§ 2. Proof of the Main Theorem. By Theorem 1 there exist positive real numbers $\varepsilon_0 = \varepsilon_0(d, D) < 1$, $L_0 = L_0(d, D)$ such that for $L > L_0$ we have

$$\begin{aligned}\Sigma^* &= \sum_{p \leqslant n} N[F = n-p] \\ &= \left(h_f^{-1} + O(\varepsilon_0^L)\right) \sum_{m \in \mathcal{M}} \sum_{l \mid m} \chi(l) + O\Big(\sum_{C \in I^d / I_1^d} \sum_{m \in \mathcal{M}_{C,L}} \sum_{l \mid m} \chi(l)\Big).\end{aligned}$$

Hence for $L_0 < L < \log\log n$ we obtain from Theorem 2 and 3

$$\begin{aligned}&\Sigma^* - \frac{2}{h_f} L(1, \chi) \prod_{p \mid d(D-dn)} \left(1 - \frac{\chi(p)}{p}\right) \prod_{p \nmid d(D-dn)} \left(1 + \frac{\chi(p)}{p(p-1)}\right) \frac{n}{\log n} \\ &\quad \ll \varepsilon_0^L \frac{n}{\log n} \log\log n + \frac{n}{\log^{1+\delta} n} \log^5 \log n + \frac{n}{\log^{1+1/h_f} n} \left(\frac{A}{L} \log\log n\right)^L.\end{aligned}$$

On putting $L = \dfrac{1}{A h_f^2} \log\log n$ we get the required estimate.

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The proof of Minkowski's conjecture concerning the critical determinant of the region $|x|^p + |y|^p < 1$ for $p \geqslant 6$

by

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1. Introduction. Let $p > 1$ be a real number, $\mathscr{D}_p$ be the convex region

$$|x|^p + |y|^p < 1$$

and $\Delta(\mathscr{D}_p)$ be the critical determinant of $\mathscr{D}_p$ (for definition of the necessary notions from the geometry of numbers see Cassels [1]). Let us consider two $\mathscr{D}_p$-admissible lattices $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$. $\Lambda_p^{(0)}$ as well as $\Lambda_p^{(1)}$ has six points on the boundary of $\mathscr{D}_p$ and $(1, 0) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$. (The lattices $\Lambda_p^{(0)}$, $\Lambda_p^{(1)}$ are defined uniquely under those conditions.) We write $\Delta_p^{(0)}$, $\Delta_p^{(1)}$ for $d(\Lambda_p^{(0)})$, $d(\Lambda_p^{(1)})$. Minkowski [4] had conjectured that

$$\Delta(\mathscr{D}_p) = \min(\Delta_p^{(0)}, \Delta_p^{(1)}), \tag{1}$$

all critical lattices of $\mathscr{D}_p$ being contained among the lattices $\Lambda_p^{(0)}$, $\Lambda_p^{(1)}$ and among those which are symmetrical to $\Lambda_p^{(0)}$, $\Lambda_p^{(1)}$ with respect to lines $x = 0$, $y = 0$, $x = y$, $x = -y$.

Papers [2], [3], [5]–[9] are devoted to this conjecture. Watson [6] has proved that there exists a constant p_0, with $2.57 < p_0 < 2.58$, such that

$$\Delta_{p_0}^{(0)} = \Delta_{p_0}^{(1)} \tag{2}$$

and

$$\begin{aligned}\Delta_p^{(1)} &< \Delta_p^{(0)} \quad \text{for} \quad 1 < p < 2,\ p > p_0, \\ \Delta_p^{(0)} &< \Delta_p^{(1)} \quad \text{for} \quad 2 < p < p_0.\end{aligned} \tag{3}$$

Therefore the conjectural equality (1) can be written as

$$\Delta(\mathscr{D}_p) = \begin{cases} \Delta_p^{(1)} & \text{for} \quad 1 \leqslant p \leqslant 2,\ p \geqslant p_0, \\ \Delta_p^{(0)} & \text{for} \quad 2 \leqslant p \leqslant p_0. \end{cases} \tag{4}$$

