§ 2. Proof of the Main Theorem. By Theorem 1 there exist positive real numbers \( \xi_0 = \xi_0(d, D) < 1 \), \( L_0 = L_0(d, D) \) such that for \( L > L_0 \) we have

\[
\sum' = \sum_{p \leq L} \{ F = n - p \} \sim \left( \frac{9}{2} + O(\varepsilon^2 \varepsilon) \right) \sum_{p \leq L} \chi(p) + O \left( \sum_{p \leq L} \sum_{m \leq d} \sum_{l \leq n} \sum \chi(l) \right).
\]

Hence for \( L_0 < L \leq \log \log n \) we obtain from Theorem 2 and 3

\[
\sum' = \frac{2}{k_f} L(1, \chi) \prod_{p \in (D-\varepsilon n)} \left( 1 - \frac{\chi(p)}{p} \right) \left( 1 + \frac{\chi(p)}{p(p-1)} \right) \frac{n}{\log n} \leq \frac{\varepsilon^2 L}{\log n} \log \log n + \frac{n}{\log(1+\varepsilon n)} \log^2 \log n + \frac{1}{L} \left( \frac{A}{L} \log \log n \right)^f.
\]

On putting \( L = \frac{1}{A k_f} \log \log n \) we get the required estimate.

References


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Cohn [2] introduced a convenient parametrization of the problem (which we also use; see §2) and gave a sketch of the proof of Minkowski’s conjecture when $p$ is “sufficiently large”. Mordell [5] has proved this conjecture for $p = 4$ (the cases $p = 2$ and $p = 1$ are trivial). Kukharev [8] (the prelimentary report—[7]) has worked out a method for the examination of Minkowski’s conjecture for every concrete $p$ (except $p$ near 1, 2 and $p_0$) and using a computer has proved this conjecture for $p = 1.3; 1.4; 1.5; 1.6; 1.7; 2.2; 2.3; 3; 4; 5$.

The aim of this work is to prove Minkowski’s conjecture for $p \geq 6$. That is, we are to prove the following statement.

**Theorem 1.** For $p \geq 6$ we have

$$
\Delta(\mathcal{B}_p) = \Delta_p^{(1)}
$$

and the set of critical lattices of $\mathcal{B}_p$ consists of $\Delta_p^{(1)}$ and lattices which are symmetrical with respect to coordinate axes and their bisectors.

**2. Reformulation of the problem.** Let for fixed $p > 1$ a value $\sigma$ increase from 1 to $\sigma_p = \left(\frac{2^p - 1}{p^p}\right)$. The equation

$$
(1 + \sigma^p)^{-1/2} - (1 + \sigma^p)^{-1/2} \sigma + (1 + \sigma^p)^{-1/2} = 1
$$

determines a function $\tau = \tau(\sigma, p) \geq 0$, which is decreasing from $\tau_p$ to 0; here $\tau_p$ is defined by

$$
2(1 - \sigma_p)^p = 1 + \tau_p^p, \quad 0 < \tau_p < 1,
$$

(see [3]). We introduce the function

$$
\Delta(\sigma, p) = (1 + \sigma)(1 + \sigma^p)^{-1/2} - (1 + \sigma^p)^{-1/2}
$$

where $\tau = \tau(\sigma, p)$. Then $\Delta(\sigma, p)$ is the determinant of $\mathcal{B}_p(\sigma)$, which has six points on the boundary of $\mathcal{B}_p$, one of them having tangent coefficient $-\sigma$; under these conditions $A_2(1) = A^{(1)}_p$, $A(1, p) = A^{(1)}_p$, $A_p(\sigma) = A^{(1)}_p$, $A(\sigma, p) = A^{(1)}_p$ (see [2]). Theorem 1 is equivalent to the following statement.

