On a paper by A. Baker on the approximation of rational powers of $e$

by

Kurt Mahler (Canberra)

In an important paper of 1965 (Canadian Journal of Mathematics, 17, pp. 616–626), A. Baker for the first time established lower bounds for products of the form

\[(I) \quad |a_1 a_2 \cdots a_k (x_1 E_1 + x_2 E_2 + \cdots + x_k E_k)|\]

and

\[y_k |y_k E_k - y_1 \cdots |y_k E_{k-1} - y_{k-1} E_{k-1}|.\]

Here $E_1, E_2, \ldots, E_k$ are distinct rational powers of $e$, with $E_k = 1$ in the second expression; the $x_i$'s are distinct integers not zero, while the $y_i$'s are integers where $y_i > 0$, and $k \geq 2$. These lower bounds involve positive constants depending only on $k$ and the $E_i$'s and are not given explicitly. The method depends on an ingenious generalization of that by C. L. Siegel in his classical paper in the Abhandlungen der Preussischen Akademie der Wissenschaften of 1929, No. 1.

I try in the present paper to carry Baker's investigations a little further by establishing lower bounds for the expressions $(I)$ which are completely explicit and do not involve any unknown constants; the results are contained in the Theorems 1 and 2 and their corollaries. It is highly probable that better estimates can be proved if explicit formulae for Baker's approximation polynomials are used. Such formulae have been obtained recently by A. van der Poorten at the University of New South Wales.

1. This paper makes use of the following well known theorem.

**Lemma 1.** Let

\[(g_{ij}) \quad (i = 1, 2, \ldots, M; j = 1, 2, \ldots, N),\]

where $M < N$, be a matrix of integers, and let

\[G_i = \sum_{j=1}^{N} |g_{ij}| \quad (i = 1, 2, \ldots, M).\]
Then there exist integers \( x_1, \ldots, x_N \) not all zero such that

\[
\sum_{j=1}^{N} g_j x_j = 0 \quad \text{for} \quad i = 1, 2, \ldots, M;
\]

\[
\max(|x_1|, \ldots, |x_N|) \leq (G_1 \cdots G_M)^{N(M-N)}.
\]

Proof. Put

\[ G = [(G_1 \cdots G_M)^{1/[N(N-M)]}], \]

where \([s]\) as usual denotes the integral part of \( s \). There are then \((G+1)^N\) distinct vectors \( x = (x_1, \ldots, x_N) \) with integral components \( a_1, \ldots, a_N \) satisfying

\[ 0 \leq a_j \leq G \quad (j = 1, 2, \ldots, N). \]

With each such vector \( x \) associate a second integral vector \( y = (y_1, \ldots, y_M) \) where

\[ y_i = \sum_{j=1}^{N} g_j x_j \quad (i = 1, 2, \ldots, M). \]

Further define for each suffix \( i = 1, 2, \ldots, M \) two non-negative integers \( n_i \) and \( p_i \) by

\[ n_i = \sum_{g_j < 0} |g_j|, \quad p_i = \sum_{g_j > 0} |g_j| \quad (i = 1, 2, \ldots, M). \]

Then evidently

\[ G_i = n_i + p_i \quad (i = 1, 2, \ldots, M) \]

and for all vectors \( y \),

\[ -n_i G \leq y_i \leq +p_i G \quad (i = 1, 2, \ldots, M). \]

This means that each component \( y_i \) has at most \( n_i G + p_i G + 1 = G_i G + 1 \) possibilities, hence that the vector \( y \) has at most \((G_1 G + 1) \cdots (G_M G + 1)\) possibilities. But

\[ (G+1)^N = (G+1)^{N-N} = (G+1)^{N} G_1 \cdots G_M \geq (G_1 G + 1) \cdots (G_M G + 1). \]

Hence there are more distinct vectors \( x \) than there are distinct vectors \( y \). It follows that a certain pair of distinct \( x \)-vectors, \( x' \) and \( x'' \), say, generate the same vector \( y \). This implies that their difference \( x = x' - x'' \) is not itself the zero vector, but generates the zero vector \( y = (0, \ldots, 0) \). Since the components \( a_1, \ldots, a_N \) of \( x \) evidently lie between \(-G\) and \( +G\), the vector \( x \) has the asserted properties.

2. Let \( a_1, \ldots, a_k \), where \( k \geq 2 \), be finitely many distinct integers, and let \( a \) be a positive integer satisfying

\[ (a, a_1, \ldots, a_k) = 1; \]

let further

\[ A = \max(|a_1|, \ldots, |a_k|) \quad \text{and} \quad B = A + a, \]

so that

\[ A \geq 1, \quad B \geq 2. \]

Put

\[ E_i = 2^{a_1 a_i}, \quad E_k = 2^{a_k a_i}. \]

Then \( E_1, \ldots, E_k \) are distinct positive numbers, and hence the exponential functions

\[ E_1^2, \ldots, E_k^2 \]

are linearly independent over the field of rational functions of \( x \).

Next denote by \( r_1, \ldots, r_k, \) \( R \) variable positive integers, and put

\[ r = \max(r_1, \ldots, r_k), \quad r_0 = \min(r_1, \ldots, r_k), \]

\[ m = r_1 + \cdots + r_k - k - E, \quad n = r_1 + \cdots + r_k + k = m + R. \]

It will be assumed that

\[ k \leq R \leq r_1 + \cdots + r_k - k - 1, \quad \text{hence that} \quad 1 \leq m \leq r_1 + \cdots + r_k \leq kr. \]

Since the following three expressions will occur frequently, the following abbreviations will be used,

\[ E = \frac{k(k-1)}{2}, \quad m = \frac{m(m-1)}{2}, \quad E' = \frac{1}{R}. \]

3. With each pair of suffixes \((i, j)\) satisfying \( 1 \leq i \leq k, j \geq 0 \) associate two coefficients \( p(i, j) \) related by the equation

\[ p(i, j) = \frac{r_i}{j!} p(i, j). \]

Both coefficients are assumed equal to zero whenever \((i, j)\) does not belong to the set \( S \) of all pairs \((i, j)\) satisfying \( 1 \leq i \leq k, r_0 - r_k \leq j \leq r \).

With these coefficients form now the \( k \) polynomials

\[ P_k(x) = r! \sum_{j=0}^{r} p(i, j) x^j = \sum_{j=0}^{r} p(i, j) x^j \quad (i = 1, 2, \ldots, k). \]
and the entire function

\[ F(z) = \sum_{i=1}^{k} P_i(z) E_i^r, \]

say with the power series

\[ F(z) = r! \sum_{k=0}^{m} f_k z^k \frac{a^h}{h!} \]

where the coefficients \( f_k \) are defined by

\[ a^h f_k = \sum_{i=1}^{h} \binom{h}{j} a_{i}^h a^j \rho_{ij} \quad (h = 0, 1, 2, \ldots). \]

Denote by \( G_{h+1} \) the sum of the absolute values of the coefficients of all the \( p_{ij} \) in this equation. Thus

\[ G_{h+1} = \sum_{i=1}^{h} \sum_{j=0}^{h} \binom{h}{j} |a_i|^h a^j = \sum_{i=1}^{h} (|a_i|^h + a)^h \]

and therefore

\[ G_{h+1} \leq kB^h \quad (h = 0, 1, 2, \ldots), \]

whence, in particular,

\[ G_1 \ldots G_m \leq k^m B^{m^2}. \]

4. Apply now Lemma 1 to the system of \( m \) homogeneous linear equations

\[ f_h = 0 \quad (h = 0, 1, \ldots, m - 1) \]

for the \( n \) unknowns \( p_{ij} \) for which \((i, j)\) lies in \( S \). In the notation of the lemma, \( M = m \) and \( N = n \), while the maxima \( G_i \) satisfy the inequalities (2) and (3). Since \( n - m = p \), the lemma shows that

There exist integers \( p_{ij} \) not all zero, but equal to zero whenever \((i, j)\) does not lie in \( S \), such that

\[ f_h = 0 \quad \text{for} \quad 0 \leq h \leq m - 1; \quad \max_{i,j} |p_{ij}| \leq (k^n B^{m^n})^{(n^2)}, \]

Next, in the sum defining \( P_i(z) \),

\[ \frac{r!}{j!} = \binom{r}{j} (r-j)! \]

where it suffices to allow \( j \) to run over the interval \( r - n_i \leq j \leq r \) and therefore

\( (r-j)! \leq r! \).

