ACTA ARITHMETICA
XXVII (1975)

Euler constants for arithmetical progressions

by

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Dedicated to the memory of Yu. V. Linnik

1. Introduction. Euler's constant γ is defined by

$$\gamma = \lim_{m\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}-\log m\right\} = .5772156649.$$

In this paper we study the properties of the corresponding limit $\gamma(r, k)$ obtained by considering the sum of the reciprocals of the terms of the arithmetic progression

$$r, r+k, r+2k, \ldots \qquad (0<r\leqslant k).$$

In § 2, $\gamma(r, k)$ is defined precisely and shown to exist. In § 3 we show that $\gamma(r, k)$ differs from γ/k by a linear combination of logarithms of cyclotomic integers in the field of kth roots of unity. From this a formula for $\gamma(r, k)$ involving only real numbers is deduced and specialized for certain small k. In § 4 a study is made of the $\varphi(k)$ primitive $\gamma(r, k)$ in which r and k are coprime. In § 5 the connection is made between $\gamma(r, k)$ and the logarithmic derivative ψ of the Gamma function at the point r/k. The results of § 3 are now seen to give a really elementary proof of Gauss' theorem on $\psi(z)$ for rational z. In § 6 we make applications of $\gamma(r, k)$ to certain infinite series. In particular we develop the connection between $\gamma(r, k)$ and the class number of the quadratic fields $Q(\sqrt{\pm k})$. The final § 7 contains some comments on the numerical evaluation of $\gamma(r, k)$.

2. Definition and existence of $\gamma(r, k)$. We denote by $H(x, r, k)$ the general harmonic sum associated with the arithmetical progression $r, r+$ $+k, r+2k, \ldots$ namely

$$H(x, r, k) = \sum_{\substack{0<n\leqslant x\\ n\equiv r \pmod{k}}} \frac{1}{n}.$$

By this definition

$$H(x, r \pm k, k) = H(x, r, k).$$

We now define $\gamma(r, k)$ by

$$\gamma(r, k) = \lim_{x\to\infty}\left\{H(x, r, k) - \frac{1}{k}\log x\right\}, \tag{1}$$

so that

$$\gamma(0, 1) = \gamma.$$

Also

$$\gamma(r \pm k, k) = \gamma(r, k).$$

In other words, $\gamma(r, k)$ is a periodic function of r of period k.

Since

$$H(x, 0, k) = \frac{1}{k}H\left(\frac{x}{k}, 0, 1\right),$$

$$\gamma(0, k) = \frac{1}{k}\lim_{x\to\infty}\left\{H\left(\frac{x}{k}, 0, 1\right) - \log\frac{x}{k}\right\} - (\log k)/k$$

or

$$\gamma(0, k) = (\gamma - \log k)/k. \tag{2}$$

To see that $\gamma(r, k)$ exists for $r \not\equiv 0 \pmod{k}$ we can note that, for $0 < r < k$,

$$U_n \equiv \frac{1}{r+nk} - \frac{1}{k}\log\frac{r+(n+1)k}{r+nk} = \int_0^1 \frac{kt}{(r+nk)(r+nk+t)}\,dt = O(n^{-2}).$$

Hence the infinite series

$$\sum_{n=0}^{\infty} U_n = \lim_{x\to\infty}\sum_{n=0}^{[x/k]} U_n = \lim_{x\to\infty}\left\{H(x, r, k) - \frac{1}{k}\log x\right\}$$

converges to a limit which we call $\gamma(r, k)$.

From the definition (1) we see that

$$\sum_{r=0}^{k-1}\gamma(r, k) = \gamma \tag{3}$$

and more generally

$$\sum_{\lambda=0}^{k-1}\gamma(r+\lambda m, mk) = \gamma(r, k). \tag{4}$$

In particular ($m = 2$) we have

$$\gamma(r, 2k) + \gamma(r+k, 2k) = \gamma(r, k). \tag{5}$$

Also from the definition, $\gamma(r, k)$ is a strictly decreasing function of r. More precisely

$$\gamma(1, k) > \gamma(2, k) > \ldots > \gamma(k-1, k) > \gamma(k, k) = \gamma(0, k).$$

3. A closed form for $\gamma(r, k)$. We begin with

THEOREM 1. *For $k > 1$,*

$$k\gamma(r, k) = \gamma - \sum_{j=1}^{k-1} e^{-2\pi rij/k}\log(1 - e^{2\pi ij/k}). \tag{6}$$

Proof. For $r = 0$ this is easy. The right hand side of (6) becomes

$$\gamma - \sum_{j=1}^{k-1}\log(1 - e^{2\pi ij/k}) = \gamma - \log\left\{\prod_{j=1}^{k-1}(1 - e^{2\pi ij/k})\right\} = \gamma - \log F(1)$$

where

$$F(x) = \prod_{j=1}^{k-1}(x - e^{2\pi ij/k}) = \frac{x^k - 1}{x - 1}.$$

Since $F(1) = k$, the theorem for $r = 0$ now follows from (2).