**Theorem 2.** If $p \geq 6, 1 < \sigma \leq \sigma_p$, then

$$
\Delta(\sigma, p) > \Delta(1, p) = A^{(1)}_p.
$$

We write

$$
(1 + \sigma^p)^{-1/2} - (1 + \sigma^p)^{-1/2} = A(\sigma, p) = A,
$$

(11)

$$
\tau(1 + \sigma^p)^{-1/2} + (1 + \sigma^p)^{-1/2} = B(\sigma, p) = B,
$$

where $\tau = \tau(\sigma, p)$ is defined by (6), which can now be rewritten as

$$
\Delta + B^p = 1.
$$

Differentiating (12) we find

$$
\frac{\partial \tau}{\partial \sigma} = \frac{(1 + \tau^p)^{-1/2} - (1 + \sigma^p)^{-1/2}}{(1 + \sigma^p)^{-1/2} - (1 + \tau^p)^{-1/2}} = \frac{g(\sigma, p)}{(1 + \sigma^p)^{1/2}(1 + \tau^p)^{-1/2}}
$$

and so

$$
\frac{\partial \Delta(\sigma, p)}{\partial \sigma} = \frac{g(\sigma, p)}{(1 + \sigma^p)^{1/2}(1 + \tau^p)^{-1/2}}
$$

where

$$
g(\sigma, p) = (1 + \sigma^p)^{-1/2}(1 - \sigma^2) + (1 + \sigma^p)^{-1/2}(1 - \sigma^2) - (1 + \tau^p)^{-1/2}(1 - \sigma^2)
$$

(15)

From (14) it follows that

$$
\text{sign} \frac{\partial \Delta(\sigma, p)}{\partial \sigma} = -\text{sign} g(\sigma, p).
$$

We are beginning to prove Theorem 2 (from which Theorem 1 follows). After some lemmas from §3, we shall prove that

$$
\frac{\partial g(\sigma, p)}{\partial \sigma} < 0 \quad \text{for} \quad p \geq 6, \quad 1 \leq \sigma < 1 + \frac{1}{10^p}
$$

(§4, Theorem 3); that

$$
g(\sigma, p) < 0 \quad \text{for} \quad p \geq 6, \quad 1 + \frac{1}{10^p} \leq \sigma \leq 1 + \frac{1.37}{10^p}
$$

(§5, Theorem 4); that

$$
\Delta(\sigma, p) > \Delta_0^{(1)} \quad \text{for} \quad p \geq 6, \quad 1 + \frac{1.37}{10^p} \leq \sigma \leq \sigma_p
$$

(§6, Theorem 5); that

$$
g(1, p) = 0 \quad \text{for} \quad p > 1
$$

(§7, Theorem 6). It follows from (17)-(20) that (9) holds for $p \geq 6, 1 < \sigma < 1$.

3. **Lemmas.**

**Lemma 1.** When $p$ is fixed, while $\sigma$ increases from 1 to $\sigma_p$;

(a) $(1 + \sigma^p)^{-1/2}$ decreases from $2^{-1/2}$ to $1$;

(b) $\sigma(1 + \sigma^p)^{-1/2}$ increases from $2^{-1/2}$ to $\frac{1}{2}\sigma_p$;

(c) $\tau$ decreases from $\tau_p$ to 0;

(d) $(1 + \tau^p)^{-1/2}$ increases from $2^{-1/2}$ to $2^{-1/2}(1 - \tau_p)^{-1}$;

(e) $(1 + \tau^p)^{-1/2}$ decreases from $2^{-1/2}$ to $2^{-1/2}(1 - \tau_p)^{-1}$ to 0;

(f) $A$ increases from $\tau_p(1 + \tau_p^p)^{-1/2}$ to $2^{-1/2}(1 - \tau_p)^{-1}$ to $\frac{1}{2}\sigma_p$;

(g) $B$ decreases from $2^{-1/2}(1 - \tau_p)^{-1}$ to $\frac{1}{2}\sigma_p$.

---

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The proof of Minkowski's conjecture

These statements follow from (6) and (12). The boundary values can be obtained by applying (7).

**Lemma 2.** For every\(^{(1)}\) \(p\) and \(\sigma\)

\[
\tau < 1 - (1 + \sigma^{-p})^{-1/p},
\]

As it follows from Lemma 1,

\[
B < (1 + \tau_p)^{-1/p}, \quad \tau \left(1 + \frac{\tau}{\tau_p}\right)^{1/p} + \sigma(1 + \sigma^{-p})^{-1/p}(1 + \frac{\tau}{\tau_p})^{1/p} \leq 1,
\]

\[
\tau + \sigma(1 + \sigma^{-p})^{-1/p} < 1.
\]

**Corollary 1.** For every \(p\) and \(\sigma\)

\[
(1 + \sigma^{-p})^{-1/p} < \frac{1 - \tau}{\tau}.
\]

It is evident that (22) is the same as (21).

