p)}{}' \left(\frac{h}{p}\right) e\left(-\frac{nh}{p}\right) \right\} \Sigma_2.$$

For $(p, n_1) = 1$ it is known (see [10], p. 60) that

$$\sum_{h \,(\mathrm{mod}\, p^\nu)}{}' \left(\frac{h}{p}\right)^{\varepsilon} e\left(-\frac{p^w n_1 h}{p^\nu}\right) = \begin{cases} 0 & \text{if } \nu > w+1, \\ -p^{\nu-1} & \text{if } \nu = w+1,\ \varepsilon = 0, \\ (p-1)p^{\nu-1} & \text{if } \nu < w+1,\ \varepsilon = 0, \\ 0 & \text{if } \nu < w+1,\ \varepsilon = 1, \\ \left(\frac{-n_1}{p}\right) i^{\left(\frac{p-1}{2}\right)^2} p^{w+\frac{1}{2}} & \text{if } \nu = w+1,\ \varepsilon = 1. \end{cases} \tag{47}$$

Therefore,

$$|B(p, 0)| \leqslant \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_k = 0,1 \\ k - \sum_j \varepsilon_j \equiv 0 \,(\mathrm{mod}\, 2)}} p^{\frac{k}{2} - \frac{3}{2}\sum_j \varepsilon_j} + p^{1/2} \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_k = 0,1 \\ k - \sum_j \varepsilon_j \equiv 1 \,(\mathrm{mod}\, 2)}} p^{\frac{k}{2} - \frac{3}{2}\sum_j \varepsilon_j}.$$

Hence

$$|G_p(f, n) - 1| \leqslant \tfrac{1}{2}(1-p^{-2})^{-k} p^{-k/2} \{[(p^{-3/2}+1)^k + (p^{-3/2}-1)^k] + p^{1/2}[(p^{-3/2}+1)^k - (p^{-3/2}-1)^k]\} = \xi_0(k, p).$$

A numerical calculation shows that $\xi_0(k, p)$ decreases with k and p, $\xi_0(4, 3) < 1$ and for $k \geqslant 4$ there is a constant c_1 such that

$$\xi_0(k, p) < c_1 p^{-2}.$$

Hence

$$\prod_{\substack{p \in P_f \\ p \nmid n}} G_p(f, n) > c_2 \prod_{\substack{p \in P_f \\ p \nmid n, p > \sqrt{c_1}}} \left(1 - \frac{c_1}{p^2}\right) > c_3 > 0.$$

(b) $w = 1$. In this case

$$G_p(f, n) = 1 + (1-p^{-2})^{-k}[p^{-k} B(p, 1) + p^{-2k} B(p^2, 1)],$$

$$S(hf_{p^{\varepsilon_1}, \ldots, p^{\varepsilon_k}}, p^2) = \prod_{j=1}^{k} p^{1+\varepsilon_j}.$$

In the same way as in (a) we have

$$|G_p(f, n) - 1| \leqslant \tfrac{1}{2}(1-p^{-2})^{-k} p^{-k/2}(p-1)[(p^{-3/2}+1)^k + (p^{-3/2}-1)^k] + \\ + (1-p^{-2})^{-k}(p-1)^k p^{1-2k} = \xi_1(k, p).$$

(c) $w = 2$. We have

$$|G_p(f, n) - 1| \leqslant \xi_2(k, p),$$

where

$$\xi_2(k, p) = \tfrac{1}{2}(1-p^{-2})^{-k} p^{-k/2}(p-1)[(p^{-3/2}+1)^k + (p^{-3/2}-1)^k] + \\ + (1-p^{-2})^{-k}(p-1)^{k+1} p^{1-2k} + \\ + \tfrac{1}{2}(1-p^{-2})^{-k} p^{2-\frac{3}{2}k}[(p^{-1/2}+1)^k + (p^{-1/2}-1)^k] + \\ + \tfrac{1}{2}(1-p^{-2})^{-k} p^{\frac{5}{2}-\frac{3}{2}k}[(p^{-1/2}+1)^k - (p^{-1/2}-1)^k].$$

(d) $w \geqslant 3$. We have

$$|G_p(f, n) - 1| \leqslant \xi_3(k, p),$$

where

$$\xi_3(k, p) = \tfrac{1}{2}(1-p^{-2})^{-k} p^{-k/2}(p-1)[(p^{-3/2}+1)^k + (p^{-3/2}-1)^k] + \\ + (1-p^{-2})^{-k}(p-1)^{k+1} p^{1-2k} + \\ + \tfrac{1}{2}(1-p^{-2})^{-k} p^{2-\frac{3}{2}k}(p-1)[(p^{-1/2}+1)^k + (p^{-1/2}-1)^k].$$

