From (74), (75), (76) it follows that generally $c_7 > 0$, except merely the case with $\chi(2) = -1$ (whence $\Delta \equiv 5 \pmod 8$; see [12], I, p. 51), $\Delta_1 \equiv 12 \pmod{16}$ and $\Delta c_1 \equiv 3 \pmod 4$, in which case $f_2 = 0$ and simultaneously $c_7 = 0$. In this exceptional case $\varphi_1(4) = 1$, by (55). Therefore we have either $\Delta \equiv 1 \pmod 4$ (whence $c_1 \equiv 3$, $-c_1 \equiv \Delta$) or $\Delta \equiv 3 \pmod 4$ (whence $c_1 \equiv 1$, $-c_1 \equiv \Delta$). In both cases $-c_1$ is an odd number congruent mod 4 to a norm of some ideal of the class $\mathfrak{K}_1$. This completes the proof of the lemma.

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The exceptional set in Goldbach's problem

by

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Dedicated with deepest respect
to the memory of
Academician Yu. V. Linnik

1. Introduction. Goldbach stated, in a letter to Euler (c. 1742), that every even integer exceeding 2 can be written as a sum of two primes. If we let $E(X)$ denote the number of even numbers not exceeding X which cannot be written as a sum of two primes, then Goldbach's conjecture can be formulated as the assertion that $E(X) = 1$ for $X \geqslant 2$. Goldbach's problem remains unsettled, but Vinogradov's fundamental work ([20], [21]) on three primes inspired others [1], [4], [17] to show that $E(X) = o(X)$, so that almost all even numbers can be expressed as a sum of two primes. Recently Vaughan [18] sharpened the earlier results by showing that

$$E(X) < X\exp(-c\log^{1/2} X).$$

We improve on this by establishing the following theorem.

THEOREM 1. *There is a positive (effectively computable) constant δ such that for all large X*

$$E(X) < X^{1-\delta}.$$

Hardy and Littlewood [6] introduced the approach by which one shows that most even integers are sums of two primes; they showed that if the Generalized Riemann Hypothesis (GRH) is true then one may take $\delta = \frac{1}{2} - \varepsilon$ in the above. We avoid the GRH by appealing to a recent result of Gallagher [5] which reflects considerable knowledge of the distribution of the zeros of L-functions. To indicate the depth of Gallagher's result (our Lemma 4.3), we note that one may easily derive from it the celebrated theorem of Linnik ([9], [10]) concerning the least prime in an arithmetic progression. A recent form of the Linnik–Rényi large sieve, Turán's method, and the Deuring–Heilbronn phenomenon all play essential roles in Gallagher's proof.

While we expect that Goldbach's conjecture is true, it nevertheless might be the case that it is false. Indeed there might even be long intervals containing no sum of two primes, although upper bounds are known for the possible length of such intervals. Linnik [13] showed that if the Riemann Hypothesis (RH) is true then for large X the interval $(X, X+\log^{3+\varepsilon}X)$ contains a sum of two primes. From Huxley's theorem [8] on the gaps between primes it is obvious that the interval $(X, X+X^{7/12+\varepsilon})$ contains a sum of two primes, and Ramachandra [15] has proved a more precise result of this sort. We sharpen these estimates by proving

THEOREM 2. *For all large X the interval $(X, X+X^{7/72+\varepsilon})$ contains a sum of two prime numbers. If the Riemann Hypothesis is true then there is a $C>0$ such that for all X the interval $(X, X+C\log^2 X)$ contains a sum of two primes* (1).

Both Linnik and Ramachandra employed the Hardy–Littlewood–Vinogradov method in obtaining their results. In § 9 we derive Theorem 2 simply by appealing to known results concerning primes in short intervals.

We are happy to record our gratitude to Professor Patrick Gallagher for his kind assistance. In particular, the proof we give of Theorem 1 incorporates a number of substantial simplifications suggested by Gallagher.

2. Notation and dissection of the unit interval. Throughout $t, u, v, x, y, \alpha, \eta, \theta, \varkappa, \sigma$ denote real variables, while H, N, P, Q, T, X, Y denote large positive real numbers. The parameter δ is a small positive real variable which is eventually taken to be a small positive absolute constant. We assume that X is larger than some $X_0(\delta)$. We let $a, b, d, h, j, k, n, q, r$ denote natural numbers, while m is an arbitrary integer, p is a prime number, and s is the complex variable $s=\sigma+it$. The constants $C, c, c_1, c_2, \ldots$, as well as all implicit constants are positive, absolute, and effectively computable.

We let χ denote a Dirichlet character, and unless the contrary is indicated, χ is a character (mod q). We let χ^* denote the primitive character which induces χ. We let χ_0 denote the principal character (mod q), while $\tilde{\chi}$ is the primitive (possibly non-existent) exceptional character (2), of modulus $\tilde{r}$, whose L-function $L(s, \tilde{\chi})$ vanishes at $\tilde{\beta}$. The expressions

$$\sum_{\chi}, \quad \sum_{\chi}^{*}, \quad \sum_{a}{}'$$

denote, respectively, a sum over all χ (mod q), a sum over all primitive χ (mod q), and a sum over all reduced residue classes a (mod q). In Lemma 4.3 we attach a special significance to the symbol $\sum^{\#}$.

As usual, we let $\|y\|$ denote the distance from y to the nearest integer, $e(\alpha)=e^{2\pi i\alpha}$, and

$$c_q(m) = \sum_{h=1}^{q}{}' e\left(\frac{hm}{q}\right)$$

is Ramanujan's sum. Analogously we let

$$c_\chi(m) = \sum_{h=1}^{q} \chi(h) e\left(\frac{hm}{q}\right); \tag{2.1}$$

thus the Gaussian sum occurs as $\tau(\chi)=c_\chi(1)$.