Cohn [2] introduced a convenient parametrization of the problem (which we also use, see § 2) and gave a sketch of the proof of Minkowski's conjecture when p is "sufficiently large". Mordell [5] has proved this conjecture for $p = 4$ (the cases $p = 2$ and $p = 1$ are trivial). Kukharev [8] (the preliminary report—[7]) has worked out a method for the examination of Minkowski's conjecture for every concrete p (except p near 1, 2 and p_0) and using a computer has proved this conjecture for $p = 1.3; 1.4; 1.5; 1.6; 1.7; 2.2; 2.3; 3; 4; 5$.

The aim of this work is to prove Minkowski's conjecture for $p \geqslant 6$. That is, we are to prove the following statement.

THEOREM 1. *For $p \geqslant 6$ we have*

$$\Delta(\mathscr{D}_p) = \Delta_p^{(1)} \tag{5}$$

and the set of critical lattices of $\mathscr{D}_p$ consists of $\Lambda_p^{(1)}$ and lattices which are symmetrical with respect to coordinate axes and their bisectors.

2. Reformulation of the problem. Let for fixed $p > 1$ a value σ increase from 1 to $\sigma_p = (2^p-1)^{1/p}$. The equation

$$\{(1+\tau^p)^{-1/p}-(1+\sigma^p)^{-1/p}\}^p+\{\tau(1+\tau^p)^{-1/p}+\sigma(1+\sigma^p)^{-1/p}\}^p = 1 \tag{6}$$

determines a function $\tau = \tau(\sigma, p) \geqslant 0$, which is decreasing from τ_p to 0; here τ_p is defined by

$$2(1-\tau_p)^p = 1+\tau_p^p, \quad 0 < \tau_p < 1, \tag{7}$$

(see [2]). We introduce the function

$$\Delta(\sigma, p) = (\tau+\sigma)(1+\tau^p)^{-1/p}(1+\sigma^p)^{-1/p} \tag{8}$$

where $\tau = \tau(\sigma, p)$. Then $\Delta(\sigma, p)$ is the determinant of $\mathscr{D}_p$-admissible lattice $\Lambda_p(\sigma)$, which has six points on the boundary of $\mathscr{D}_p$, one of them having tangent coefficient $-\sigma$; under those conditions $\Lambda_p(1) = \Lambda_p^{(1)}$, $\Delta(1, p) = \Delta_p^{(1)}$; $\Lambda_p(\sigma_p) = \Lambda_p^{(0)}$, $\Delta(\sigma_p, p) = \Delta_p^{(0)}$ (see [2]). Theorem 1 is equivalent to the following statement.

THEOREM 2. *If $p \geqslant 6, 1 < \sigma \leqslant \sigma_p$, then*

$$\Delta(\sigma, p) > \Delta(1, p) = \Delta_p^{(1)}. \tag{9}$$

We write

$$(1+\tau^p)^{-1/p}-(1+\sigma^p)^{-1/p} = A(\sigma, p) = A, \tag{10}$$

$$\tau(1+\tau^p)^{-1/p}+\sigma(1+\sigma^p)^{-1/p} = B(\sigma, p) = B, \tag{11}$$

where $\tau = \tau(\sigma, p)$ is defined by (6), which can now be rewritten as:

$$A^p+B^p = 1. \tag{12}$$

Differentiating (12) we find

$$\frac{\partial\tau}{\partial\sigma} = -\frac{(1+\tau^p)^{1+1/p}}{(1+\sigma^p)^{1+1/p}}\,\frac{B^{p-1}+\sigma^{p-1}A^{p-1}}{B^{p-1}-\tau^{p-1}A^{p-1}} \tag{13}$$

and so

$$\frac{\partial\Delta(\sigma, p)}{\partial\sigma} = -\frac{g(\sigma, p)}{(1+\sigma^p)^{1+1/p}(B^{p-1}-\tau^{p-1}A^{p-1})} \tag{14}$$

where

$$g(\sigma, p) = (1+\sigma^p)^{-1/p}(1-\sigma\tau^{p-1})(B^{p-1}+\sigma^{p-1}A^{p-1})-\\ -(1+\tau^p)^{-1/p}(1-\tau\sigma^{p-1})(B^{p-1}-\tau^{p-1}A^{p-1}). \tag{15}$$