Since

\[ \sum_{j=0}^{r} \binom{r}{j} = 2^r, \]

it follows then from (4) that also

\[ \sum_{j=0}^{r} |p(i, j)| \leq 2^n B^r \quad (i = 1, 2, \ldots, k). \]

From their construction, the \( p(i, j) \) likewise are integers, and they vanish whenever \((i, j)\) does not lie in \( S \).

From (1), (2), (4), and the definition of \( G_{h+1} \) it finally follows that

\[ |f_h| \leq k(B/a)^h (k^n B^{m^n})^{(n^2)} \quad \text{for} \quad h \geq m. \]

5. By construction, not all the polynomials \( P_i(z) \) vanish identically. Denote by \( i_1, \ldots, i_K \), where \( 1 \leq K \leq k \), all the distinct suffices \( i \) for which \( p_i(z) \neq 0 \). Then, by what was said in \( \S \) 2 about the exponential functions \( E_1^r, \ldots, E_K^r \), the \( K \) functions

\[ g_1(z) = P_{i_1}(z) E_{i_1}^r, \quad \ldots, \quad g_K(z) = P_{i_K}(z) E_{i_K}^r \]

are linearly independent over the complex number field so that the Wronskian determinant

\[ W(z) = \begin{vmatrix} g_1(z) & g_2(z) & \cdots & g_K(z) \\ g'_1(z) & g'_2(z) & \cdots & g'_K(z) \\ \vdots & \vdots & \ddots & \vdots \\ g^{(K-1)}_1(z) & g^{(K-1)}_2(z) & \cdots & g^{(K-1)}_K(z) \end{vmatrix} \]

does not vanish identically.

Let now \( D \) be the differential operator

\[ D = \frac{d}{dz}. \]

By the definition of \( E_i \) and by a well known symbolic relation,

\[ g^{(i)}_1(z) = \left( \frac{d}{dz} \right)^i \left( P_{i_1}(z) E_{i_1}^r \right) = E_{i_1}^r \left( D + \frac{a_{i_1}}{a} \right)^i P_{i_1}(z). \]

Put therefore

\[ P_{ij}(z) = \left( D + \frac{a_{i_1}}{a} \right)^j P_i(z) \quad (i = 1, 2, \ldots, k; \ j = 0, 1, 2, \ldots), \]

so that

\[ g^{(i)}_j(z) = P_{i_j}(z) E_{i_j}^r \quad (i = 1, 2, \ldots, K; \ j = 0, 1, 2, \ldots). \]

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It follows that

\[ W(z) = (E_1 z E_2 \ldots E_{K} z) w(z) \]

where \( w(z) \) denotes the new determinant

\[
\begin{vmatrix}
E_{ij}(z) & E_{ik}(z) & \cdots & E_{ik}(z) \\
E_{ij}(z) & E_{ij}(z) & \cdots & E_{ij}(z) \\
\cdots & \cdots & \cdots & \cdots \\
E_{ij}(z) & E_{ij}(z) & \cdots & E_{ij}(z)
\end{vmatrix}
\]

which naturally also is not identically zero.

In this determinant \( w(z) \) multiply, for \( l = 1, 2, \ldots, K \), the \( l \)th column by the factor \( E_{ij}(z) \), and afterwards add the 2nd, 3rd, \ldots, \( K \)th new columns to the first new column. This leads to the formula

\[
w(z)E_{ij} = \begin{vmatrix} F(z) & P_{1i}(z) & \cdots & P_{Ki}(z) \\
F(z) & P_{11}(z) & \cdots & P_{11}(z) \\
\cdots & \cdots & \cdots & \cdots \\
P_{K-1}(z) & P_{K-1}(z) & \cdots & P_{K-1}(z)
\end{vmatrix}
\]

because

\[
F^{(j)}(z) = \sum_{l=1}^{K} P_{lj}(z) E_{lj}^{(j)} \quad (j = 0, 1, 2, \ldots).
\]

On multiplying in this determinant the successive rows by the factors \( 1, z, z^2, \ldots, z^{K-1} \), respectively, we finally arrive at the equation

\[
\begin{align*}
\sum_{l=1}^{K} & P_{lj}(z) E_{lj}^{(j)} = \begin{vmatrix} F(z) & P_{1j}(z) & \cdots & P_{Kj}(z) \\
F(z) & P_{1j}(z) & \cdots & P_{1j}(z) \\
\cdots & \cdots & \cdots & \cdots \\
P_{Kj-1}(z) & P_{Kj-1}(z) & \cdots & P_{Kj-1}(z)
\end{vmatrix} \\
& \qquad \times z^{K-1} P_{i1}(z) P_{i2}(z) \cdots P_{iK}(z) \times z^{K-1} P_{i1}(z) P_{i2}(z) \cdots P_{iK}(z).
\end{align*}
\]

6. By (7), all the \( P_{ij}(z) \) are polynomials in \( z \) at most of degree \( r_i \), and hence \( w(z) \) is a polynomial in \( z \) at most of degree \( Kr \). On the other hand, in the determinant (8), all elements of the first column have at \( z = 0 \) a zero at least of order \( m \), while, for \( l = 2, 3, \ldots, K \), all elements of the \( l \)th column have at \( z = 0 \) a zero at least of order \( r - r_i \), respectively. Hence \( w(z) \) itself has at \( z = 0 \) a zero of order not less than

\[
\omega = m + \sum_{i=2}^{K} (r_i - r_i) - \frac{K(K-1)}{2}.
\]

Since \( w(z) \neq 0 \) is at most of degree \( Kr \), it follows that we can write

\[
w(z) = z^\omega P(z)
\]

where \( P(z) \neq 0 \) is a polynomial of degree \( s = Kr - \omega \). Naturally, \( s \) cannot be negative.

Let us for the moment, without loss of generality, assume that \( r = r_1 \); this assumption can always be satisfied by a suitable renumbering of the pairs of integers \((a_1, r_1), \ldots, (a_k, r_k)\). Let us further from now on always assume that

\[
(\text{A}) \quad r_k \geq R + k^* - k + 1.
\]

The first assumption insures that, in explicit form,

\[
s = Kr - \left[ r + \sum_{i=2}^{k} r_i - R \right] - \sum_{i=2}^{K} \left( (r - r_i) + \frac{K(K-1)}{2} \right)
\]

\[
= \sum_{i=2}^{k} r_i - \sum_{i=2}^{K} r_i - k + R + \frac{K(K-1)}{2}.
\]

The hypothesis (A) implies that

\[ K = k. \]

For if \( k \leq k - 1 \), then there exists a suffix \( I \) in the interval \( 2 \leq i \leq k \) such that

\[
\sum_{i=2}^{k} r_i - \sum_{i=2}^{K} r_i \leq -r_i \leq -R + k^* + k - 1,
\]

and hence it follows from (9) that \( s \leq -1 \) which is absurd.