Much of our analysis is concerned with the sum

$$S(\alpha) = \sum_{P<p\leqslant X} (\log p) e(p\alpha), \tag{2.2}$$

and the associated sum

$$S(\chi, \eta) = \sum_{P<p\leqslant X} (\log p)\chi(p) e(p\eta). \tag{2.3}$$

To dissect the unit interval, we now put

$$P = X^{6\delta}, \qquad Q = X^{1-6\delta}, \tag{2.4}$$

so that $PQ=X$. For $1\leqslant a\leqslant q\leqslant P$, $(a,q)=1$, we let $\mathfrak{M}(q,a)$ be the major arc $\left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right]$. The major arcs are non-overlapping, since

$$\left|\frac{a}{q}-\frac{a'}{q'}\right| \geqslant \frac{1}{qq'} > \frac{2P}{qq'Q} \geqslant \frac{q+q'}{qq'Q} = \frac{1}{qQ}+\frac{1}{q'Q}.$$

We let $\mathfrak{M}$ be the union of all the major arcs, and we let $\mathfrak{m}$ denote the set of those α, $Q^{-1}<\alpha<1+Q^{-1}$, not lying in $\mathfrak{M}$.

3. The minor arcs. Let $R(n)$ be the coefficient of $e(\alpha n)$ in the exponential sum $S(\alpha)^2$; we note that if $R(n)>0$ then n is a sum of two primes. Clearly

$$R(n) = R_1(n)+R_2(n), \tag{3.1}$$

where

$$R_1(n) = \int_{\mathfrak{M}} S(\alpha)^2 e(-n\alpha)\,d\alpha,$$

$$R_2(n) = \int_{\mathfrak{m}} S(\alpha)^2 e(-n\alpha)\,d\alpha.$$

The sets $\mathfrak{M}$ and $\mathfrak{m}$ are even (mod 1), so $R_1(n)$ and $R_2(n)$ are real. Our object is to show that $R_1(n)$ is large with few exceptions for $\frac{1}{2}X<n\leqslant X$, and

(1) Note added in proof. It has come to our attention that Kátai [8a] has anticipated the conditional result stated here.

(2) The precise delineation of what constitutes an exceptional character is given in Lemma 4.1.

that $R_2(n)$ is small, with few exceptions. This latter is achieved in a standard way by showing that

$$\sum_{n\leqslant X} R_2(n)^2 \ll X^3P^{-1}\log^{35}X. \tag{3.2}$$

The first step in proving this is to observe that Parseval's identity implies that

$$\sum_n R_2(n)^2 = \int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leqslant \big(\max_{\mathfrak{m}} |S(\alpha)|\big)^2 \int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha.$$

We extend the range of integration in this second integral and apply Parseval's identity again to find that

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha \leqslant \int_{Q^{-1}}^{1+Q^{-1}} |S(\alpha)|^2 d\alpha = \sum_{P<p\leqslant X} \log^2 p \ll X\log X.$$

Thus to obtain (3.2) it suffices to establish that

$$\max_{\mathfrak{m}} |S(\alpha)| \ll XP^{-1/2}\log^{17}X. \tag{3.3}$$

We now appeal to Vinogradov's fundamental lemma, which we state in the following form.

LEMMA 3.1. *If* $Y\leqslant q\leqslant XY^{-1}$, $1\leqslant Y\leqslant X^{1/4}$, $(a,q)=1$, $\left|\alpha-\frac{a}{q}\right| \leqslant q^{-2}$, *then*

$$S(\alpha) \ll XY^{-1/2}\log^{17}X.$$

This is essentially a consequence of Theorems 1 and 3 of Vinogradov ([22], Chapter IX). Linnik [11], [12] and Čudakov [2] found that similar results could be derived from zero density estimates for L-functions. Recently these estimates have been greatly improved, facilitating this approach. A derivation of Lemma 3.1 from zero density estimates is found in Chapter 16 of Montgomery [14]. Recently Vaughan [19] discovered a very simple proof of Lemma 3.1 with the condition $Y\leqslant X^{1/4}$ weakened to read $Y\leqslant X^{1/3}$.

Suppose $\alpha\in\mathfrak{m}$. By Dirichlet's theorem on Diophantine approximation there exist $q\leqslant Q$ and a, $1\leqslant a\leqslant q$, $(a,q)=1$, such that $\left|\alpha-\frac{a}{q}\right|\leqslant q^{-1}Q^{-1}$. This would imply that $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$ if it were the case that $q\leqslant P$. Thus $q>P$, and we may take $Y=P$ in Lemma 3.1. This gives (3.3).

4. Analytic lemmas.

We now state the basic properties of exceptional characters.

LEMMA 4.1. *There is a constant $c_1>0$ such that $L(\sigma,\chi)\neq 0$ whenever*

$$\sigma\geqslant 1-\frac{c_1}{\log P},$$

for all primitive characters χ of modulus $q\leqslant P$, with the possible exception of at most one primitive character $\tilde\chi(\bmod\,\tilde r)$. If it exists, the character $\tilde\chi$ is quadratic, and the (unique) exceptional real zero $\tilde\beta$ of $L(s,\tilde\chi)$ satisfies

$$\frac{c_2}{\tilde r^{1/2}\log^2\tilde r}\leqslant 1-\tilde\beta\leqslant\frac{c_1}{\log P}. \tag{4.1}$$

These assertions are established by Davenport ([3], § 14).

The following is Lemma 1 of Gallagher [5].

LEMMA 4.2. *Let $u_1, u_2, \ldots, u_N$ be arbitrary real numbers. Then for any $\varkappa>0$*

$$\int_{-\varkappa}^{\varkappa}\Big|\sum_{n\leqslant N} u_n e(n\eta)\Big|^2 d\eta \ll \int_{-\infty}^{+\infty}\Big|\varkappa\sum_{x}^{x+(2\varkappa)^{-1}} u_n\Big|^2 dx.$$

The following lemma forms the crux of our treatment of the error terms which arise in estimating $R_1(n)$.

LEMMA 4.3. *For suitable (small) positive absolute constants c_3, c_4,*

$$\sum_{q\leqslant P}\sum_{\chi}{}^{*}\max_{x\leqslant N}\max_{h\leqslant N}\Big(h+\frac{N}{P}\Big)^{-1}\Big|\sum_{x-h}^{x}{}^{\#}\chi(p)\log p\Big| \ll \exp\Big(-c_3\frac{\log N}{\log P}\Big) \tag{4.2}$$

provided $\exp(\log^{1/2}N)\leqslant P\leqslant N^{c_4}$. *Here $\sum^{\#}$ indicates that the term with $q=1$ is to be*

$$\sum_{x-h}^{x}\log p - \sum_{\substack{x-h<n\leqslant x\\ n>0}} 1,$$

and that if there is an exceptional character $\tilde\chi$ then the corresponding term is

$$\sum_{x-h}^{x}\tilde\chi(p)\log p + \sum_{\substack{x-h<n\leqslant x\\ n>0}} n^{\tilde\beta-1}.$$

If the exceptional character occurs then the right hand side of (4.2) may be reduced by a factor of $(1-\tilde\beta)\log P$.