From (14) it follows that

$$\operatorname{sign}\frac{\partial\Delta(\sigma, p)}{\partial\sigma} = -\operatorname{sign} g(\sigma, p). \tag{16}$$

We are beginning to prove Theorem 2 (from which Theorem 1 follows). After some lemmas from § 3, we shall prove that

$$\frac{\partial g(\sigma, p)}{\partial\sigma} < 0 \quad \text{for} \quad p \geqslant 6,\ 1 \leqslant \sigma \leqslant 1+\frac{1}{5p} \tag{17}$$

(§ 4, Theorem 3); that

$$g(\sigma, p) < 0 \quad \text{for} \quad p \geqslant 6,\ 1+\frac{1}{5p} \leqslant \sigma \leqslant 1+\frac{1.37}{\sqrt{p}} \tag{18}$$

(§ 5, Theorem 4); that

$$\Delta(\sigma, p) > \Delta_p^{(1)} \quad \text{for} \quad p \geqslant 6,\ 1+\frac{1.37}{\sqrt{p}} \leqslant \sigma \leqslant \sigma_p \tag{19}$$

(§ 6, Theorem 5); that

$$g(1, p) = 0 \quad \text{for} \quad p > 1 \tag{20}$$

(§ 7, Theorem 6). It follows from (17)–(20) that (9) holds for $p \geqslant 6, 1 < \sigma \leqslant \sigma_p$.

3. Lemmas.

LEMMA 1. *When p is fixed, while σ increases from 1 to σ_p:*

(a) $(1+\sigma^p)^{-1/p}$ *decreases from* $2^{-1/p}$ *to* $\frac{1}{2}$;

(b) $\sigma(1+\sigma^p)^{-1/p}$ *increases from* $2^{-1/p}$ *to* $\frac{1}{2}\sigma_p$;

(c) τ *decreases from* τ_p *to* 0;

(d) $(1+\tau^p)^{-1/p}$ *increases from* $(1+\tau_p^p)^{-1/p} = 2^{-1/p}(1-\tau_p)^{-1}$ *to* 1;

(e) $\tau(1+\tau^p)^{-1/p}$ *decreases from* $\tau_p(1+\tau_p^p)^{-1/p} = 2^{-1/p}\tau_p(1-\tau_p)^{-1}$ *to* 0;

(f) A *increases from* $\tau_p(1+\tau_p^p)^{-1/p} = 2^{-1/p}\tau_p(1-\tau_p)^{-1}$ *to* $\frac{1}{2}$;

(g) B *decreases from* $(1+\tau_p^p)^{-1/p} = 2^{-1/p}(1-\tau_p)^{-1}$ *to* $\frac{1}{2}\sigma_p$.

These statements follow from (6) and (12). The boundary values can be obtained by applying (7).

LEMMA 2. *For every*[1] *p and σ*

$$\tau < 1-(1+\sigma^{-p})^{-1/p}. \tag{21}$$

As it follows from Lemma 1,

$$B \leqslant (1+\tau_p)^{-1/p}, \quad \tau\left(\frac{1+\tau_p^p}{1+\tau^p}\right)^{1/p} + \sigma(1+\sigma^p)^{-1/p}(1+\tau_p^p)^{1/p} \leqslant 1,$$
$$\tau+\sigma(1+\sigma^p)^{-1/p} < 1.$$

COROLLARY 1. *For every p and σ*

$$(1+\sigma^p)^{-1/p} < \frac{1-\tau}{\sigma}. \tag{22}$$

It is evident that (22) is the same as (21).