Since then \( K = k \), and since by our notation we may take \( i_1 = 1 \), we obtain

\[
\sum_{i=2}^{k} r_i = \sum_{i=2}^{K} r_i,
\]

so that the relation (9) leads to the following result.

**Lemma 2.** Assume that the condition (A) is satisfied. Then none of the polynomials

\[
P_1(z), \ldots, P_k(z), w(z), P(z)
\]

vanishes identically. Here \( w(z) \) is the determinant

\[
w(z) = \begin{vmatrix} P_{10}(z) & P_{11}(z) & \cdots & P_{1k}(z) \\
P_{10}(z) & P_{11}(z) & \cdots & P_{1k}(z) \\
\cdots & \cdots & \cdots & \cdots \\
P_{1k-1}(z) & P_{1k-1}(z) & \cdots & P_{1k-1}(z)
\end{vmatrix}
\]

\[
P_{11}(z) & P_{12}(z) & \cdots & P_{1k}(z) \\
P_{11}(z) & P_{12}(z) & \cdots & P_{1k}(z) \\
\cdots & \cdots & \cdots & \cdots \\
P_{1k-1}(z) & P_{1k-1}(z) & \cdots & P_{1k-1}(z)
\end{vmatrix}
\]
and
\[ w(x) = e^{\sigma \Pi(x)} \]
where \( \Pi(x) \) is a polynomial at most of degree
\[ s = R + k^* - k. \]

7. The polynomials \( P_q(x) \) have been defined by the equations (7). These equations show that they have rational coefficients, hence that the values \( P_q(1) \) are rational numbers. In terms of these polynomials, the derivatives
\[ F_i^{(j)}(x) = \sum_{j=1}^{k} P_q(x) E_i^j \quad (j = 0, 1, 2, \ldots) \]
are linear forms in the \( k \) exponential functions \( E_1^1, \ldots, E_k^k \).

By Lemma 2, the determinant \( w(x) \) of the first \( k \) of these linear forms is not identically zero and has at \( s = 1 \) a zero at most of order
\[ s = R + k^* - k. \]
Let it in fact have a zero of the exact order \( \sigma \) so that
\[ w(1) = w'(1) = \ldots = w^{(\sigma-1)}(1) = 0, \quad w^{(\sigma)}(1) \neq 0, \quad \text{where} \quad 0 \leq \sigma \leq s. \]

On solving the first \( k \) linear forms
\[ F_i^{(j)}(x) = \sum_{j=1}^{k} P_q(x) E_i^j \quad (j = 0, 1, \ldots, k-1) \]
for \( E_i^j \), we obtain equations of the form
\[ w(x) E_i^j = \sum_{s=0}^{k-1} q_s(x) F^s_i(x) \quad (i = 1, 2, \ldots, k) \]
where the \( q_s(x) \) are cofactors of the determinant \( w(x) \) and hence are again polynomials in \( x \) with rational coefficients.

Differentiate these \( k \) equations \( \sigma \) times. Then
\[ \sum_{s=0}^{\sigma} q_s(x)(a_i/a)^{s-h} E_i^j = \sum_{s=0}^{k-1} Q_s(x) F^s_i(x) \quad (i = 1, 2, \ldots, k) \]
where also the \( Q_s(x) \) are polynomials in \( x \) with rational coefficients.

Here finally put \( s = 1 \). Then, by (10),
\[ w^{(\sigma)}(1) E_i^j = \sum_{s=0}^{k-1} Q_s(1) F^s_i(1) \quad (i = 1, 2, \ldots, k). \]
The \( k + \sigma \) expressions
\[ F_i^{(j)}(1) = \sum_{j=1}^{k} P_q(1) E_i^j \quad (j = 0, 1, \ldots, k + \sigma - 1) \]
on the right-hand sides of these equations are linear forms in \( E_1, \ldots, E_k \) with rational coefficients, and these \( k + \sigma \) linear forms can, by \( w^{(\sigma)}(1) \neq 0 \), be solved for each of the \( E_i \).

It follows that there exist \( k \) distinct suffices \( J = J(1), J(2), \ldots, J(k) \) in the interval \( 0 \leq J \leq k + \sigma - 1 = R + k^* - 1 \) for which the corresponding linear forms
\[ F_i^{(J)}(1) = \sum_{s=1}^{k} P_q(x_j(1)) E_i^j \quad (j = 1, 2, \ldots, k) \]
in \( E_1, \ldots, E_k \) are linearly independent. Hence the determinant of these forms
\[ \Omega = \begin{vmatrix} P_{1,x_j(1)} \ldots P_{k,x_j(1)} \\ \vdots \ldots \vdots \\ P_{1,x_{j+k}(1)} \ldots P_{k,x_{j+k}(1)} \end{vmatrix} \]
is distinct from zero.

8. The new polynomials
\[ a^j P_q(x) = (aD + a_j)^j P_q(x) \quad (i = 1, 2, \ldots, k; \ j = 0, 1, 2, \ldots) \]
are again at most of degree \( r \), but have integral rather than rational coefficients, say
\[ a^j P_q(x) = \sum_{h=0}^{r} p[h, i, j] x^h \quad (i = 1, 2, \ldots, k; \ j = 0, 1, 2, \ldots). \]

From
\[ P_i(x) = \sum_{h=0}^{r} p(i, j) x^h \]
it follows that \( a^j P_q(x) \) has the explicit form
\[ a^j P_q(x) = \sum_{h=0}^{r} \frac{1}{h!} a^j q_i^{j-1} p(i, h) h(h-1) \ldots (h-l+1) x^{h-l} \quad (i = 1, 2, \ldots, k; \ j = 0, 1, 2, \ldots). \]
Here
\[ \sum_{h=0}^{r} \frac{1}{h!} a^j q_i^{j-1} \leq B^j \quad \text{and} \quad h(h-1) \ldots (h-l+1) \leq h^l \leq r^l, \]
so that
\[ \sum_{h=0}^{r} |p[h, i, j]| \leq (rB)^j \sum_{k=0}^{r} |p[i, k]|. \]
and therefore, by (5),

\[(12) \quad \sum_{j=0}^{r} |p[k, i, j]| \leq (rB)^{j}2^{r}r_{1}!(k^{m}B^{n})^{x^{r}}
\]

\[(i = 1, 2, \ldots, k; j = 0, 1, 2, \ldots).\]

Put now

\[g_{ij} = a^{ij}P_{i,j} \text{ for } (i, j) \text{ in(1)} \]

Then all the numbers \(g_{ij}\) are integers, and by §7 their determinant

\[g = \begin{vmatrix} g_{11} & \ldots & g_{1k} \\ \vdots & \ddots & \vdots \\ g_{k1} & \ldots & g_{kk} \end{vmatrix} = a^{k}Q\]

does not vanish.