This is Theorem 7 of Gallagher [5], with two modifications. In the first place Gallagher did not have the $\max_x\max_h$ in (4.2); to introduce this we have only to note that

$$\max_{x\leqslant N}\max_{h\leqslant N}\Big(h+\frac{N}{P}\Big)^{-1} x^{\beta-1}\min(x,h) \ll \Big(\frac{N}{P}\Big)^{\beta-1},$$

and that

$$\max_{x\leqslant N}\max_{h\leqslant N}\Big(h+\frac{N}{P}\Big)^{-1} xT^{-1}\log^2 x \ll PT^{-1}\log^4 P.$$

In the second place Gallagher appeals to Siegel's theorem, which renders his theorem non-effective. However, an appeal to the effective lower bound (4.1) will suffice if we take $T = P^6$ instead of $T = P^5$. Gallagher's proof is effective in all other aspects, so Lemma 4.3 is effective.

5. Arithmetic lemmas. We begin by recalling several well-known results.

LEMMA 5.1. *If χ is a primitive character* (mod q) *then* $|\tau(\chi)| = q^{1/2}$. *If χ is a primitive quadratic character* (mod q) *then*

$$\tau(\chi)^2 = \chi(-1)q,$$

and $q/(4, q)$ is square-free.

Our object in the next three lemmas is to establish a formula for $c_\chi(m)$ in terms of $\tau(\chi^*)$.

LEMMA 5.2. *Let χ be a character* (mod k), *induced by the primitive character* χ^* (mod r). *Then $r|k$, and*

$$\tau(\chi) = \mu\left(\frac{k}{r}\right)\chi^*\left(\frac{k}{r}\right)\tau(\chi^*).$$

This is well-known; for example Davenport ([3], p. 148) provides a proof.

LEMMA 5.3. *Suppose that the above hypotheses hold, and that* $(m, k) = 1$. *Then*

$$c_\chi(m) = \bar{\chi}^*(m)\mu\left(\frac{k}{r}\right)\chi^*\left(\frac{k}{r}\right)\tau(\chi^*).$$

Proof. Clearly $\chi^*(m)c_\chi(m) = \chi(m)c_\chi(m) = c_\chi(1) = \tau(\chi)$.

We now use the above to prove a result which includes Lemmas 5.2 and 5.3 as special cases.

LEMMA 5.4. *Let χ be a character* (mod q), *induced by a primitive character* χ^* (mod r). *For an arbitrary integer m put $q_1 = q/(q, |m|)$. If $r\nmid q_1$ then $c_\chi(m) = 0$. If $r|q_1$ then*

$$c_\chi(m) = \bar{\chi}^*\left(\frac{m}{(q, |m|)}\right)\frac{\varphi(q)}{\varphi(q_1)}\mu\left(\frac{q_1}{r}\right)\chi^*\left(\frac{q_1}{r}\right)\tau(\chi^*). \tag{5.1}$$

While many special cases of this lemma are familiar, we have been unable to locate (5.1) in the literature(³). For convenience of reference we note that $c_{\chi_0}(m) = c_q(m)$, so that the Ramanujan sum is

$$c_q(m) = \mu(q_1)\frac{\varphi(q)}{\varphi(q_1)}. \tag{5.2}$$

(³) Note added in proof. This result is given on pages 449–450 of Hasse [7].

Clearly $c_\chi(m)$ is periodic with a period which divides q, so in proving Lemma 5.4 we may assume that m is positive. Write $q = q_1q_2$, put $h = aq_1+b$, and set

$$\frac{m}{q} = \frac{m_1}{q_1},$$

with $(m_1, q_1) = 1$. Then

$$c_\chi(m) = \sum_{h=1}^{q}\chi(h)e\left(\frac{m_1h}{q_1}\right) = \sum_{b=1}^{q_1}e\left(\frac{bm_1}{q_1}\right)\sum_{a=1}^{q_2}\chi(aq_1+b).$$

The outer sum can be restricted to reduced residue classes, since $\chi(aq_1+b) = 0$ if $(b, q_1) > 1$. Thus

$$c_\chi(m) = {\sum_{b=1}^{q_1}}' e\left(\frac{bm_1}{q_1}\right)S(b), \tag{5.3}$$

say. We now consider two cases.

Case 1. $r\nmid q_1$. We show that $S(b) = 0$ whenever $(b, q_1) = 1$. For any d,

$$\chi(d)S(b) = \sum_{a=1}^{q_2}\chi(adq_1+bd);$$

if $(d, q) = 1$ then this is

$$= \sum_{a=1}^{q_2}\chi(aq_1+bd);$$

if $d \equiv 1 \pmod{q_1}$ then this is

$$= \sum_{a=1}^{q_2}\chi(aq_1+b) = S(b).$$

If in addition $\chi(d) \neq 1$ then we deduce that $S(b) = 0$. We now show that there is a d with the three required properties. Since $r\nmid q_1$, χ is not periodic with period q_1 among reduced residue classes. Thus there are d_1, d_2 such that $(d_1, q) = (d_2, q) = 1$, $d_1 \equiv d_2 \pmod{q_1}$, but $\chi(d_1) \neq \chi(d_2)$. Then $d \equiv d_1d_2^{-1} \pmod{q}$ has the required properties.

The argument that we have just given is known in the case $r = q$; see Davenport [3], p. 68.

Case 2. $r|q_1$. Now $\chi(aq_1+b)$ is either $\chi^*(b)$ or 0. Thus if $(b, q_1) = 1$ then

$$S(b) = \chi^*(b)\sum_{\substack{a=1\\(aq_1+b,q)=1}}^{q_2}1 = \chi^*(b)q_2\prod_{\substack{p|q_2\\p\nmid q_1}}\left(1-\frac{1}{p}\right) = \chi^*(b)\frac{\varphi(q)}{\varphi(q_1)}.$$

Continuing from (5.3), we see that

$$c_\chi(m) = \frac{\varphi(q)}{\varphi(q_1)} \sum_{b=1}^{q_1}{}' \chi^*(b) e\left(\frac{bm_1}{q_1}\right) = \frac{\varphi(q)}{\varphi(q_1)} c_{\chi_1}(m_1),$$

where χ_1 is the character (mod q_1) induced by χ^*. But $(m_1, q_1) = 1$, so (5.1) now follows from Lemma 5.3.