COROLLARY 2. *For every p and σ*

$$\tau < \frac{\log(1+\sigma^{-p})}{p} < \frac{1}{p\sigma^p}. \tag{23}$$

If we take into account that for $x > 0$

$$x\left(1-\frac{x}{2}\right) < 1-e^{-x} < x, \tag{24}$$

then for $\alpha > 0$, $\beta > 0$

$$1-(1+\alpha)^{-\beta} < \beta\log(1+\alpha), \tag{25}$$

so the first inequality follows from (21). The second inequality follows from the first because for $x > 0$ we have

$$\log(1+x) < x. \tag{26}$$

COROLLARY 3. *For $p > 1$*

$$\tau \leqslant \tau_p < 1-2^{-1/p} < \frac{\log 2}{p} < \frac{0.7}{p}. \tag{27}$$

See (21) and (23).

LEMMA 3. *For $p \geqslant 6$*

$$(1+\tau^p)^{-1/p} > 1-\frac{\tau^p}{p} > 1-\frac{2\cdot 10^{-6}}{p}, \tag{28}$$

$$(1+\tau^p)^{-1-1/p} \geqslant (1+\tau^p)^{-1-2/p} > 1-\left(1+\frac{2}{p}\right)\tau^p > 1-\frac{2\cdot 10^{-5}}{p}. \tag{29}$$

[1] From now on we suppose that $p > 1$, $1 \leqslant \sigma \leqslant \sigma_p$.

Applying (25) and (27) we find that

$$(1+\tau^p)^{-1/p} > 1-\frac{\tau^p}{p} > 1-\frac{\tau_p^p}{p} > 1-\frac{(0.11)^6}{p} > 1-\frac{2\cdot 10^{-6}}{p};$$

$$(1+\tau^p)^{-1-2/p} > 1-\left(1+\frac{2}{p}\right)\tau^p > 1-\left(1+\frac{2}{6}\right)\frac{\log 2}{p}\tau_6^5 > 1-\frac{2\cdot 10^{-5}}{p}.$$

LEMMA 4. *For every p and σ*

$$A < 1+(1+\sigma^p)^{-1/p} < \frac{\log(1+\sigma^p)}{p}. \tag{30}$$

See (10) and (25).

LEMMA 5. *If $1 \leqslant \sigma \leqslant 1+\frac{1}{5p}$ then*

$$1 \leqslant \sigma^{p-3} \leqslant \sigma^{p-2} \leqslant \sigma^{p-1} \leqslant \sigma^p < e^{1/5} < 1.2215. \tag{31}$$

It follows from the inequality that

$$\left(1+\frac{\alpha}{p}\right)^p < e^{\alpha}, \quad \alpha > 0,\ p \geqslant 1. \tag{32}$$

COROLLARY. *If $1 \leqslant \sigma \leqslant 1+\frac{1}{5p}$ then*

$$A < \frac{\log(1+e^{1/5})}{p} < \frac{0.8}{p}. \tag{33}$$

See (30) and (31).

LEMMA 6. *For every p and σ*

$$1 > B \geqslant B^{p-2} \geqslant B^{p-1} \geqslant B^p = 1-A^p > 1-\left\{\frac{\log(1+\sigma^p)}{p}\right\}^p. \tag{34}$$

See (12) and (30).

COROLLARY. *If $p \geqslant 6$, $1 \leqslant \sigma \leqslant 1+\frac{1}{5p}$ then*

$$1 \geqslant B \geqslant B^{p-2} \geqslant B^{p-1} \geqslant B^p > 1-\frac{3.4\cdot 10^{-5}}{p} > 1-10^{-5}. \tag{35}$$

See (34) and (33).

LEMMA 7. *For every p and σ*

$$B^{p-1}+\sigma^{p-1}A^{p-1} < 1+\frac{\{\sigma\log(1+\sigma^p)\}^{p-1}}{p^{p-1}}. \tag{36}$$

See (34) and (30).

COROLLARY. *If* $p \geq 6$, $1 \leq \sigma \leq 1+\dfrac{1}{5p}$ *then*

$$B^{p-1}+\sigma^{p-1}A^{p-1} < 1+\frac{3.1\cdot 10^{-4}}{p} < 1+5.2\cdot 10^{-5}. \tag{37}$$

See (36) and (31).