This is just routine to prove that $\xi_j(k, p)$ decreases with k and p, $\xi_j(4, 3) < 1$ $(j = 1, 2, 3)$ and there is a constant $c_4 = c_4(k)$ such that

for $k \geqslant 5$

$$\xi_j(k, p) < \frac{c_4}{p^2}, \qquad \xi_j(4, p) < \frac{c_4}{p} \qquad (j = 1, 2, 3).$$

Hence, for $k \geqslant 5$

$$\prod_{\substack{p \notin P_f \\ p \mid n}} G_p(f, n) > c_5$$

and for $k = 4$

$$\prod_{\substack{p \notin P_f \\ p \mid n}} G_p(f, n) > c_6 \prod_{\substack{p \mid n \\ p > c_4}} \left(1 - \frac{c_4}{p}\right) > c_7 n^{-\varepsilon}.$$

The lemma is therefore proved.

THEOREM 2. *Let N_p be as in (42). If for every prime $p \in P_f$ it is soluble congruences*

$$f(x_1, \ldots, x_k) \equiv n \pmod{p^{N_p}}$$

in integers $x_1, \ldots, x_k$ not divisible by p^2 then there is a constant $G_\varepsilon^{(k)}$ which depends only on f and ε such that

$$G(f, n) > \begin{cases} G_\varepsilon^{(4)} n^{-\varepsilon} & \text{if} \quad k = 4, \\ G_\varepsilon^{(k)} & \text{if} \quad k \geqslant 5. \end{cases}$$

Otherwise $G(f, n) = 0$.

Proof. By Lemma 14

$$G(f, n) = \left(\frac{6}{\pi^2}\right)^k \prod_p G_p(f, n) = \left(\frac{6}{\pi^2}\right)^k \prod_{p \in P_f} G_p(f, n) \prod_{p \notin P_f} G_p(f, n)$$

and by Lemma 18

$$\prod_{p \notin P_f} G_p(f, n) > \begin{cases} c_\varepsilon^{(4)} n^{-\varepsilon} & \text{if} \quad k = 4, \\ c_\varepsilon^{(k)} & \text{if} \quad k \geqslant 5. \end{cases}$$

Let now $p \in P_f$, then by Lemma 17

$$G_p(f, n) = p^{-(k-1)N_p}(1-p^{-2})^{-k} \varrho(f, p^{N_p}, n),$$

hence $G_p(f, n) = G(f, n) = 0$, if the congruence (43) is insoluble in integers not divisible by p^2. Otherwise

$$G_p(f, n) \geqslant p^{-(k-1)(w_p(f)+5)}(1-p^{-2})^{-k}$$

and the result follows if we put

$$G_s^{(k)} = \left(\frac{6}{\pi^2}\right)^k c_s^{(k)} \prod_{p\in P_f} p^{-(k-1)(w_p(f)+5)}(1-p^{-2})^{-k}.$$

COROLLARY. *For all sufficiently large integers n are representabl by the quadratic form f provided f, n satisfy conditions of Theorem 2.*

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О некоторых арифметических задачах с числами, имеющими малые простые делители

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В статье рассматривается ряд проблем аналитической теории чисел (см. [1]) в числах, имеющих малые простые делители. Это позволяет использовать для их решения p-адический метод, первые применения которого в тригонометрических суммах были даны Ю. В. Линником [6]. Об одной из этих проблем, именно, о возможности получения асимптотической формулы для числа представлений достаточно большого натурального числа суммой n-х степеней чисел с малыми простыми делителями и числом слагаемых порядка $n\ln n$ (аналог асимптотической формулы в проблеме Варинга), говорил Ю.В. Линник в 1971 году на Международной конференции по теории чисел в Москве. Введем определение и ряд обозначений, необходимых для дальнейшего.

Определение. Пусть $g(x)$ — монотонно возрастающая функция, причем $g(x) \geqslant \ln\ln x$ при $x \geqslant x_0 > 0$ и

$$\lim_{x\to+\infty} \frac{g(x)}{\ln x} = 0.$$

Натуральное число m называется *числом с малыми простыми делителями класса* E_g, если для каждого простого делителя p числа m выполняется неравенство $\ln p \leqslant g(m)$.

Число чисел m с малыми простыми делителями класса E_g, не превосходящих P, будем обозначать $\boldsymbol{P}$; таким образом,

$$\boldsymbol{P} = \boldsymbol{P}(P, g) = \sum_{\substack{m\in E_g \\ m\leqslant P}} 1.$$

Подобно тому, как это делается в [2], можно показать, что при $P\to+\infty$

$$\boldsymbol{P} \sim Pe^{-\omega\ln\omega}, \quad \omega = \frac{\ln P}{g(P)}.$$