Later we shall also require

LEMMA 5.5. *Let χ_i be primitive characters* (mod r_i), $i = 1, 2$. *Then for $m \neq 0$,*

(5.4) $$\sum_q \varphi(q)^{-2} |c_{\chi_1\chi_2\chi_0}(m) \tau(\bar{\chi}_1\chi_0)\tau(\bar{\chi}_2\chi_0)| \ll \frac{|m|}{\varphi(|m|)},$$

where the sum is over all q which are divisible by both r_1 and r_2; here χ_0 is the principal character (mod q).

Proof. We may assume that $m > 0$. Let r_3 be the conductor of the primitive character that induces $\chi_1\chi_2$, let r_4 be the least common multiple of r_1 and r_2, $r_4 = [r_1, r_2]$, and let $r_5 = (r_4, m)$. We let $a_i = a_i(p)$ be defined so that $p^{a_i} \| r_i$, $1 \leq i \leq 5$. Clearly $r_3 | r_4$, so that $a_4 = \max(a_1, a_2, a_3)$ for any prime p. As $r_4 | q$, we write $q = r_4 k$. If q gives rise to a non-zero term in the sum then $(q/r_i, r_i) = 1$ and $\mu(q/r_i)^2 = 1, i = 1, 2$, from which we deduce that $(k, r_4) = 1$ and $\mu(k)^2 = 1$. Thus

$$\varphi(q) = \varphi(r_4)\varphi(k), \quad \varphi\big(q/(q, m)\big) = \varphi\big(k/(k, m)\big)\varphi(r_4/r_5).$$

We may assume that $r_3 \Big| \dfrac{r_4}{r_5}$, for otherwise $c_{\chi_1\chi_2\chi_0}(m) = 0$ for all q; thus $r_3 \leq r_4/r_5$. These observations lead to the conclusion that the sum under consideration is

(5.5) $$\ll (r_1 r_2 r_4/r_5)^{1/2} \varphi(r_4)^{-1} \varphi(r_4/r_5)^{-1} \sum_{\substack{k=1 \\ (k, r_4)=1}}^{\infty} \mu(k)^2 \varphi(k)^{-1} \varphi\big(k/(k, m)\big)^{-1}.$$

Here the sum is

$$= \prod_{\substack{p \nmid r_4 \\ p \nmid m}} \left(1 + \frac{1}{(p-1)^2}\right) \prod_{\substack{p \nmid r_4 \\ p | m}} \left(1 + \frac{1}{p-1}\right) \ll \prod_{\substack{p \nmid r_4 \\ p | m}} \left(1 - \frac{1}{p}\right)^{-1}.$$

The other factor of (5.5) may be written as $\Pi_1 \cdot \Pi_2$, where

$$\Pi_1 = \frac{(r_4/r_5)^{1/2}}{\varphi(r_4/r_5)} \prod_{p | r_4, p \nmid m} p^{\frac{1}{2}a_1 + \frac{1}{2}a_2 - a_4} \left(1 - \frac{1}{p}\right)^{-1} \ll \prod_{p | r_4, p \nmid m} p^{-1/2} \left(1 - \frac{1}{p}\right)^{-2} \ll 1,$$

and

$$\Pi_2 = \prod_{p | r_5} p^{\frac{1}{2}a_1 + \frac{1}{2}a_2 - a_4} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{p | r_5} \left(1 - \frac{1}{p}\right)^{-1}.$$

We combine these three estimates to obtain the desired result.

6. The major arcs. For α in the major arc $\mathfrak{M}(q, a)$ we write $\alpha = \dfrac{a}{q} + \eta$. We have $q \leq P$, so if $p > P$ then $(p, q) = 1$, and it follows that

$$e(p\alpha) = \varphi(q)^{-1} \sum_\chi \chi(pa)\tau(\bar{\chi}) e(p\eta).$$

Thus in the notation of (2.2), (2.3) we find that

(6.1) $$S(\alpha) = \varphi(q)^{-1} \sum_\chi \chi(a)\tau(\bar{\chi}) S(\chi, \eta).$$

Note that the harmless condition $p > P$ ensures that $S(\chi, \eta) = S(\chi^*, \eta)$.

In general we expect $S(\chi, \eta)$ to be small, but if $\chi = \chi_0$ or $\chi = \tilde{\chi}\chi_0$ then we approximate to $S(\chi, \eta)$ by the corresponding expression

$$T(\eta) = \sum_{P < n \leq X} e(n\eta), \quad \tilde{T}(\eta) = -\sum_{P < n \leq X} n^{\tilde{\beta}-1} e(n\eta).$$

Of course $\tilde{T}(\eta)$ is defined only if there is an exceptional zero $\tilde{\beta}$. Put

$$S(\chi_0, \eta) = T(\eta) + W(\chi_0, \eta), \quad S(\tilde{\chi}\chi_0, \eta) = \tilde{T}(\eta) + W(\tilde{\chi}\chi_0, \eta),$$

and

$$S(\chi, \eta) = W(\chi, \eta) \quad (\chi \neq \chi_0, \chi \neq \tilde{\chi}\chi_0).$$

Thus also $W(\chi, \eta) = W(\chi^*, \eta)$ for any χ. By Lemma 5.2 we see that $\tau(\bar{\chi}_0) = \mu(q)$, so the above definitions give

(6.2) $$S(\alpha) = \frac{\mu(q)}{\varphi(q)} T(\eta) + \frac{1}{\varphi(q)} \sum_\chi \chi(a)\tau(\bar{\chi}) W(\chi, \eta),$$

unless there is an exceptional character of modulus $\tilde{r}$, in which case if $\tilde{r} | q$ then we obtain an additional term

(6.$\tilde{2}$) $$\frac{\tilde{\chi}(a)\tau(\tilde{\chi}\chi_0)}{\varphi(q)} \tilde{T}(\eta)$$

on the right hand side of (6.2).