LEMMA 8. *For every* p *and* σ

$$\begin{aligned} B^{p-1}-\tau^{p-1}A^{p-1} &> 1-\left\{\frac{\log(1+\sigma^p)}{p}\right\}^p-\tau^{p-1}\left\{\frac{\log(1+\sigma^p)}{p}\right\}^{p-1} \\ &> 1-(1+\tau^{p-1})\left\{\frac{\log(1+\sigma^p)}{p}\right\}^{p-1}. \end{aligned} \tag{38}$$

See (34), (30).

COROLLARY. *If* $p \geq 6$, $1 \leq \sigma \leq 1+\dfrac{1}{5p}$ *then*

$$B^{p-1}-\tau^{p-1}A^{p-1} > 1-\frac{4\cdot 10^{-5}}{p} > 1-10^{-5}. \tag{39}$$

See (38), (31).

LEMMA 9. *For* $p \geq 6$

$$\tau+\sigma(1+\sigma^p)^{-1/p}-\frac{3\cdot 10^{-7}}{p} < B < \tau+\sigma(1+\sigma^p)^{-1/p}. \tag{40}$$

See (11) and (28).

LEMMA 10. *For* $p \geq 6, 1 \leq \sigma \leq 1+\dfrac{1}{5p}$

$$(1+\sigma^p)^{-1-1/p} < \left|\frac{\partial\tau}{\partial\sigma}\right| < (1+\sigma^p)^{-1-1/p}\left(1+\frac{4\cdot 10^{-4}}{p}\right) \tag{41}$$

$$< \tfrac{1}{2}(1+10^{-4}). \tag{42}$$

See (13), (27), (37) and (39).

4. The case $1 \leq \sigma \leq 1+\dfrac{1}{5p}$**.**

THEOREM 3. *If* $p \geq 6, 1 \leq \sigma \leq 1+\dfrac{1}{5p}$ *then*

$$\frac{\partial g(\sigma, p)}{\partial\sigma} < 0. \tag{43}$$

Proof. Differentiating (15) and applying (13) we find that

$$\frac{\partial g(\sigma, p)}{\partial\sigma} < u(\sigma, p)+r(\sigma, p), \tag{44}$$

where

$$\begin{aligned} u(\sigma, p) = &-\sigma^{p-1}(1+\sigma^p)^{-1-1/p}B^{p-1}+(p-1)(1+\sigma^p)^{-1-2/p}B^{p-2}- \\ &-(p-1)(1+\tau^p)^{-1-1/p}(1+\sigma^p)^{-1/p}B^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|- \\ &-\sigma^{p-1}(1+\sigma^p)^{-1-1/p}B^{p-1}+(p-1)\tau\sigma^{p-2}B^{p-1}- \\ &-(p-1)(1+\tau^p)^{-1/p}(1+\sigma^p)^{-1-1/p}B^{p-2}+ \\ &+(p-1)\tau\sigma^{p-1}(1+\sigma^p)^{-1-1/p}B^{p-2}+(p-1)B^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|- \\ &-(p-1)\tau\sigma^{p-1}(1+\tau^p)^{-1-2/p}B^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|, \end{aligned} \tag{45}$$

$$\begin{aligned} r(\sigma, p) = &\tau^{p-1}B^{p-1}+(p-1)\tau^{p-2}(B^{p-1}+\sigma^{p-1}A^{p-1})\left|\frac{\partial\tau}{\partial\sigma}\right|+ \\ &+(p-1)\tau^{p-1}B^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|+(p-1)\sigma^{p-3}A^{p-1}+(p-1)\sigma^{p-4}A^{p-2}+ \\ &+(p-1)\tau^{p-1}A^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|+(p-1)\tau^{p-1}A^{p-2}+ \\ &+(p-1)\tau^{2(p-1)}A^{p-2}\left|\frac{\partial\tau}{\partial\sigma}\right|. \end{aligned} \tag{46}$$