Since none of the suffixes \(J(j)\) exceeds \(R + k^{*} - 1\), we deduce immediately from the estimate (12) that

\[(13) \quad |g_{ij}| \leq C_{1}r_{1}! \quad (i, j = 1, 2, \ldots, k)\]

where \(C_{1}\) denotes the expression

\[(14) \quad C_{1} = 2^{x}(rB)^{R+k^{*}-1}(k^{m}B^{n})^{x^{r}}.\]

9. In analogy to the integers \(g_{ij}\) put

\[L_{j} = a^{(0)}P^{(0)}(1) \quad (j = 1, 2, \ldots, k),\]

so that \(L_{j}\) is the linear form

\[L_{j} = g_{1j}E_{1} + \ldots + g_{kj}E_{k} \quad (j = 1, 2, \ldots, k)\]

in \(E_{1}, \ldots, E_{k}\). An upper estimate for \(|L_{j}|\) is obtained as follows.

The hypothesis

\[(A) \quad r_{c} \geq R + k^{*} - k + 1\]

implies that

\[m = r_{1} + \ldots + r_{k} + k - R \geq k(R + k^{*} - k + 1) + k - R
\]

\[= (k-1)R + (k-2)k^{*} + k,
\]

hence, by \(k \geq 2\), that

\[(15) \quad m > R + k^{*} - 1.
\]

From

\[F(x) = r! \sum_{h=m}^{\infty} f_{h} \frac{x^{h}}{h!}
\]

it follows further that

\[F^{(0)}(x) = r! \sum_{h=m}^{\infty} f_{h} \frac{x^{h}}{(h-j)!}.
\]

Here we proved already the estimate

\[(6) \quad |f_{j}| \leq k(B/a)^{k}(k^{m}B^{n})^{x^{r}} \quad \text{for} \quad h \geq m.
\]

Hence it follows that

\[|a^{k}P^{(0)}(1)| \leq a^{k}r! \sum_{h=m}^{\infty} k(B/a)^{k}(k^{m}B^{n})^{x^{r}} \frac{1}{(h-j)!}.
\]

Here substitute \(h = m + l\) in the infinite series; the right-hand side assumes then the form

\[\frac{a^{k}r!}{(m-j)!} \frac{(B/a)^{m}(k^{m}B^{n})^{x^{r}}}{(m-j+1)(m-j+2) \ldots (m-j+l)}\]

where for \(j \leq m\) the infinite series satisfies the inequality

\[\sum_{j=m}^{\infty} \leq a^{B^{a}}.
\]

Finally let \(j\) run over the suffixes \(J(1), \ldots, J(k)\). These suffixes do not exceed \(R + k^{*} - 1\), hence by (15) are less than \(m\). Thus we obtain the estimate

\[(16) \quad |L_{j}| \leq a^{R+k^{*}-1}e^{B/a}r_{1}! \frac{(B/a)^{m}(k^{m}B^{n})^{x^{r}}}{(m-R-k^{*}-1)!} \quad (j = 1, 2, \ldots, k).
\]

Assume now again, just as in §6, that \(r_{1}\) is the largest of the integers \(r_{1}, \ldots, r_{k}\), thus that \(r = r_{1}\). By (16),

\[|L_{j}r_{1}! \ldots r_{k}!| \leq \frac{r_{1}! \ldots r_{k}!}{(m-R-k^{*}-1)!} k^{m}B^{n}e^{R+k^{*}-1-m}(k^{m}B^{n})^{x^{r}}.
\]

Here, by (15),

\[a^{R+k^{*}-1-m} \leq 1.
\]

Further

\[0 < m - R - k^{*} - 1 = (r_{1} + \ldots + r_{k}) - (2R + k^{*} - k - 1)
\]

and \(r_{1} + \ldots + r_{k} \leq kr\),

hence

\[(m - R - k^{*} - 1)! \geq (r_{1} + \ldots + r_{k})!(kr)^{(2R+k^{*}-k-1)}.
\]
Also
\[
\frac{r_1! \ldots r_k!}{(r_1 + \ldots + r_k)!} \leq 1
\]
because the reciprocal of this fraction is an integer. It follows then that
\[
|L_j r_1! \ldots r_k!| \leq C_2 \quad (j = 1, 2, \ldots, k)
\]
where \( C_2 \) denotes the expression
\[
C_2 = \frac{\kappa e^{B/2} (k\pi)^{2k} + e^{B\kappa} (e^{k\pi} / 4k)^{k/4}}{4k^{3/4}}.
\]
The results so proved in this and the preceding section may be combined into the following lemma.

**Lemma 3.** Let the notation be as in §2 and assume in addition that
\[
r = r_1 \quad \text{and} \quad r_\infty > B + k^* + k + 1.
\]

Then there exist \( k \) linearly independent linear forms
\[
L_j = g_{j1} E_1 + \ldots + g_{jk} E_k \quad (j = 1, 2, \ldots, k)
\]
with integral coefficients \( g_{jk} \) such that
\[
|g_{j1}| \leq C_1 r_j! \quad (i, j = 1, 2, \ldots, k),
\]
\[
|L_j r_1! \ldots r_k!| \leq C_4 \quad (j = 1, 2, \ldots, k),
\]
where \( C_1 \) and \( C_4 \) are defined by (14) and (18), respectively.

**10.** Lemma 3 will now be applied to the study of a general linear form. Denote by
\[
L = x_1 E_1 + \ldots + x_k E_k
\]
a linear form in \( E_1, \ldots, E_k \) with integral coefficients not all zero, and put
\[
a'_j = 1 \quad \text{if} \quad x_j = 0, \quad \text{and} \quad a'_j = a_j \quad \text{if} \quad x_j \neq 0 \quad (j = 1, 2, \ldots, k),
\]
and
\[
x = \max(|x_1|, \ldots, |x_k|) = \max(|a'_1|, \ldots, |a'_k|).
\]
We shall now choose the parameters \( r_1, \ldots, r_k, B \) of Lemma 3 as functions of \( a'_1, \ldots, a'_k \) by the following construction.

Put
\[
C = C(r) = k^3 r ((\log B) (\log r)^{1/2}),
\]
and define a function \( f(r) \) of the positive integer \( r \) by
\[
f(r) = e^{-\frac{C(r)}{r}}.
\]
A well known form of Sterling's formula states that
\[
r! = \sqrt{2\pi e} r^r e^{-r+\frac{1}{12r}} \quad \text{where} \quad 0 < \rho(r) < \frac{1}{12r}.
\]
It follows that
\[
\frac{\log f(r)}{r} = \log r - 2k^3 ((\log B) (\log r)^{1/2} - 1 + \sigma(r),
\]
where \( \sigma(r) \) denotes the expression
\[
\sigma(r) = \frac{\log r}{2r} + \frac{\log 2\pi}{2r} + \rho(r).
\]
Here, for \( r \geq 2 \), it is easily verified that
\[
0 < \sigma(r) < 1,
\]
hence that
\[
(19) \quad \log r - 2k^3 ((\log B) (\log r)^{1/2} - 1 < \frac{\log f(r)}{r} < \log r - 2k^3 ((\log B) (\log r)^{1/2}).
\]

The definition of \( f(r) \) and this inequality show immediately that
\[
(20) \quad f(1) = 1, \quad f(r) < 1 \quad \text{if} \quad 2 \leq r \leq B^{k^*}
\]
It is also obvious that
\[
(21) \quad C(r-1) < C(r) \quad \text{if} \quad r \geq 2.
\]
By definition, \( x \) is a positive integer. There exists therefore a smallest positive integer \( r \) such that
\[
f(r) > x,
\]
and this integer necessarily has the further properties
\[
(22) \quad f(r-1) < x < f(r),
\]
so that by (21) also
\[
(23) \quad (r-1)! < e^{C(r)} x < r!.
\]
Define similarly the integers \( r_1, \ldots, r_k \) by the inequalities
\[
(24) \quad (r_j-1)! < e^{C(r)} |a'_j| < r_j! \quad (j = 1, 2, \ldots, k).
\]
Then by (23) and (24) and in agreement with the hypothesis of §2,
\[
r = \max(r_1, \ldots, r_k).
\]
Without loss of generality, let from now on
\[
x = |a'_1| = |a_1|
be the largest of the integers \(|x|_1, \ldots, |x|_n\). The formulae (23) and (24) imply then that also

\[ r = r_1 \]

is the largest of the integers \(r_1, \ldots, r_k\), in agreement with the previous assumption.