Assume for the moment that the exceptional character does not occur. Then

$$(6.3)\qquad S(a)^2 = \left(\frac{\mu(q)T(\eta)}{\varphi(q)}\right)^2 + 2\mu(q)\varphi(q)^{-2}\sum_{\chi}\chi(a)\tau(\bar{\chi})T(\eta)W(\chi,\eta) + \\ + \varphi(q)^{-2}\sum_{\chi,\chi'}\chi\chi'(a)\tau(\bar{\chi})\tau(\bar{\chi}')\,W(\chi,\eta)W(\chi',\eta).$$

Hence

$$(6.4)\qquad \sum_a{}' \int_{\mathfrak{M}(q,a)} S(a)^2 e(-na)\,da = \mu(q)^2\varphi(q)^{-2}c_q(-n)\int_{-1/qQ}^{1/qQ} T(\eta)^2 e(-n\eta)\,d\eta + \\ +2\mu(q)\varphi(q)^{-2}\sum_{\chi} c_\chi(-n)\tau(\bar{\chi})\int_{-1/qQ}^{1/qQ} T(\eta)W(\chi,\eta)e(-n\eta)\,d\eta + \\ +\varphi(q)^{-2}\sum_{\chi,\chi'} c_{\chi\chi'}(-n)\tau(\bar{\chi})\tau(\bar{\chi}')\int_{-1/qQ}^{1/qQ} W(\chi,\eta)W(\chi',\eta)e(-n\eta)\,d\eta.$$

Here the first term contributes to our main term, and the others are remainder terms which we now estimate. Suppose that $\chi(\bmod q)$ is induced by $\chi^*(\bmod r)$. Put

$$(6.5)\qquad W(\chi) = \Big(\int_{-1/rQ}^{1/rQ} |W(\chi,\eta)|^2 d\eta\Big)^{1/2}.$$

We note that $W(\chi) = W(\chi^*)$, so the total (over $q \leqslant P$) major arc remainder is bounded by

$$(6.6)\qquad 2X^{1/2}\sum_{q\leqslant P}\mu(q)^2\varphi(q)^{-2}\sum_{\chi}|c_\chi(-n)\tau(\bar{\chi})|\,W(\chi^*) + \\ + \sum_{q\leqslant P}\varphi(q)^{-2}\sum_{\chi,\chi'}|c_{\chi\chi'}(-n)\tau(\bar{\chi})\tau(\bar{\chi}')|\,W(\chi^*)W(\chi'^*).$$

Here we have used the Cauchy–Schwarz inequality and the fact that

$$(6.7)\qquad \int_{-1/qQ}^{1/qQ}|T(\eta)|^2 d\eta \leqslant \int_0^1 |T(\eta)|^2 d\eta = \sum_{P<n\leqslant X} 1 \leqslant X.$$

We now group terms arising from fixed χ^* and χ'^*; by Lemma 5.5 we see that our error terms are

$$(6.8)\qquad \ll \frac{n}{\varphi(n)}(WX^{1/2}+W^2),$$

where

$$(6.9)\qquad W = \sum_{q\leqslant P}\sum_{\chi}{}^{*} W(\chi).$$

We now consider the first term on the right of (6.4). The integral in this term satisfies the trivial bound (6.7), but now we require a more precise estimate. Clearly $T(\eta) \ll \|\eta\|^{-1}$, so

$$(6.10)\qquad \int_{1/qQ}^{1/2} |T(\eta)|^2 d\eta \ll qQ.$$

Thus the integral under consideration is

$$(6.11)\qquad = \int_0^1 T(\eta)^2 e(-n\eta)\,d\eta + O(qQ) = n + O(qQ)$$

for $n \leqslant X$, so our total major arc main term is

$$(6.12)\qquad \sum_{q\leqslant P}\mu(q)^2\varphi(q)^{-2}c_q(-n)\big(n+O(qQ)\big).$$

Here by (5.2) the error term is

$$(6.13)\qquad \ll Q\sum_{q\leqslant P} q\varphi(q)^{-1}\varphi\big(q/(q,n)\big)^{-1} \ll Q\sum_{d|n} d\varphi(d)^{-1}\sum_{r\leqslant P} r\varphi(r)^{-2} \\ \ll Qn\varphi(n)^{-1}d(n)\log P \ll X^{1+\delta}P^{-1}.$$

In (6.12) the main term can be written as a sum over all $q \geqslant 1$, with an error of

$$(6.14)\qquad \ll n\sum_{q>P}\varphi(q)^{-1}\varphi\big(q/(q,n)\big)^{-1} \ll n\sum_{d|n}\varphi(d)^{-1}\sum_{r>P/d}\varphi(r)^{-2} \\ \ll nP^{-1}d(n)n\varphi(n)^{-1} \ll X^{1+\delta}P^{-1}$$

for $n \leqslant X$. Thus the first term on the right of (6.4) summed over $q \leqslant P$ becomes

$$(6.15)\qquad \mathfrak{S}(n)n + O(X^{1+\delta}P^{-1}),$$

where $\mathfrak{S}(n)$ is the singular series

$$(6.16)\qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty}\mu(q)^2\varphi(q)^{-2}c_q(-n) = \prod_{p\nmid n}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{p|n}\Big(1+\frac{1}{p-1}\Big),$$

by (5.2). Combining (6.4), (6.8), and (6.15), we find altogether that

$$(6.17)\qquad R_1(n) = \mathfrak{S}(n)n + O(X^{1+\delta}P^{-1}) + O\big(n\varphi(n)^{-1}(WX^{1/2}+W^2)\big)$$

for $n \leqslant X$, provided that the exceptional term does not occur.