Due to (28), (29), (40), (41) and (27) we can write that

$$u(\sigma, p) < B^{p-2}v(\sigma, p)+s(\sigma, p), \tag{47}$$

where

$$\begin{aligned} v(\sigma, p) = &-2\sigma^p(1+\sigma^p)^{-1-2/p}+(p-1)\tau\sigma^{p-1}(1+\sigma^p)^{-1/p}- \\ &-2\tau\sigma^{p-1}(1+\sigma^p)^{-1-1/p}+(p-1)\tau^2\sigma^{p-2}, \end{aligned} \tag{48}$$

$$\begin{aligned} s(\sigma, p) = &\frac{3\cdot 10^{-7}}{p}+(p-1)\frac{2\cdot 10^{-5}}{p}\cdot\frac{1}{2}+\frac{3\cdot 10^{-7}}{p}+(p-1)\frac{2\cdot 10^{-6}}{p}\cdot\frac{1}{2}+ \\ &+(p-1)\cdot\frac{1}{2}\cdot\frac{10^{-3}}{p}+(p-1)\cdot\frac{0.7}{p}\cdot\frac{2\cdot 10^{-5}}{p} < 10^{-3}. \end{aligned} \tag{49}$$

Since due to (27) and (31) we have

$$-2\tau\sigma^{p-1}(1+\sigma^p)^{-1-1/p}+(p-1)\tau^2\sigma^{p-2} < \frac{\tau\sigma^{p-2}}{(1+\sigma^p)^{1+1/p}}\{-2\cdot 1+0.7\cdot 1.2215^{7/6}\}<0,$$

so that

$$v(\sigma,p) < -2\sigma^p(1+\sigma^p)^{-1-2/p}+0.7\,\sigma^{p-1}(1+\sigma^p)^{-1/p}=w(\sigma,p).$$

We have

$$\frac{\partial w(\sigma,p)}{\partial\sigma}=\sigma^{p-2}(1+\sigma^p)^{-2-2/p}h(\sigma,p),$$

where

$$h(\sigma,p) = -2p\sigma+4\sigma^{p+1}+0.7(1+\sigma^p)^{1+1/p}\{(p-1)(\sigma^p+1)-\sigma^p\} > -2p\left(1+\frac{1}{5p}\right)+4+0.7\cdot 2\cdot(2p-3)>0.$$

Therefore,

$$v(\sigma,p)<w(\sigma,p)\leqslant w\left(1+\frac{1}{5p},p\right) \tag{50}$$
$$<\left(1+\frac{1}{5p}\right)^{p-1}\left\{1+\left(1+\frac{1}{5p}\right)^p\right\}^{-1/p}\{-2\cdot 1\cdot(1+e^{1/5})^{-1-1/6}+0.7\}$$
$$< -\left(1+\frac{1}{30}\right)^5(1+e^{1/5})^{-1/6}\cdot 0.085< -0.085.$$

From (46), (27), (34), (37), (42), (31) and (33) we conclude that

$$r(\sigma,p)<3.5\cdot 10^{-3}. \tag{51}$$

Finally, we can obtain from (44), (51), (47), (49), (50) and (35)

$$\frac{\partial g(\sigma,p)}{\partial\sigma}< -0.08.$$

Therefore, Theorem 3 is proved.

5. The case $1+\dfrac{1}{5p}\leqslant\sigma\leqslant 1+\dfrac{1.37}{\sqrt{p}}$.

THEOREM 4. *If* $p\geqslant 6$, $1+\dfrac{1}{5p}\leqslant\sigma\leqslant 1+\dfrac{1.37}{\sqrt{p}}$ *then*

$$g(\sigma,p)<0. \tag{52}$$

Proof. As $(1-\tau\sigma^{p-1})(1+\tau^{p-1})\leqslant(1-\tau)(1+\tau)\leqslant 1$ $(p\geqslant 2)$, taking into account (22), (36), (28), (38), (23), (27), it follows from (15) that