**Corollary 2.** For every \(p\) and \(\sigma\)

\[
\tau < \frac{\log(1 + \sigma^{-p})}{p} < \frac{1}{p \sigma p^{1/p}}.
\]

If we take into account that for \(\sigma > 0\)

\[
\sigma \left(1 - \frac{\sigma}{2}\right) < 1 - \sigma^{-\sigma} < \sigma,
\]

then for \(\alpha > 0, \beta > 0\)

\[
1 - (1 + \alpha)^{-\beta} < \beta \log(1 + \alpha),
\]

so the first inequality follows from (21). The second inequality follows from the first because for \(\sigma > 0\) we have

\[
\log(1 + \sigma) < \sigma.
\]

**Corollary 3.** For \(p > 1\)

\[
\tau < \tau_p < 1 - 2^{-1/p} < \frac{\log 2}{p} < 0.7.
\]

See (21) and (23).

**Lemma 3.** For \(p \geq 6\)

\[
(1 + \tau_p)^{-1/p} > 1 - \frac{\tau_p}{p} > 1 - \frac{2 \cdot 10^{-4}}{p},
\]

\[
(1 + \tau_p)^{-1/p} > (1 + \tau_p)^{-1/p} > 1 - \left(1 + \frac{2}{p}\right)^{\tau_p} > 1 - \frac{2 \cdot 10^{-4}}{p}.
\]

\(^{(1)}\) From now on we suppose that \(p > 1, 1 < \sigma < \sigma_0\).

Applying (25) and (27) we find that

\[
(1 + \tau_p)^{-1/p} > 1 - \frac{\tau_p}{p} > 1 - \frac{(0.11)^{1/p}}{p} > 1 - \frac{2 \cdot 10^{-4}}{p};
\]

\[
(1 + \tau_p)^{-1/p} > 1 - \left(1 + \frac{2}{6}\right)^{\tau_p} > 1 - \left(1 + \frac{2}{6}\right)^{\log 2} > 1 - \frac{2 \cdot 10^{-4}}{p}.
\]

**Lemma 4.** For every \(p\) and \(\sigma\)

\[
A < 1 + (1 + \sigma^{-p})^{-1/p} < \frac{\log(1 + \sigma^p)}{p}.
\]

See (10) and (25).

**Lemma 5.** If \(1 \leq \sigma < 1 + \frac{1}{5p}\) then

\[
1 < \sigma^{p-1} < \sigma^{p-2} < \sigma^{p} < \sigma^{p} < 1.2215.
\]

It follows from the inequality that

\[
\left(1 + \frac{a}{p}\right)^{p} < a^{p}, \quad a > 0, p \geq 1.
\]

**Corollary.** If \(1 \leq \sigma < 1 + \frac{1}{5p}\) then

\[
A < \frac{\log(1 + \sigma^p)}{p} < 0.8.
\]

See (30) and (31).

**Lemma 6.** For every \(p\) and \(\sigma\)

\[
B > B^0 > B^{p-2} > B^{p-1} > B^p = 1 - A^p > 1 - \frac{\log(1 + \sigma^p)^p}{p}.
\]

See (12) and (30).

**Corollary.** If \(p > 6, 1 \leq \sigma < 1 + \frac{1}{5p}\) then

\[
1 > B > B^{p-2} > B^{p-1} > B^p > 1 - \frac{3.4 \cdot 10^{-4}}{p} > 1 - 10^{-4}.
\]

See (34) and (33).

**Lemma 7.** For every \(p\) and \(\sigma\)

\[
B^{p-1} + \sigma^{p-1} A^{p-1} < 1 + \frac{\log(1 + \sigma^p)^{p-1}}{p^{p-1}}.
\]

See (34) and (30).
Corollary. If $p \geq 6$, $1 \leq \sigma \leq 1 + \frac{1}{5p}$ then

$$B^{p-1} + \sigma^{p-1} A^{p-1} < 1 + \frac{3.1 \cdot 10^{-4}}{p} < 1 + 3.2 \cdot 10^{-4}.$$  

See (36) and (31).

Lemma 8. For every $p$ and $\sigma$

$$B^{p-1} - \tau^{p-1} A^{p-1} > 1 - \frac{(\log(1 + \sigma^p))}{p} p^{p-1} \frac{(\log(1 + \sigma^p))}{p} p^{p-1} > 1 - \frac{(1 + \tau^p)}{p} \frac{(1 + \tau^p)}{p}.$$  

See (34), (30).