We now suppose that the exceptional term exists, and proceed to determine the effect that the term (6.2) has in (6.2). Clearly the right hand side of (6.4) must be augmented by the amount

$$(6.\tilde{4})\qquad \tau(\tilde\chi\chi_0)^2\varphi(q)^{-2}c_q(-n)\int_{-1/qQ}^{1/qQ}\tilde T(\eta)^2e(-n\eta)\,d\eta+$$
$$+2\mu(q)c_{\tilde\chi\chi_0}(-n)\tau(\tilde\chi\chi_0)\varphi(q)^{-2}\int_{-1/qQ}^{1/qQ}T(\eta)\tilde T(\eta)e(-n\eta)\,d\eta+$$
$$+2\varphi(q)^{-2}\sum_{\chi}c_{\chi\tilde\chi}(-n)\tau(\bar\chi)\tau(\tilde\chi\chi_0)\int_{-1/qQ}^{1/qQ}W(\chi,\eta)\tilde T(\eta)e(-n\eta)\,d\eta.$$

We now treat this last term in the same way that we dealt with the second and third terms in (6.4). The total contribution of this term is no more than

$$(6.\tilde{6})\qquad 2X^{1/2}\sum_{\substack{q\le P\\ \tilde r|q}}\varphi(q)^{-2}\sum_{\chi}|c_{\chi\tilde\chi}(-n)\tau(\bar\chi)\tau(\tilde\chi\chi_0)|W(\chi^*),$$

since

$$(6.\tilde{7})\qquad \int_{-1/qQ}^{1/qQ}|\tilde T(\eta)|^2\,d\eta\leqslant X.$$

Applying Lemma 5.5 as before, we find that (6.$\tilde{6}$) is

$$\ll n\varphi(n)^{-1}X^{1/2}W,$$

which is absorbed by (6.8).

We now investigate the first two terms in (6.$\tilde{4}$). By partial summation we see that $\tilde T(\eta)\ll\|\eta\|^{-1}$, so in addition to (6.10) we have

$$(6.\tilde{10})\qquad \int_{1/qQ}^{1/2}|\tilde T(\eta)|^2\,d\eta\ll qQ,\qquad \int_{1/qQ}^{1/2}|T(\eta)\tilde T(\eta)|\,d\eta\ll qQ.$$

We now write the integrals in (6.$\tilde{4}$) as $\tilde I(n)+O(qQ)$ and $\tilde J(n)+O(qQ)$, where

$$\tilde I(n)=\int_0^1\tilde T(\eta)^2e(-n\eta)\,d\eta,\qquad \tilde J(n)=\int_0^1T(\eta)\tilde T(\eta)e(-n\eta)\,d\eta.$$

As in (6.7) we find that

$$(6.18)\qquad \tilde I(n)\leqslant X,\qquad |J(n)|\leqslant X.$$

(Later we estimate $\tilde I(n)$ more precisely.) Regarding (6.12), we find that we must introduce the terms

$$(6.\tilde{12})\qquad \sum_{\substack{q\le P\\ \tilde r|q}}\tau(\tilde\chi\chi_0)^2c_q(-n)\varphi(q)^{-2}\big(\tilde I(n)+O(qQ)\big)+$$
$$+2\sum_{\substack{q\le P\\ \tilde r|q}}\mu(q)\tau(\tilde\chi\chi_0)c_{\tilde\chi\chi_0}(-n)\varphi(q)^{-2}\big(\tilde J(n)+O(qQ)\big).$$

Here we treat the error terms as in (6.13). They are

$$(6.19)\qquad \ll\tilde rQ\sum_{\substack{q\le P\\ \tilde r|q}}q\varphi(q)^{-1}\varphi\big(q/(n,q)\big)^{-1}\ll X^{1+\delta}P^{-1}(n,\tilde r).$$

We now extend the sums in the main terms to include all $q\geqslant 1$. This introduces a further error, which in view of (6.14) and (6.18) is

$$\ll X\tilde r\sum_{\substack{q>P\\ \tilde r|q}}\varphi(q)^{-1}\varphi\big(q/(q,n)\big)^{-1}\ll X^{1+\delta}P^{-1}(n,r).$$

The first infinite sum is

$$(6.\tilde{16})\qquad \tilde{\mathfrak S}(n)=\sum_{\substack{q=1\\ \tilde r|q}}^{\infty}\tau(\tilde\chi\chi_0)^2c_q(-n)\varphi(q)^{-2}$$
$$=\tilde\chi(-1)\mu\big(\tilde r/(\tilde r,n)\big)\tilde r\varphi(\tilde r)^{-1}\varphi\big(\tilde r/(\tilde r,n)\big)^{-1}\prod_{\substack{p\nmid\tilde r\\ p\nmid n}}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\nmid\tilde r\\ p|n}}\Big(1+\frac{1}{p-1}\Big),$$

by Lemma 5.1 and (5.2). The second sum in (6.$\tilde{12}$), extended to infinity, is

$$(6.20)\qquad \sum_{\substack{q=1\\ \tilde r|q}}^{\infty}\mu(q)\tau(\tilde\chi\chi_0)c_{\tilde\chi\chi_0}(-n)\varphi(q)^{-2}$$
$$=\mu(\tilde r)\tilde\chi(n)\tilde r\varphi(r)^{-2}\prod_{\substack{p\nmid\tilde r\\ p\nmid n}}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\nmid\tilde r\\ p|n}}\Big(1+\frac{1}{p-1}\Big)$$
$$\ll\tilde\chi(n)^2\tilde r\varphi(\tilde r)^{-2}n\varphi(n)^{-1}.$$

Collecting our estimates (6.17)–(6.20), we find that if the exceptional term occurs then instead of (6.17) we have

$$(6.\tilde{17})\qquad R_1(n)=\mathfrak S(n)n+\tilde{\mathfrak S}(n)\tilde I(n)+O\Big(\frac{\tilde\chi(n)^2\tilde rnX}{\varphi(\tilde r)^2\varphi(n)}\Big)+$$
$$+O\big(X^{1+\delta}P^{-1}(n,\tilde r)\big)+O\big(n\varphi(n)^{-1}(X^{1/2}W+W^2)\big),$$

for $n\leqslant X$.