$$g(\sigma,p)<\frac{1-\tau}{\sigma}\cdot 1\cdot\left\{1+\left(\frac{\sigma\log(1+\sigma^p)}{p}\right)^{p-1}\right\}- \tag{53}$$
$$-\left(1-\frac{\tau_p^p}{p}\right)(1-\tau\sigma^{p-1})\left\{1-(1+\tau^{p-1})\left(\frac{\log(1+\sigma^p)}{p}\right)^{p-1}\right\}$$
$$<\frac{1}{\sigma p}h(\sigma,p),$$

where

$$h(\sigma,p) = -p(\sigma-1)+1-\sigma^{-p}+\frac{(\sigma^p+\sigma)\{\log(1+\sigma^p)\}^{p-1}}{p^{p-2}}+\frac{2\cdot(0.7)^p}{p^p}. \tag{54}$$

As $\dfrac{\partial^2 h(\sigma,p)}{\partial\sigma^2}$ increases with σ, then for $p\geqslant 6$, $1+\dfrac{1}{5p}\leqslant\sigma\leqslant 1+\dfrac{1.37}{\sqrt{p}}$

$$h(\sigma,p)\leqslant\max\left\{h\left(1+\frac{1}{5p},p\right),\ h\left(1+\frac{1.37}{\sqrt{p}},p\right)\right\}. \tag{55}$$

If $p\geqslant 6$ then

$$h\left(1+\frac{1}{5p},p\right) \tag{56}$$
$$< -\frac{1}{5}+1-e^{-1/5}+\frac{1}{6^4}\left(e^{1/5}+1+\frac{1}{30}\right)\{\log(1+e^{1/5})\}^5+\frac{0.25}{6^6}< -0.017<0;$$

$$h\left(1+\frac{1.37}{\sqrt{p}},p\right)<u(p), \tag{57}$$

where

$$u(p) = -1.37\sqrt{p}+1+\frac{1}{p^{p-2}}\left(1+\frac{1.37}{\sqrt{p}}\right)^p\left\{\log\left(1+\left(1+\frac{1.37}{\sqrt{p}}\right)^p\right)\right\}^{p-1}+$$
$$+\frac{1}{p^{p-2}}\left(1+\frac{1.37}{\sqrt{6}}\right)\left\{\log\left(1+\left(1+\frac{1.37}{\sqrt{p}}\right)^p\right)\right\}^{p-1}+\frac{0.25}{6^6}.$$

As the function $u(p)$ for $p\geqslant 6$ is the decreasing function (it can be verified by differentiating),

$$u(p)\leqslant u(6)< -0.4<0. \tag{58}$$

Due to (55), (53), (56), (57) and (58) we have (52).

The proof of Theorem 4 is completed.

6. The case $1+\frac{1.37}{\sqrt{p}} \leqslant \sigma \leqslant \sigma_p$.

THEOREM 5. *If* $p \geqslant 6$, $1+\frac{1.37}{\sqrt{p}} \leqslant \sigma \leqslant \sigma_p$ *then*

(59) $$\Delta(\sigma, p) > \Delta_p^{(1)}.$$

Proof. From (7) and (8) it follows that

$$\Delta(\sigma, p) \geqslant 2^{-1/p}(1-\tau_p)^{-1}(1+\sigma^{-p})^{-1/p},$$
$$\Delta_p^{(1)} = \Delta(1, p) = 4^{-1/p}(1+\tau_p)(1-\tau_p)^{-1},$$

hence (59) will result from the following inequality

(60) $$\left\{1+\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p}\right\}^{-1/p} > 2^{-1/p}(1+\tau_p).$$

Applying (25) and (27), we get

$$\left\{1+\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p}\right\}^{-1/p} > 1-\frac{\log\left(1+\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p}\right)}{p},$$
$$1+\tau_p < 2-2^{-1/p},$$

and (60) follows from

(61) $$\frac{1}{p}\log\left(1+\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p}\right) < (1-2^{-1/p})^2.$$

Inequality (61) follows from that of (24), (26) and

(62) $$p\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p} < \log^2 2\left(1-\frac{\log 2}{2p}\right)^2.$$

For $p \geqslant 6$ we have

(63) $$\log^2 2\left(1-\frac{\log 2}{2p}\right) \geqslant \log^2 2\left(1-\frac{\log 2}{12}\right) > 0.425,$$

(64) $$p\left(1+\frac{1.37}{\sqrt{p}}\right)^{-p} < 6\left(1+\frac{1.37}{\sqrt{6}}\right)^{-6} < 0.419,$$

as the function $p(1+1.37p^{-1/2})^{-p}$ is decreasing (it can be verified by differentiating).