Corollary. If $p \geq 6$, $1 \leq \sigma \leq 1 + \frac{1}{5p}$ then

$$B^{p-1} - \tau^{p-1} A^{p-1} > 1 - \frac{4 \cdot 10^{-5}}{p} > 1 - 10^{-5}.$$  

See (38), (31).

Lemma 9. For $p \geq 6$

$$\tau + \sigma (1 + \sigma^p)^{-1/p} - \frac{3 \cdot 10^{-7}}{p} < B < \tau + \sigma (1 + \sigma^p)^{-1/p}.$$  

See (11) and (28).

Lemma 10. For $p \geq 6$, $1 \leq \sigma \leq 1 + \frac{1}{5p}$

$$(1 + \sigma^p)^{-1-1/p} < \frac{\partial \tau}{\partial \sigma} < (1 + \sigma^p)^{-1-1/p} \left(1 + \frac{4 \cdot 10^{-4}}{p}\right)$$  

$$(1 + \sigma^p)^{-1-1/p} < \frac{\partial \tau}{\partial \sigma} < \frac{1}{2} (1 + 10^{-4}).$$  

See (13), (37), (37) and (39).

4. The case $1 \leq \sigma \leq 1 + \frac{1}{5p}$.

Theorem 3. If $p \geq 6$, $1 \leq \sigma \leq 1 + \frac{1}{5p}$ then

$$\frac{\partial g(\sigma, p)}{\partial \sigma} < 0.$$  

Proof. Differentiating (15) and applying (13) we find that

$$\frac{\partial g(\sigma, p)}{\partial \sigma} < u(\sigma, p) + \tau(\sigma, p),$$  

where

$$u(\sigma, p) = \sigma^p \left(1 + \sigma^p\right)^{-1-1/p} B^{p-1} + (p-1) (1 + \sigma^p)^{-1-1/p} B^{p-1} -$$

$$- (p-1) (1 + \tau^p)^{-1-1/p} (1 + \sigma^p)^{-1-1/p} B^{p-1} +$$

$$- (p-1) (1 + \tau^p)^{-1-1/p} (1 + \sigma^p)^{-1-1/p} B^{p-1} +$$

$$+(p-1) \tau^p (1 + \sigma^p)^{-1-1/p} B^{p-1} + (p-1) B^{p-1} \frac{\partial \tau}{\partial \sigma}$$

$$(1 + \sigma^p)^{-1-1/p} B^{p-1}$$

$$+(p-1) \tau^p (1 + \sigma^p)^{-1-1/p} B^{p-1} +$$

Due to (28), (29), (40), (41) and (27) we can write that

$$u(\sigma, p) < B^{p-1} \tau^p + (p-1) \tau^{p-2} (B^{p-1} + \sigma^p A^{p-1}) \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \tau^p B^{p-2} \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \sigma^p A^{p-1} \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \tau^p A^{p-1} \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \tau^{p-1} A^{p-1} \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \tau^{p-2} A^{p-1} \frac{\partial \tau}{\partial \sigma} +$$

$$+(p-1) \tau^{p-3} A^{p-1} \frac{\partial \tau}{\partial \sigma} +$$

Due to (28), (29), (40), (41) and (27) we can write that

$$u(\sigma, p) < B^{p-1} \tau^p + s(\sigma, p),$$  

where

$$\nu(\sigma, p) = -2 \sigma^p (1 + \sigma^p)^{-1-1/p} + (p-1) \tau^p (1 + \sigma^p)^{-1-1/p} -$$

$$-2 \tau^p (1 + \sigma^p)^{-1-1/p} + (p-1) \tau^2 \sigma^p -$$

$$s(\sigma, p) = \frac{3 \cdot 10^{-7}}{p} +$$

$$+ (p-1) \frac{2 \cdot 10^{-4}}{p} +$$

$$+ (p-1) \frac{3 \cdot 10^{-7}}{p} +$$

$$+ (p-1) \frac{2 \cdot 10^{-7}}{p} +$$

$$+ (p-1) \frac{1 \cdot 10^{-3}}{p} +$$

$$+ (p-1) \frac{0.7 \cdot 2 \cdot 10^{-5}}{p} < 10^{-3}.$$
Since due to (37) and (31) we have

\[-2\sigma^{p-1}(1+\sigma^{p})^{-1-2p} + (p-1)\frac{\tau \sigma^{p-1}}{(1+\sigma^{p})^{1+2p}} [\frac{-2\cdot 1 + 0.7 \cdot 1.2215^{7p}}{1}] < 0,
\]

so that

\[w(\sigma, p) < -2\sigma^{p}(1+\sigma^{p})^{-1-2p} + 0.7 \sigma^{p-1}(1+\sigma^{p})^{-1-2p} = w(\sigma, p).
\]