To complete our description of the main terms arising from the major arcs it remains to derive a sharp upper bound for $\tilde I(n)$. Clearly

$$\tilde I(n)=\sum_{P<k<n-P}\big(k(n-k)\big)^{\tilde\beta-1}\leqslant n\cdot n^{\tilde\beta-1}=n^{\tilde\beta},$$

and

$$n-n^{\tilde\beta} \geqslant \int\limits_{\max(\tilde\beta, 1-\frac{1}{\log X})}^{1} n^u(\log n)\,du$$
$$\geqslant \Big(1-\max\Big(\tilde\beta, 1-\frac{1}{\log X}\Big)\Big) n\exp\Big(-\frac{\log n}{\log X}\Big)\log n$$
$$\geqslant \begin{cases}(1-\tilde\beta)n\log n & (\tilde\beta \geqslant 1-\frac{1}{\log X}),\\ n\frac{\log n}{\log X} & (\tilde\beta < 1-\frac{1}{\log X}).\end{cases}$$

Thus, by (4.1),

(6.21) $$n-n^{\tilde\beta} > c_5(1-\tilde\beta)n\log P \quad (\tfrac12 X < n \leqslant X).$$

7. The major arc error terms. We now estimate

$$W = \sum_{q\leqslant P}\sum_{\chi}{}^{*} W(\chi),$$

where $W(\chi)$ is defined in (6.5). One reasonable approach would be to use an explicit formula to relate $R(\chi,\eta)$ to zeros of $L(s,\chi)$, and then appeal to an appropriate zero density estimate (namely Theorem 6 of Gallagher [5]). We choose a route which is conceptually more sophisticated but technically simpler.

By Lemma 4.2 we see that

$$W(\chi) \ll \Big(\int\limits_0^{2X}\Big|\frac{1}{qQ}\sum_{\substack{P<p\leqslant X\\ x-\frac12 qQ<p\leqslant x}}{}^{\#}\chi(p)\log p\Big|^2 dx\Big)^{1/2}$$
$$\ll X^{1/2}\max_{x\leqslant 2X}\max_{0<h\leqslant X}(h+XP^{-1})^{-1}\Big|\sum_{x-h}^{x}{}^{\#}\chi(p)\log p\Big|$$

in the notation of Lemma 4.3, an application of which gives

(7.1) $$W \ll X^{1/2}\exp\Big(-c_6\frac{\log X}{\log P}\Big)$$

if there is no exceptional term. If the exceptional term occurs then

(7.$\tilde 1$) $$W \ll X^{1/2}(1-\tilde\beta)\exp\Big(-c_6\frac{\log X}{\log P}\Big)\log P.$$

8. Completion of the proof of Theorem 1. We have already observed that n is a sum of two primes if $R(n) > 0$. Now $R(n) \geqslant R_1(n) - |R_2(n)|$, so n is representable if

(8.1) $$R_1(n) > |R_2(n)|.$$

We now show that this inequality holds for even n, $\frac12 X < n \leqslant X$, with the exception of at most

(8.2) $$\ll XP^{-1/3}\log^{35}X = X^{1-2\delta}\log^{35}X$$

values of n, $\frac12 X < n \leqslant X$. Then Theorem 1 is immediate.

From (3.2) we see that the number of $n \leqslant X$ for which $|R_2(n)| > XP^{-1/3}$ is at most $\ll XP^{-1/3}\log^{35}X$. We may discard such n, in view of (8.2), so that $|R_2(n)| \leqslant XP^{-1/3}$ for our remaining n. We now show that

(8.3) $$R_1(n) > XP^{-1/3}$$

for even n, $\frac12 X < n \leqslant X$, with the exception of $\ll XP^{-1/3}$ values of n. This suffices to complete the proof, since the exceptional n can be absorbed in (8.2).

We suppose first that there is no exceptional character. Then from (6.17) and (7.1) it follows that

$$R_1(n) = \mathfrak{S}(n)n + O\big(n\varphi(n)^{-1}X\exp(-c_7\delta^{-1})\big) \gg n\varphi(n)^{-1}X \gg X$$

for even n, $\frac12 X < n \leqslant X$, supposing that δ is sufficiently small. This gives (8.3) without exception.

If there is an exceptional character then we appeal to (6.$\tilde{17}$) and (7.$\tilde 1$). If $(n,\tilde r) = 1$ then, by (6.$\tilde{16}$),

$$\tilde{\mathfrak{S}}(n) \ll n\varphi(n)^{-1}\tilde r\varphi(r)^{-2} = o(1),$$

since it follows from Lemma 4.1 that $\tilde r \gg \log P$. Thus $R_1(n) \gg X$ for all even n with $(n,\tilde r) = 1$, $\frac12 X < n \leqslant X$. If $(n,\tilde r) > 1$ then the first error term in (6.$\tilde{17}$) vanishes, but now the second error term may be large. To cope with this we now discard those even n for which $(n,\tilde r) > P^{1/2}$. Then for the remaining n this error term is

$$\ll X^{1+\delta}P^{-1}(n,\tilde r) \ll X^{1+\delta}P^{-1/2} \ll XP^{-1/3}.$$

Moreover, the number of discarded n is

$$\sum_{\substack{d|\tilde r\\ d>P^{1/2}}}\sum_{\substack{n\leqslant X\\ d|n}}1 \ll XP^{-1/2}d(\tilde r) \ll XP^{-1/3},$$

which is admissible, in view of (8.2). Thus it remains to treat those n with $1 < (n,\tilde r) \leqslant P^{1/2}$. For these, by (6.$\tilde{17}$), (7.1) and (2.4),

(8.4) $$R_1(n) = \mathfrak{S}(n)n + \tilde{\mathfrak{S}}(n)\tilde I(n) + O(XP^{-1/3}) + O\big(n\varphi(n)^{-1}(1-\tilde\beta)X\exp(-c_7\delta^{-1})\log P\big).$$

We now consider $\tilde{\mathfrak{S}}(n)$. From (6.16) and (6.15), we see that

$$|\tilde{\mathfrak{S}}(n)| \leqslant \mathfrak{S}(n) \prod_{\substack{p|\tilde r \\ p\nmid n \\ p>3}} (p-2)^{-1}. \tag{8.5}$$

If the product is non-empty then as before

$$R_1(n) \gg n\varphi(n)^{-1}X \gg X$$

for even n, $\frac{1}{2}X < n \leqslant X$. On the other hand, if the product is empty then, by Lemma 5.1, $(n, \tilde r) \geqslant \frac{1}{24}\tilde r$. But the n under consideration satisfy $(n, \tilde r) \leqslant P^{1/2}$, so the present case arises only if

$$\tilde r \ll P^{1/2}. \tag{8.6}$$

By (6.21) and (8.5) we deduce that

$$\mathfrak{S}(n)n + \tilde{\mathfrak{S}}(n)\tilde I(n) \geqslant c_5\mathfrak{S}(n)(1-\tilde\beta)n\log P \geqslant c_8 n\varphi(n)^{-1}(1-\tilde\beta)X\log P,$$

for even n, $\frac{1}{2}X < n \leqslant X$. The last error term in (8.4) is less than half this size if δ is sufficiently small, so

$$R_1(n) \geqslant c_9 n\varphi(n)^{-1}(1-\tilde\beta)X\log P - c_{10}XP^{-1/3}.$$

By Lemma 4.1 and (8.6) we see that

$$1-\tilde\beta \gg \tilde r^{-1/2}\log^{-2}\tilde r \gg P^{-1/4}\log^{-2}P.$$

Thus

$$R_1(n) \gg XP^{-1/4}\log^{-1}P > XP^{-1/3},$$

as required.