By (22), \(f(r)\) is greater than \(x \geq 1\). Hence, by (20) necessarily

\[ r \geq B^{2k^4}+1. \]

11. Having fixed \(r, r_1, \ldots, r_k\) in this manner, define now \(R\) by

\[ R = \left[ kr \left( \frac{\log B}{\log r} \right)^{1/2} \right] + 1, \]

so that

\[ kr \left( \frac{\log B}{\log r} \right)^{1/2} < R \leq kr \left( \frac{\log B}{\log r} \right)^{1/2} + 1. \]

By (25) and \(k \geq 2\) this choice implies that

\[ R < \frac{k}{2k^4} r + 1 < r + k - 1, \]

and since \(r(\log r)^{-1/2}\) is an increasing function of \(r\) when (25) holds, it also follows from \(B \geq 2\) that

\[ R > \frac{B^{2k^4}}{2k} \geq \frac{2^{2k^4}}{2k} > \frac{4k^4}{2k} > \max(k, k^*). \]

Hence the condition

\[ k \leq R \leq r_1 + \ldots + r_k + k - 1 \]

of § 2 is certainly satisfied. It further follows that

\[ R + k^* - k + 1 < 2R. \]

The former hypothesis

(A)

\[ r_0 \geq R + k^* - k + 1 \]

does then certainly hold if

\[ r_j < 2R \quad (j = 1, 2, \ldots, k). \]

That this set of inequalities is in fact satisfied will now be proved indirectly. Assume there exists a suffix \(j\) for which

\[ r_j < 2R. \]

Then, by (26) and (27),

\[ r_j < 2R \leq 2kr \left( \frac{\log B}{\log r} \right)^{1/2} + 2 < 3kr \left( \frac{\log B}{\log r} \right)^{1/2}, \]

whence

\[ \log r_j < r_j \log r_j < 3kr \left( \frac{\log B}{\log r} \right)^{1/2} \log \left( 3kr \left( \frac{\log B}{\log r} \right)^{1/2} \right). \]

Here, again by (25) and by \(k \geq 2\),

\[ 3k \left( \frac{\log B}{\log r} \right)^{1/2} < \frac{3k}{2k^3} < 1. \]

Hence

\[ \log r_j < 3kr \left( \frac{\log B}{\log r} \right)^{1/2} \cdot \log r < 3kr \left( \log B(\log r) \right)^{1/2}, \]

and so, once more by \(k \geq 2\),

\[ r_j < e^{10r_j}, \]

contrary to the definition (24) of \(r_j\) because \(|x|_j\) is at least 1.

We have thus proved that the definitions (22), (24), and (26) of \(r, r_1, \ldots, r_k, \) and \(R\), together with a notation such that \(x = |x|_l\) and hence also \(r = r_1\), satisfy all the conditions of § 2 and of Lemma 3. We are thus allowed to apply this lemma.

12. This means that, in addition to the given linear form

\[ L = x_1 E_1 + \ldots + x_k E_k, \]

there exist the \(k\) linearly independent linear forms

\[ L_j = g_1 E_1 + \ldots + g_k E_k \quad (j = 1, 2, \ldots, k) \]

of the lemma which have integral coefficients such that

\[ |g_i| \leq C_i r_j \quad (i = 1, 2, \ldots, k), \]

\[ |L_j r_1 r_2 \ldots r_k| \leq C_i \quad (j = 1, 2, \ldots, k). \]

The form \(L\) is then linearly independent of certain \(k - 1\) of the forms \(L_j\). To fix the ideas, assume that the \(k\) forms

(29)

\[ L, L_2, \ldots, L_k \]

are linearly independent. Hence their determinant

\[
\begin{vmatrix}
x_1 & \ldots & x_k \\
g_1 & \ldots & g_k \\
\vdots & \ldots & \vdots \\
g_{1k} & \ldots & g_{kk}
\end{vmatrix}
= \Delta,
\]

is not zero.
does not vanish. This determinant is an integer, and so

$$|D| \geq 1.$$  

On solving the $k$ forms (29), say for $E_3$, we find that

$$\begin{bmatrix} L & a_2 & \ldots & a_k \\ L_2 & g_2 & \ldots & g_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ L_k & g_{k2} & \ldots & g_{kk} \end{bmatrix} = \Delta E_1.$$  

It follows that

$$\Delta E_j^{(b)} = LM + L_2 M_3 + \ldots + L_k M_k,$$

where $M, M_3, \ldots, M_k$ denote the cofactors of the elements $L, L_2, \ldots, L_k$ in the first column of the determinant for $\Delta E_j$, respectively.

By (28),

$$|M| \leq (k-1)! C_1^{k-1} r_4 \ldots r_k,$$

and since $\sigma_j \leq [\sigma_j]$,  

$$|M_j| \leq (k-2)! C_1^{k-2} r_4 \ldots r_k \sum_{j=3}^k \frac{[\sigma_j]}{\sigma_j}$$

$$\text{(j = 2, 3, \ldots, k)}.$$

The identity (30) implies therefore the inequality

$$1 \leq U + V,$$

where

$$U = (k-1)! e^{-a_2/2} C_1^{k-1} r_4 \ldots r_k |L|, \quad V = (k-1)! e^{-a_2/2} C_1^{k-2} r_2 \sum_{j=3}^k \frac{[\sigma_j]}{\sigma_j}.$$

We shall next establish upper estimates for $U$ and $V$.

13. Since $r_j \leq r$ for all $j$, by (24),

$$r_2 \ldots r_k \leq r^{k-1} e^{-a_2/2} |a_2| \ldots |a_k|,$$

and therefore

$$U \leq (k-1)! e^{-a_2/2} C_1^{k-1} e^{-a_2/2} |a_2| \ldots |a_k| L,$$

$$V \leq \sum_{j=3}^k \frac{[\sigma_j]}{\sigma_j} e^{-a_2/2} L.$$

Here, by the definitions (14) and (18),

$$C_1 = 2^r (rB)^{2r+k-1} (B^m B^{m^*})^{Br}, \quad C_2 = \frac{B}{B} (kr)^{2r+k-1} (B^m B^{m^*})^{Br}.$$

It follows that

$$U \leq C_1 |a_2| \ldots |a_k| L, \quad V \leq C_4,$$

where $C_3$ and $C_4$ are defined by

$$C_3 = (k-1)! e^{-a_2/2} (2^r (rB)^{2r+k-1} (B^m B^{m^*})^{Br}) e^{-a_2/2} \leq (k-1)! e^{-a_2/2}.$$  

$$C_4 = \frac{B}{B^m B^{m^*}}.$$  

Here it is convenient to split off the factors of maximal size from $C_3$ and $C_4$ and to write these expressions as

$$C_3 = C_5 e^{-a_2/2} \log(k^{k_1} B^{k_1} B^m B^{m^*}), \quad C_4 = C_6 e^{-a_2/2} \log(k^{k_1} B^{k_1} B^m B^{m^*}),$$

where the new factors $C_5$ and $C_6$ are given by

$$C_5 = (k-1)! e^{-a_2/2} g(k-1, k_1, k_1, k_1, k_1) \leq (k-1)! e^{-a_2/2},$$

and

$$C_6 = \frac{B}{B^m B^{m^*}}.$$  

We shall next establish upper estimates for $2C_3$ and $2C_6$ and hence also for $2C_5$ and $2C_6$.