Evaluations (63) and (64) consequently lead to (62), (60), (61) and (59).

Therefore, Theorem 5 is proved.

7. Completion of the proof of the main theorem.

THEOREM 6. *For any* $p > 1$

(65) $$g(1, p) = 0.$$

The equality (65) follows from (15), Lemma 1 and (7).

Proof of Theorem 2. When (65), (43), (52) and (16) are carried out then for $p \geqslant 6$ and $1 < \sigma \leqslant \frac{1.37}{\sqrt{p}}$

$$g(\sigma, p) < 0, \quad \frac{\partial \Delta(\sigma, p)}{\partial \sigma} > 0, \quad \Delta(\sigma, p) > \Delta(1, p) = \Delta_p^{(1)}.$$

These inequalities together with (59) result in the inequality (9). Therefore, Theorem 2 is proved.

Theorem 1 (Minkowski's conjecture for $p \geqslant 6$) is equivalent to Theorem 2.

The proposed proof means that for $p \geqslant 6$ we can disregard values of the order of $p^{-(p-3)}$ (Δ^{p-3} for $1 \leqslant \sigma \leqslant \frac{1}{5p}$, τ^{p-3} etc.), which greatly simplifies our estimations of $\Delta(\sigma, p)$, $g(\sigma, p)$ and $\frac{\partial g(\sigma, p)}{\partial \sigma}$. Apparently, using a computer we can prove Minkowski's conjecture for $1 < p < 6$. However, it requires more exact estimations, up to the third derivative of $\Delta(\sigma, p)$ in some cases, and special research into the case p near $p = 1$ and $p = 2$ (using a method similar to that of Watson's second paper [6]).

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On the representation of the integer by positive quadratic forms with square-free variables

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1. Introduction. Let

$$f = f(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} a_{ij}x_ix_j \quad (a_{ij} = a_{ji},\ 1 \leqslant i, j \leqslant k)$$

be a positive quadratic form with integral coefficients $a_{11}, \ldots, a_{kk}$, $2a_{12}, \ldots, 2a_{k-1,k}$ and determinant $D = \det(a_{ij}) \neq 0$. $R(f, n)$ denotes the number of representations of the positive integer n by the quadratic form f with square-free variables, i.e. the number of solutions of the equation

$$f(x_1, \ldots, x_k) = n \tag{1}$$

in square-free integers $x_1, \ldots, x_k$. Estermann [1] has obtained the asymptotic value of $R(f, n)$ for $k \geqslant 5$ and $f = x_1^2 + \ldots + x_k^2$; he has also considered the singular series (see also [3]). In [11] improvement has been obtained for the error term in the Estermann formula(1).

In the present paper we consider the asymptotic value of $R(f, n)$ in the case when f is an arbitrary positive quadratic form in $k \geqslant 4$ variables. We deduce the following

THEOREM 1. *Let* $k \geqslant 4$, $\alpha = \dfrac{k-3}{4(k+1)}$, $\varepsilon > 0$ — *an arbitrary positive number. Then*

$$R(f, n) = \frac{\pi^{k/2}}{D^{1/2}\Gamma(k/2)} G(f, n) n^{k/2-1} + O(n^{k/2-1-\alpha+\varepsilon}) \tag{2}$$

where $G(f, n)$ *is the singular series:*

$$G(f, n) = \sum_{t_1, \ldots, t_k=1}^{\infty} \frac{\mu(t_1)\ldots\mu(t_k)}{t_1^2 \ldots t_k^2} H(f_{t_1,\ldots,t_k}, n);$$

(1) Unfortunatly, issues [5], [8] have been found to be mistaken (see [11]).