We have

\[
\frac{\partial w(\sigma, p)}{\partial \sigma} = \sigma^{p-2}(1+\sigma^{p})^{-2-2p} h(\sigma, p),
\]

where

\[h(\sigma, p) = -2p\sigma + 4\sigma^{p+1} + 0.7(1+\sigma^{p})^{4+4p}(p-1)(\sigma^{p}+1) - \sigma^{p},
\]

\[> -2p \left(1 + \frac{1}{5p}\right)^{2p-3} + 0.7 \cdot 2 \cdot (2p - 3) > 0.
\]

Therefore,

\[
(50) \quad w(\sigma, p) < w(\sigma, p) \leq \frac{1}{5p} \left(1 + 1 \right)^{p-1} \left(1 + \frac{1}{5p}\right)_{1-2p} \left(1 - 2 \cdot 1 \cdot (1+\sigma^{p})^{-1-2p} + 0.7 \right)
\]

\[< \left(1 + \frac{1}{5p}\right)^{p-1} \left(1 + \frac{1}{5p}\right)_{1-2p} \left(1 - 2 \cdot 1 \cdot (1+\sigma^{p})^{-1-2p} + 0.7 \right) - 0.038.
\]

From (46), (27), (34), (37), (42), (31) and (33) we conclude that

\[(51) \quad w(\sigma, p) < 3.5 \cdot 10^{-4}.
\]

Finally, we can obtain from (44), (51), (47), (49), (60) and (35)

\[
\frac{\partial g(\sigma, p)}{\partial \sigma} < -0.08.
\]

Therefore, Theorem 3 is proved.

5. The case \(1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}\).

**Theorem 4.** If \(p \geq 6, 1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}\) then

\[(52) \quad g(\sigma, p) < 0.
\]

Proof. As \(1 - \pi^{p-1}(1 + \pi^{p})^{-1} \leq (1 - \pi)(1 + \pi) \leq 1 \) \((p \geq 2)\), taking into account (22), (36), (28), (38), (23), (27), it follows from (15) that

\[
g(\sigma, p) \leq \frac{1}{\sigma} \cdot \left(1 + \left(\frac{(1+\sigma^{p})^{p-1}}{p}\right) \right) - \left(1 - \frac{2\pi}{p}\right) \left(1 - \pi^{p-1}\right) \left(1 - \pi^{p-1}\right) \left(\frac{\log(1+\sigma^{p})}{p}\right)^{p-1}
\]

\[< \frac{1}{\sigma} \ h(\sigma, p),
\]

where

\[
(54) \quad h(\sigma, p) = -p(\sigma - 1) + \sigma^{-p} + \left(\sigma^{p} - \pi^{p}ight) \left(\frac{\log(1+\sigma^{p})}{p}\right)^{p-1} + \frac{2 \cdot 0.7^{p}}{p^{2}}.
\]

As \(\frac{\partial^{2} h(\sigma, p)}{\partial \sigma^{2}}\) increases with \(\sigma\), then for \(p \geq 6\),

\[
1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}.
\]

If \(p \geq 6\) then

\[
(56) \quad h(\sigma, p) \leq \max\left\{ h\left(1 + \frac{1}{5p}, p\right), h\left(1 + \frac{1.37}{\sqrt[p]{p}}, p\right) \right\}.
\]

From (46), (27), (34), (37), (42), (31) and (33) we conclude that

\[
(51) \quad w(\sigma, p) < 3.5 \cdot 10^{-4}.
\]

Finally, we can obtain from (44), (51), (47), (49), (60) and (35)

\[
\frac{\partial g(\sigma, p)}{\partial \sigma} < -0.08.
\]

Therefore, Theorem 3 is proved.