14. Firstly, by the definition (26) of $R$,

$$R - 1 \leq \frac{\log B}{\log r} \leq R,$$

while

$$m \leq kr \quad \text{and therefore} \quad m^* \leq \frac{k^2 r^2}{2}.$$  

The second factors of $C_5$ and $C_4$ in (34) have therefore the upper bounds

$$e^{\log(k-1)(R-1)^{2r+k-1}} \leq \frac{1}{k^2 - 1},$$

and

$$e^{-a_2/2} \log(k^{k_1} B^{k_1} B^m B^{m^*}) \leq e^{-a_2/2}.$$  

To deal with the first factors $C_3$ and $C_4$, we first note that

$$2(k-1)! \leq k^2, \quad k \cdot k \leq k^2, \quad (k-1)(k^2 + 1) \leq k^3 / 2,$$

$$(k-1)k^2 \leq k^3 / 2,$$

and

$$e^{-a_2/2} \leq e^B, \quad e^{2-a_2/2} \leq e^B.$$
Therefore
\[ 2C_5 \leq k^6 e^{p^2 r^2} k^{x^2 r^2} k^{x^2 r^2} B^{x^2 r} = e^{C_5} \text{ say,} \]
and
\[ 2C_6 \leq k^6 e^{p^2 r^2} k^{x^2 r^2} k^{x^2 r^2} k^{x^4 r^2} k^{x^4 r^2} B^{x^4 r^2} = e^{C_6} \text{ say.} \]

Here
\[ k \geq 2, \quad \log k \leq e^{-1} k < k, \quad B \geq 2, \quad \log B \leq e^{-1} B < B, \quad 2 \log B > 1. \]

Also, by (25),
\[ r > B^{4kr} > 2^{64} > e^{42}, \quad \log r > 4k^4 \log B > 2^{64} > 32. \]

Therefore
\[ (\log B)(\log r)^{1/2} > 2k^4 \log B > k^4, \]
whence
\[ C(r) = k^4 r (\log B)(\log r)^{1/2} > k^4 r > k^4 B^{4k^4} > 2^{64}. \]

We also note that, for the values of \( r \) considered, the function
\[ \frac{\log r}{r} \]
is strictly decreasing and hence satisfies the inequality
\[ \frac{\log r}{r} < \frac{4k^4 \log B}{B^{4k^4}} = \frac{4k^4}{B^{4k^4-1}} \leq \frac{2^{64}}{2^{64}} = 2^{-32}. \]

Thus the following upper estimates for the successive terms of \( C_5 \) and \( C_6 \) are obtained.
\[ C(r)^{-1} \cdot k \cdot \log k < \frac{k^3}{k^4 r} < \frac{1}{2^{32}} < 2^{-32}; \]
\[ C(r)^{-1} \cdot 2B < \frac{2B}{k^4 B^{4k^4}} \leq \frac{1}{k^4 B^{4k^4-1}} \leq \frac{1}{2^{32}} < 2^{-32}; \]
\[ C(r)^{-1} \cdot kr \cdot \log 2 < \frac{\log 2}{k((\log B)(\log r))^{1/2}} < \frac{\log 2}{k \cdot 2^{2k^4} \log B} \leq \frac{1}{2^{k^3}} \leq \frac{1}{16}; \]
\[ C(r)^{-1} \cdot 2R \cdot \log k < \frac{3k^2}{k^4 r ((\log B)(\log r))^{1/2}} = \frac{3}{k^4 \log r} \leq \frac{3}{64} < 2^{-32}; \]
\[ C(r)^{-1} \cdot kr \cdot \log k < \frac{k^3 \log B}{(\log r)^{1/2}} = \frac{k^3 \log B}{(\log r)^{1/2}} < \frac{1}{k \cdot 2^{k^3}} < \frac{1}{40}; \]

On adding these results it follows at once that
\[ C_5 < \frac{1}{4} C(r) \quad \text{and} \quad C_6 < \frac{1}{4} C(r), \]
hence that
\[ 2C_5 < e^{2C(r)} \quad \text{and} \quad 2C_6 < e^{2C(r)}. \]

Therefore, by (34), (35), and (36),
\[ 2C_5 < e^{(2k^{-1} - \frac{3}{4})C(r)} < e^{(2k^{-1} - \frac{1}{4})C(r)} < e^{2C(r)} \]
and
\[ 2C_6 < e^{-\frac{1}{k} - \frac{3}{4} C(r)} < e^{-\frac{1}{4} C(r)} < 1. \]

By (33), these estimates imply that
\[ 2U < e^{2C(r)} |z_1' \ldots z_k' L| \quad \text{and} \quad 2V < 1. \]
(In fact, they imply the slightly stronger inequalities
\[ 2U < e^{2k^{-1} C(r)} |z_1' \ldots z_k' L| \quad \text{and} \quad 2V < e^{-4C(r)}. \]

Since \( 2V < 1 \), it next follows from (31) that
\[ 2U > 1. \]
This we combine with the upper bound for \( 2U \) just obtained, and we multiply both sides of the resulting inequality by the factor \( x = |a_1| \). We may then again drop the hypothesis that \( |a_1| = \max(|a_1|, \ldots, |a_k|) \) and so obtain the following result.

**Theorem 1.** Let \( a_1, \ldots, a_k \), where \( k \geq 2 \), be distinct integers, and let \( a > 0 \) be an integer satisfying \( (a, a_1, \ldots, a_k) = 1 \); let further \( x_1, \ldots, x_k \) be integers not all zero. Put

\[
B = a + \max(|a_1|, \ldots, |a_k|), \quad E_1 = e^{a_1/a}, \ldots, E_k = e^{a_k/a}
\]

and

\[
x_j = \begin{cases} 1 & \text{if } x_j = 0 \\ x_j & \text{if } x_j \neq 0 \end{cases} \quad (j = 1, 2, \ldots, k); \\
x = \max(|x_1|, \ldots, |x_k|).
\]

Also, for positive integer \( r \), put

\[
C(r) = k^2r((\log B)(\log r))^{1/2}, \quad f(r) = e^{-C(r)r}.
\]

If \( r \) denotes the smallest positive integer for which

\[
f(r-1) \leq x < f(r),
\]

then

\[
r \geq B^{1/4} + 1
\]

and

\[
|x_1 E_1 + x_2 E_2 + \ldots + x_k E_k| > xe^{-C(r)},
\]

By (39), this inequality may in fact be replaced by

\[
|x_1 E_1 + x_2 E_2 + \ldots + x_k E_k| > xe^{-C(r)},
\]

which is slightly stronger.