One should note that this concluding argument can be arranged rather differently: Take $P = X^{12\delta}$ if the exceptional term does not occur, or if it does and $\tilde r \leqslant X^{6\delta}$. On the other hand, if $X^{6\delta} < r \leqslant X^{12\delta}$ then take $P = X^{6\delta}$. In this way we ensure that $\tilde r \leqslant P^{1/2}$ whenever $\tilde r \leqslant P$. Then the treatment of the exceptional case is somewhat simplified.

9. Proof of Theorem 2. We require the following two lemmas.

LEMMA 9.1. *If $X > X_0(\varepsilon)$ and $X^{7/12+\varepsilon} < h < X$ then the interval $(X, X+h)$ contains $\sim h\log^{-1}X$ primes.*

The first result of this character was proved by Hoheisel. The present form is due to Huxley [8], whose basic result is the zero density estimate

$$N(\sigma, T) \ll T^{\frac{12}{5}(1-\sigma)}\log^9 T, \tag{9.1}$$

valid for $\frac{1}{2} \leqslant \sigma \leqslant 1$. The exponent $7/12$ arises as $1-c^{-1}$, where $c = 12/5$ is the constant in the exponent in (9.1).

LEMMA 9.2. *If the Riemann Hypothesis is true then*

$$\int_{\frac{1}{2}Y}^{Y}\Big(\sum_{y}^{y+\theta y}\log p - \theta y\Big)^2 dy \ll \theta Y^2\log^2 Y \tag{9.2}$$

for $0 \leqslant \theta \leqslant 1$. Unconditionally,

$$\int_{\frac{1}{2}Y}^{Y}\Big(\sum_{y}^{y+\theta y}\log p - \theta y\Big)^2 dy \ll \theta^2 Y^3\log^{-10} Y \tag{9.3}$$

provided that $\theta Y > Y^{1/6+\varepsilon}$.

The first assertion is due to Selberg [16]. Selberg also used a zero density estimate to establish an unconditional result; his analysis with (9.1) yields (9.3). Here the exponent $1/6$ occurs as $1-2/c$, where c is the constant in the exponent in (9.1).

We now prove Theorem 2. Suppose that the interval $(X, X+h)$ contains no sum of two prime numbers. Let $Y = X^{7/12+\varepsilon}$. Then by Lemma 9.1 the interval $(X-Y, X-\frac{1}{2}Y)$ contains $\gg Y\log^{-1}X$ primes. For such a prime p the interval $(X-p, X-p+h)$ contains no prime number. Thus the interval $(y, y+\frac{1}{2}h)$ contains no prime for a set of y, $\frac{1}{2}Y \leqslant y \leqslant Y$, with measure $\gg Y\log^{-1}X$. We take $\theta = \frac{1}{2}hY^{-1}$ in (9.3), and deduce that $h \ll Y^{1/6+\varepsilon}$. That is, $h < X^{7/72+\varepsilon}$. Here the constant $7/72$ arises as $\left(1-\dfrac{1}{c}\right)\left(1-\dfrac{2}{c}\right)$, where c is the exponent in (9.1).

Suppose now that the Riemann Hypothesis is true and that the interval $(X, X+h)$ contains no sum of two prime numbers. Then for each $y \leqslant X$ at most one of the intervals $(y, y+\frac{1}{2}h)$, $(X-y, X-y+\frac{1}{2}h)$ contains a prime number. Thus of the intervals

$$\big(\tfrac{1}{2}X+\tfrac{1}{2}kh,\ \tfrac{1}{2}X+\tfrac{1}{2}(k+1)h\big)$$

with $-\frac{1}{2}Xh^{-1} < k < \frac{1}{2}Xh^{-1}$, at least $\frac{1}{3}Xh^{-1}$ of them contain no prime number, and hence the interval $(y, y+\frac{1}{4}h)$ contains no prime for a set of y, $\frac{1}{4}X < y \leqslant X$, of measure $\gg X$. From (9.2) with $Y = X$, $\theta = \frac{1}{4}hX^{-1}$ we see that $h \ll \log^2 X$.

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On an inequality for additive arithmetic functions

by

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In memory of Yu. V. Linnik

G. H. Hardy and S. Ramanujan [2] proved that for any fixed $\delta > 0$ and all positive integers $m \leqslant n$, with a possible exception of $o(n)$ of them, the inequality

$$|\omega(m) - \ln\ln n| < (\ln\ln n)^{1/2+\delta}$$

is true. Here $\omega(m)$ denotes the number of different prime divisors of m. This is an analogue of the probabilistic weak law of large numbers. It shows the bounds between which the function $\omega(m)$ oscillates for the great majority of values of the argument.

P. Turán [5], [6] gave a very simple derivation of this statement. He proved the elementary inequality

$$\sum_{m=1}^{n} (\omega(m) - \ln\ln n)^2 \leqslant c_1 n \ln\ln n, \tag{1}$$

where c_1 is a constant, which evidently implies the result of Hardy and Ramanujan.

Naturally there arose a question of the generalization of (1) to a larger class of arithmetic functions. P. Turán [7] obtained the following theorem. Let $f(m)$ be a real-valued strongly additive function such that

$$0 \leqslant f(p) \leqslant K$$

for all primes p and a constant K and

$$M_n = \sum_{p \leqslant n} \frac{f(p)}{p} \to \infty$$

as $n \to \infty$. Then the inequality

$$\sum_{m=1}^{n} \big(f(m) - M_n\big)^2 \leqslant c_2 n M_n$$

holds, where c_2 is a constant depending on K.