5. The case \(1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}\).

**Theorem 4.** If \(p \geq 6, 1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}\) then

\[(52) \quad g(\sigma, p) < 0.
\]

Proof. As \(1 - \pi^{p-1}(1 + \pi^{p})^{-1} \leq (1 - \pi)(1 + \pi) \leq 1 \) \((p \geq 2)\), taking into account (22), (36), (28), (38), (23), (27), it follows from (15) that

\[
g(\sigma, p) \leq \frac{1}{\sigma} \cdot \left(1 + \left(\frac{(1+\sigma^{p})^{p-1}}{p}\right) \right) - \left(1 - \frac{2\pi}{p}\right) \left(1 - \pi^{p-1}\right) \left(1 - \pi^{p-1}\right) \left(\frac{\log(1+\sigma^{p})}{p}\right)^{p-1}
\]

\[< \frac{1}{\sigma} \ h(\sigma, p),
\]

where

\[
(54) \quad h(\sigma, p) = -p(\sigma - 1) + \sigma^{-p} + \left(\sigma^{p} - \pi^{p}ight) \left(\frac{\log(1+\sigma^{p})}{p}\right)^{p-1} + \frac{2 \cdot 0.7^{p}}{p^{2}}.
\]

As \(\frac{\partial^{2} h(\sigma, p)}{\partial \sigma^{2}}\) increases with \(\sigma\), then for \(p \geq 6\),

\[
1 + \frac{1}{5p} \leq \sigma \leq 1 + \frac{1.37}{\sqrt[p]{p}}.
\]

If \(p \geq 6\) then

\[
(56) \quad h\left(1 + \frac{1}{5p}, p\right)
\]

\[< -\frac{1}{5} + 1 - \pi^{p-1} + \frac{1}{6\cdot \pi^{p}} \left(\pi^{p} + 1 \right) \left(\frac{\log(1+\pi^{p})}{p}\right)^{p-1} + 0.25 \cdot \frac{0.7^{p}}{p^{2}}< -0.017 < 0;
\]

\[
(57) \quad h\left(1 + \frac{1.37}{\sqrt[p]{p}}, p\right) < u(p),
\]

where

\[
u(p) = -1.37\sqrt[p]{p} + 1 + \frac{1}{p^{p-2}} \left(1 + \frac{1.37}{\sqrt[p]{p}}\right)^{p} \left(\log(1+\pi^{p})^{p-1} + \frac{0.25}{6^{p}}\right) +
\]

\[+ \frac{1}{p^{p-2}} \left(1 + \frac{1.37}{\sqrt[p]{p}}\right)^{p} \left(\log(1+\pi^{p})^{p-1} + \frac{0.25}{6^{p}}\right) + 0.25 \cdot \frac{0.7^{p}}{p^{2}}.
\]

As the function \(u(p)\) for \(p \geq 6\) is the decreasing function (it can be verified by differentiating),

\[(58) \quad u(p) < u(6) < -0.4 < 0.
\]

Due to (35), (53), (56), (57) and (58) we have (52).
The proof of Theorem 4 is completed.

6. The case \( 1 + \frac{1.37}{V_p} \leq \sigma \leq \sigma_p \).

**Theorem 5.** If \( p \gg 6 \), \( 1 + \frac{1.37}{V_p} \leq \sigma \leq \sigma_p \) then

\[
\Delta(\sigma, p) > \Delta_p^{(3)}.
\]

**Proof.** From (7) and (8) it follows that

\[
\Delta(\sigma, p) \geq 2^{-1/p} (1 - \tau_p)^{-1} (1 + \sigma^{-p})^{-1/p},
\]

\[
\Delta_p^{(3)} = \Delta(1, p) = 4^{-1/p} (1 + \tau_p) (1 - \tau_p)^{-1},
\]

hence (59) will result from the following inequality

\[
\left( 1 + \left( 1 + \frac{1.37}{V_p} \right)^{-p} \right)^{-1/p} > 2^{-1/p} (1 + \tau_p).
\]

Applying (25) and (27), we get

\[
\left( 1 + \left( 1 + \frac{1.37}{V_p} \right)^{-p} \right)^{-1/p} > 1 - \frac{\log \left( 1 + \left( 1 + \frac{1.37}{V_p} \right)^{-p} \right)}{p},
\]

and

\[
1 + \tau_p < 2 - 2^{-1/p},
\]

and (60) follows from

\[
\frac{1}{p} \log \left( 1 + \left( 1 + \frac{1.37}{V_p} \right)^{-p} \right) < (1 - 2^{-1/p})^p.
\]

Inequality (61) follows from that of (24), (26) and

\[
p \left( 1 + \frac{1.37}{V_p} \right)^{-p} < \log^2 \left( 1 - \frac{\log 2}{2p} \right).
\]

For \( p \gg 6 \) we have

\[
\log^2 \left( 1 - \frac{\log 2}{2p} \right) \geq \log^2 \left( 1 - \frac{\log 2}{12} \right) > 0.426,
\]

\[
p \left( 1 + \frac{1.37}{V_p} \right)^{-p} < 6 \left( 1 + \frac{1.37}{V_p} \right)^{-6} < 0.419,
\]

as the function \( p(1 + 1.37p^{-1/2})^{-p} \) is decreasing (it can be verified by differentiating).