15. As a corollary to this theorem we show how it simplifies when \( x \) is very large, thus under the hypothesis that, say

\[
x \geq B^{1/4} + 1.
\]

It had been found in § 10, formula (19), that

\[
\log r - 2k^2((\log B)(\log r))^{1/2} - 1 < \frac{\log f(r)}{r} < \log r - 2k^2((\log B)(\log r))^{1/2}.
\]

Here the right-hand side implies that

\[
f(r) < r^x.
\]

Since \( f(r) > x \), it follows therefore from (41) that now

\[
r \geq B^{1/4} + 1,
\]

hence, by \( k \geq 2 \) and \( B \geq 2 \), that

\[
r > 2^{k^2}.
\]

By the first lower bound for \( r \),

\[
(\log B)^{1/2} \leq \frac{1}{4k^2},
\]

whence, by (19),

\[
\frac{f(r-1)}{r-1} > \frac{1}{2} \log (r-1) - 1.
\]

This implies that

\[
\frac{f(r-1)}{r-1} > \frac{r-1}{r} (\frac{1}{2} \log (r-1) - 1) > \frac{1}{2} \log r,
\]

as follows from the very large lower bound (42) for \( r \). Thus also

\[
f(r-1) > r^{1/2}.
\]

The integer \( r \) is then connected with \( x \) by the inequalities

\[
r^{1/2} < x < r,
\]

so that

\[
\frac{r}{3} \log r < \log x < r \cdot \log r,
\]

\[
\log r - \log 3 + \log \log 2 < \log \log x < \log r + \log \log r \leq 2 \cdot \log r.
\]

On the left-hand side, by (42), trivially

\[
\log \log r > \log 3,
\]

so that

\[
\log r < \log \log x < 2 \cdot \log r.
\]

On combining the last inequalities, it follows then that

\[
\frac{\log x}{\log \log x} < x < \frac{6 \cdot \log x}{\log \log x}.
\]

These inequalities combine to the result that

\[
C(r) < 6k^2((\log x)(\log B)(\log \log x)^{-1})^{1/2}.
\]

Theorem 1 implies therefore the following corollary.
Let $a_1, \ldots, a_k, a, E_1, \ldots, E_k, x_1, \ldots, x_k, a_1', a_2', \ldots, a_k'$, and $x$ be as in Theorem 1, but assume that now

$$x \geq B^{2^{ka+1}+1}.$$  

Then

$$|a'_1 \cdots a'_k(x_1E_1 + \cdots + x_kE_k)| > x^{2^{ka+1}+1} \Theta \left(\frac{1}{2^{k+1}}\right).$$  

This is Baker's first result, but with explicit constants.

16. Let $a_1, \ldots, a_k, a, E_1, \ldots, E_k$ be as in Theorem 1, but assume that $a_k$ and $E_k$ have now been specialized by taking

$$a_k = 0, \quad \text{hence} \quad E_k = 1.$$  

Denote by $y_1, \ldots, y_k$ positive integers such that

$$y_k \geq k$$  

and that the product

$$\omega = y_k! y_k! E_1 \cdots y_k! E_{k-1} = y_k! \cdots y_k!$$  

satisfies the inequality

$$0 < \omega < 1.$$  

Theorem 1 will enable us to establish a lower bound for $x$ in terms of $y_k$.

For this purpose put

$$\varphi_j = \omega^{(k-1)} |y_k! E_j - y_j| \quad (j = 1, 2, \ldots, k-1)$$  

and assume, without loss of generality, that the notation is such that

$$\varphi_1 \geq \varphi_2 \geq \ldots \geq \varphi_{k-1} > 0.$$  

Since evidently

$$\varphi_1 \varphi_2 \cdots \varphi_{k-1} = y_k! \geq k \geq 2,$$  

not all the $\varphi_j$ can be $\leq 1$.

If $\varphi_{k-1} > 1$ and hence $\varphi_j > 1$ for $j = 1, 2, \ldots, k-1$, put $\kappa = k-1$; otherwise denote by $\kappa$ the smallest suffix in the interval $1 \leq \kappa \leq k-2$ for which

$$\varphi_{\kappa+1} \varphi_{\kappa+2} \cdots \varphi_{k-1} \leq 1.$$  

By (45), such a suffix certainly exists.

17. Having fixed $x$ in this way, consider now the system of $\kappa+1 \leq \kappa$ linear inequalities,

$$|x_j| \leq y_j \quad (j = 1, 2, \ldots, \kappa-1),$$  

$$|x_\kappa| \leq \varphi_\kappa \varphi_{\kappa+1} \cdots \varphi_{k-1} \leq \varphi_\kappa,$$  

$$x_1 y_1 + \cdots + x_\kappa y_\kappa + x_\kappa y_\kappa < 1$$  

for $x_1, x_2, \ldots, x_\kappa, a_\kappa$. The $\kappa+1$ linear forms

$$x_1, x_2, \ldots, x_\kappa, a_\kappa y_1 + \cdots + x_\kappa y_\kappa + a_\kappa y_\kappa$$  

in $x_1, x_2, \ldots, x_\kappa, a_\kappa$ on the left-hand sides in (46) have the determinant $y_\kappa$; and the product of the right-hand sides is by (45) equal to the same value since

$$\varphi_{\kappa-1} \varphi_\kappa \varphi_{\kappa+1} \cdots \varphi_{k-1} = y_\kappa.$$  

Hence, by Minkowski's theorem on linear forms, the inequalities (46) can be satisfied by a system of $\kappa+1$ integers $x_1, x_2, \ldots, x_\kappa, a_\kappa$ not all zero. But since all the $x$'s and $y$'s are integers, the last inequality (46) implies the equation

$$x_1 y_1 + \cdots + x_\kappa y_\kappa + x_\kappa y_\kappa = 0.$$  

Hence it follows that already at least one of the integers

$$x_1, x_2, \ldots, x_\kappa$$  

does not vanish. On the other hand, it is uncertain, and in fact of no importance, whether $x_\kappa$ is or is not equal to zero.

We denote from now on by $i_1, i_2, \ldots, i_\kappa$ all the distinct suffixes $1, 2, \ldots, \kappa$

for which

$$x_i \neq 0 \quad (l = 1, 2, \ldots, K);$$

here naturally

$$1 \leq K < \kappa.$$  

18. The right-hand sides $x_i$ and $\varphi_\kappa \varphi_{\kappa+1} \cdots \varphi_{k-1}$ of the first $\kappa$ inequalities (46) all are greater than 1. It follows therefore from these inequalities and from the equation (45) that

$$|x_1 x_2 \cdots x_\kappa| \leq \varphi_1 \varphi_2 \cdots \varphi_{\kappa-1} \varphi_\kappa \varphi_{\kappa+1} \cdots \varphi_{k-1} = y_\kappa.$$  

It is also possible to give an upper bound for $x_\kappa$. For identically,

$$x_1 y_1 + \cdots + x_\kappa y_\kappa + a_\kappa y_\kappa = (x_1 E_1 + \cdots + x_\kappa E_\kappa + a_\kappa) y_\kappa - \sum_{j=1}^\kappa x_j (y_j E_j - y_j),$$

where
so that, by (47),

\[(a_1 E_1 + \ldots + a_k E_k + y_k) y_k = \sum_{j=1}^{\infty} \alpha_j (y_n E_j - y_j).\]

Here

\[|x_j| \leq \alpha_j, \quad |y_n E_j - y_j| = \frac{\omega^{l(k-1)}}{q_j} \quad (j = 1, 2, \ldots, k).\]

It follows then from (49) that

\[|x_1 E_1 + \ldots + x_k E_k + x_k| \leq n \cdot \omega^{l(k-1)}.\]

Here \(n \leq k-1, \omega < 1, \text{ and } y_k > k, \) so that also

\[(a_1 E_1 + \ldots + a_k E_k + x_k) \leq \frac{(k-1) \omega^{l(k-1)}}{y_k} < 1.\]

Thus

\[|y_k| < 1 + |a_1 E_1 + \ldots + a_k E_k|,\]

where

\[E_j = \epsilon^{q_j} \leq \epsilon^{q_j} \leq \epsilon^{D} \quad (j = 1, 2, \ldots, k).\]

Hence, by \(n \leq k-1,\)

\[|x_k| < \epsilon \omega^D \max(|a_1|, \ldots, |a_k|) = \epsilon \omega^D \max(|x_1|, \ldots, |x_k|).\]

Therefore, on noting that \(a_{k+1} = a_{k+2} = \ldots = a_{n-1} = 0\) and putting

\[\sigma = \max(|x_1|, \ldots, |x_k|) = \max(|a_1|, \ldots, |a_k|),\]

it has been proved that

\[\sigma < \epsilon \omega^D \max(|x_1|, \ldots, |x_k|).\]

All factors of the product \(a_1 a_2 \ldots a_k\) are integers not zero so that the absolute value of this product cannot be less than \(\max(|x_1|, \ldots, |x_k|).\)

The inequality (51) implies therefore that

\[|x_k a_2 \ldots a_k| > (\epsilon \omega^D)^{-1} \sigma,\]

and hence it follows from (48) that

\[\sigma \leq \epsilon \omega^D y_n.\]

Put now again

\[x_j' = \begin{cases} 1 & \text{if } x_j = 0 \\ x_j & \text{if } x_j \neq 0 \end{cases} \quad (j = 1, 2, \ldots, k).\]

Since \(|x_j| \leq \sigma, \) by (48),

\[|x_1 x_2' \ldots x_k'| \leq \sigma y_n,\]

whence, by (50),

\[(a_1 x_2' \ldots x_k' (a_1 E_1 + \ldots + a_k E_k) + x_1) \leq (k-1) \omega^{l(k-1)}.\]

19. Apply now to this inequality (53) the remark to Theorem 1. For this purpose, with a slight change of notation, denote by \(r'\) and \(r''\) the smallest positive integers satisfying

\[f(r'-1) \leq \sigma < f(r') \quad \text{and} \quad f(r''-1) \leq y_k < f(r''),\]

respectively. It follows immediately from the estimate (40) of § 14 that

\[(k-1) \omega^{l(k-1)} \geq \epsilon \omega^{D} e^{-(2\omega-1)\epsilon r'},\]

so that, by the definition of \(\omega,\)

\[y_k |y_1 E_1 - y_2| \ldots |y_k E_{k-1} - y_{k-1}| > (k-1) \omega^{l(k-1)} \epsilon \omega^{D} e^{-(2\omega-1)\epsilon r'}.\]

This formula has still the disadvantage of involving the integer \(r'\) depending on \(x\) rather than the integer \(r''\) which depends on \(y_k.\) We show now how to change over to a formula involving \(r''\).

For the moment, put

\[\beta = 2k^2 (\log B)^{1/2}.\]

In § 10 we had for every integer \(r \geq 2\) obtained the formula

\[
\frac{\log f(r)}{r} = \log r - \beta (\log r)^{1/2} - 1 + \sigma(r),
\]

where

\[\sigma(r) = \frac{\log(2\pi r)}{2r} + \frac{e(r)}{r} \quad \text{and} \quad 0 < e(r) < \frac{1}{12r}.\]

Assume now again that

\[r \geq B^{44} + 1;\]

then, by § 14,

\[r > 2^4, \quad \log r > \beta^2 > 32, \quad \log r < 3^{27}.\]

From these estimates it is easily deduced that

\[0 < e(r) < \frac{\log r}{r} < 2^{-27},\]
and that therefore
\[ \frac{\log f(r)}{r} < \log r - \beta(\log r)^{1/2} - 2^{-r}. \]

Hence, whenever \( f(r) > 1 \), then necessarily
\[ (55) \quad \log r - \beta(\log r)^{1/2} > 1 - 2^{-r} > \frac{1}{r}. \]

On the other hand, from the definition of \( f(r) \),
\[ f(r+1) = e^{\log (r+1) - \beta(\log r)^{1/2}} e^{\log (r+1) - \beta(\log r)^{1/2}}. \]

Here, by (55) applied to \( r+1 \) instead of \( r \), the first exponential factor
on the right-hand side is greater than \( e^{\beta} \). Next, by the mean value theorem
of differential calculus,
\[ (\log (r+1))^{1/2} - (\log r)^{1/2} < (2\varphi \log r)^{1/2} - 1, \]

hence, by \( \log r > \beta \),
\[ \beta r (\log (r+1))^{1/2} - (\log r)^{1/2} < 2\beta 2 = 1 \]

so that the second exponential factor is greater than \( e^{-r} \). We have thus
found the basic inequality
\[ (56) \quad f(r+1) > e^{\beta} f(r) \quad \text{if} \quad r \geq B^{1/2} + 1. \]

29. This inequality shows that \( f(r) \) is strictly increasing and that for
every pair of positive integers \( n \) and \( r \geq B^{1/2} + 1, \)
\[ f(n+1) > e^{\beta} f(n). \]

Now, by (52) and by the definitions of \( r' \) and \( r'' \),
\[ f(r'-1) \leq k e^{\beta} y_k \quad \text{and} \quad y_k > f(r''). \]

It follows that
\[ r' < r'' + 4B + 4 \log k + 1. \]

The right-hand side of the estimate (54) is then greater than
\[ (k-1)^{(b-1)} e^{-(k-1)(2k-1)} e^{(k-1)(2k-1)} e^{(k-1)(2k-1)}, = M, \]say,
for \( C(r) \) trivially is an increasing function of \( r \).

The quantity \( 4B + 4 \cdot \log k + 1 \) is negligibly small compared with
\( r'' \geq B^{1/2} + 1 \). From the definition
\[ C(r) = e^{\beta} f((\log E)(\log r)^{1/2}) \]

of \( C(r) \) we deduce then easily that
\[ M > e^{-(k-1)} \cdot \ell C(r'). \]

Hence, on writing again \( r \) for \( r'' \), the following result has been established.

**Theorem 2.** Let \( a_1, \ldots, a_{b-1} \) be distinct non-vanishing integers, and let \( a > 0 \) be an integer satisfying \( (a, a_1, \ldots, a_{b-1}) = 1 \); let
further \( y_1, \ldots, y_b \) be integers such that \( y_b > k \). Put
\[ E = a + \max \{ |a_1|, \ldots, |a_{b-1}| \}, \quad E_1 = e^{\alpha}, \ldots, E_{b-1} = e^{\alpha-1/2} \]

and define \( C(r) \) and \( f(r) \) as in Theorem 1. If \( r \) is the smallest positive integer satisfying \( f(r-1) < y_b < f(r) \), then
\[ r \geq B^{1/2} + 1 \]
and
\[ y_b/y_b E_1 - y_1 \ldots y_b E_{b-1} - y_{b-1} > e^{-2(b-1)^C(r)}. \]

21. Considerations similar to those of § 15 allow to replace this estimate
by one which, although less good, is more explicit.

We now assume that
\[ y_b \geq B^{1/2} e^{1/4}. \]

Under the same hypothesis as in Theorem 2 it follows then that
\[ y_b/y_b E_1 - y_1 \ldots y_b E_{b-1} - y_{b-1} > e^{-2(b-1)^C(r)}. \]

Apart from the explicit constants, this estimate is again due to Baker.

It is highly probable that the constants in Theorem 2 and in this
corollary can be improved by a direct application of Lemma 3 instead
of the transfer method.