Evaluations (63) and (64) consequently lead to (62), (60), (61) and (59).

Therefore, Theorem 5 is proved.

7. Completion of the proof of the main theorem.

**Theorem 6.** For any \( p > 1 \)

\[
g(1, p) = 0.
\]

The equality (65) follows from (15), Lemma 1 and (7).

**Proof of Theorem 2.** When (65), (43), (53) and (16) are carried out then for \( p \gg 6 \) and \( 1 < \sigma \leq \frac{1.37}{V_p} \)

\[
g(\sigma, p) < 0, \quad \frac{\partial \Delta(\sigma, p)}{\partial \sigma} > 0, \quad \Delta(\sigma, p) > \Delta(1, p) = \Delta_p^{(3)}.
\]

These inequalities together with (59) result in the inequality (9). Therefore, Theorem 2 is proved.

Theorem 1 (Minkowski's conjecture for \( p \gg 6 \)) is equivalent to Theorem 2.

The proposed proof means that for \( p \gg 6 \) we can disregard values of the order of \( p^{-p-1} (\Delta_p^{(2)} \log^2 (1 - \tau_p) \approx 1 \leq \sigma \leq \frac{1.37}{V_p} \), \( \sigma^{-p-1} \) etc.), which greatly simplifies our estimations of \( \Delta(\sigma, p) \), \( g(\sigma, p) \) and \( \frac{\partial g(\sigma, p)}{\partial \sigma} \). Apparently, using a computer we can prove Minkowski's conjecture for \( 1 < p < 6 \). However, it requires more exact estimations, up to the third derivative of \( \Delta(\sigma, p) \) in some cases, and special research into the case \( p \approx 1 \) and \( p = 2 \) (using a method similar to that of Watson's second paper [6]).

**References**


On the representation of the integer by positive quadratic forms with square-free variables

by

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1. Introduction. Let

\[ f = f(x_1, \ldots, x_n) = \sum_{i,j=1}^{k} a_{ij} x_i x_j \quad (a_{ij} = a_{ji}, 1 \leq i, j \leq k) \]

be a positive quadratic form with integral coefficients \(a_{11}, \ldots, a_{kk}, 2a_{12}, \ldots, 2a_{k-1,k}\) and determinant \(D = \det(a_{ij}) \neq 0\). \(R(f, n)\) denotes the number of representations of the positive integer \(n\) by the quadratic form \(f\) with square-free variables, i.e. the number of solutions of the equation

\[ f(x_1, \ldots, x_n) = n \]

in square-free integers \(x_1, \ldots, x_n\). Estermann [1] has obtained the asymptotic value of \(R(f, n)\) for \(k \geq 5\) and \(f = x_1^2 + \ldots + x_k^2\); he has also considered the singular series (see also [3]). In [11] improvement has been obtained for the error term in the Estermann formula.(1)

In the present paper we consider the asymptotic value of \(R(f, n)\) in the case when \(f\) is an arbitrary positive quadratic form in \(k \geq 4\) variables. We deduce the following

**Theorem 1.** Let \(k \geq 4\), \(\alpha = \frac{k-3}{4(k+1)}\), \(\varepsilon > 0\) — an arbitrary positive number. Then

\[ R(f, n) = \frac{n^{k \alpha}}{D^{1/2} \Gamma(k/2)} G(f, n) n^{k \alpha - 1} + O(n^{k \alpha - 1 - \alpha + \varepsilon}) \]

where \(G(f, n)\) is the singular series:

\[ G(f, n) = \sum_{t_1, \ldots, t_k = 1}^{\infty} \frac{\mu(t_1) \ldots \mu(t_k)}{t_1^{\alpha} \ldots t_k^{\alpha}} H(f(t_1, \ldots, t_k; n)); \]

(1) Unfortunately, issues [5], [8] have been found to be mistaken (see [11]).