Suppose now that $r \not\equiv 0 \pmod{k}$. For simplicity we write ε for $e^{2\pi i/k}$. Consider the finite Fourier series generated by $\gamma(r, k)$ namely

$$\sigma_j = \sum_{\lambda=0}^{k-1}\gamma(\lambda, k)\varepsilon^{\lambda j}. \tag{7}$$

By (3),

$$\sigma_0 = \gamma.$$

When $j \neq 0$

$$\sigma_j = \lim_{x\to\infty}\left\{\sum_{\lambda=0}^{k-1}\left(H(x, \lambda, k) - \frac{1}{k}\log x\right)\varepsilon^{\lambda j}\right\}.$$

Since

$$\sum_{\lambda=0}^{k-1}\varepsilon^{\lambda j} = 0, \tag{8}$$

we have

$$\sigma_j = \lim_{x\to\infty}\sum_{\lambda=0}^{k-1}H(x, \lambda, k)\varepsilon^{\lambda j} = \sum_{n=1}^{\infty}\frac{\varepsilon^{jn}}{n} = -\log(1 - \varepsilon^j). \tag{9}$$

Multiplying both members of (7) by ε^{-jr} and summing over j gives us

$$\sum_{j=0}^{k-1}\sigma_j\varepsilon^{-jr} = \sum_{\lambda=0}^{k-1}\gamma(\lambda, k)\sum_{j=0}^{k-1}\varepsilon^{j(\lambda-r)} = k\gamma(r, k)$$

in view of (8). Substituting for σ_j from (9) gives the theorem.

The simplest instances of Theorem 1 are for $k=2$. In this case $1-e^{2\pi ij/k}=2$ so that we have

$$\gamma(0,2)=\tfrac{1}{2}(\gamma-\log 2)=-.05797\ldots,$$
$$\gamma(1,2)=\tfrac{1}{2}(\gamma+\log 2)=.63518\ldots$$

These results also follow at once from (3) and (2). For $k>2$ the terms of the sum in (6) become complex. Since $\gamma(r,k)$ is real we can replace the sum by its real part. However, this leaves something to be desired, namely a simplification using real logarithms. To achieve this we prepare

LEMMA A.

(a) $$\sum_{j=1}^{k-1}\sin\frac{2\pi rj}{k}=0,$$

(b) $$\sum_{j=0}^{k-1}\cos\frac{2\pi rj}{k}=0 \quad \textit{if} \quad r\not\equiv 0 \pmod{k},$$

(c) $$\sum_{j=0}^{k-1} j\sin\frac{2\pi rj}{k}=\begin{cases}0 & \textit{if} \quad r\equiv 0 \pmod{k},\\ -\dfrac{k}{2}\cot\dfrac{\pi r}{k} & \textit{otherwise}.\end{cases}$$

Proof. The well known sums (a) and (b) are the imaginary and real parts of the geometric progression sum

$$\sum_{j=0}^{k-1} e^{2\pi irj/k}=(e^{2\pi ir}-1)/(e^{2\pi ir/k}-1)=0.$$

One way to prove (c) is via the identity

$$\sum_{j=0}^{k-1} ju^j=(1-u)^{-2}\{ku^k(u-1)-u(u^k-1)\}$$

which is established by an easy induction on k. When u is a kth root of unity ($\neq 1$) this becomes

$$\sum_{j=0}^{k-1} ju^j=-\frac{k}{1-u},$$

or

$$\sum_{j=0}^{k-1} j(u^j-u^{-j})=-k\frac{1+u}{1-u}.$$

For $r\not\equiv 0 \pmod{k}$ and $u=\varepsilon^r$ this becomes

$$\sum_{j=0}^{k-1}\sin\frac{2\pi rj}{k}=\frac{1}{2i}(-k)\frac{1+\varepsilon^r}{1-\varepsilon^r}=-\frac{k}{2}\cot\frac{\pi r}{k}.$$

This is (c).

We return now to the sum in Theorem 1. We may assume that $r\not\equiv 0 \pmod{k}$. Next we observe that

$$1-\varepsilon^j=-2ie^{\pi ij/k}\sin\frac{\pi j}{k}$$

so that

$$\log(1-\varepsilon^j)=\log 2+\log\sin\frac{\pi j}{k}+\frac{\pi i}{2k}(2j-k)$$

and

$$\varepsilon^{-rj}=\cos\frac{2\pi rj}{k}-i\sin\frac{2\pi rj}{k}.$$

Hence Theorem 1 is equivalent to

$$\text{(10)}\quad \gamma-k\gamma(r,k)=\mathrm{Re}\Big\{\sum_{j=1}^{k-1}\varepsilon^{-rj}\log(1-\varepsilon^j)\Big\}$$
$$=\sum_{j=1}^{k-1}\log\left(2\sin\frac{\pi j}{k}\right)\cos\frac{2\pi rj}{k}+\frac{\pi}{2k}\sum_{j=1}^{k-1}(2j-k)\sin\frac{2\pi rj}{k}.$$

Applying Lemma A (a) and (c) to the second sum reduces it to

$$-\frac{\pi}{2}\cot\frac{\pi r}{k}.$$

By (b) the first sum in (10) is equal to

$$\sum_{j=1}^{k-1}\Big\{\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}\Big\}-\log 2.$$

In this sum the terms corresponding to j and $k-j$ are equal and when $j=k-j$ the term is zero. Hence we can finally write, when $r\not\equiv 0 \pmod{k}$,

$$\text{(11)}\qquad k\gamma(r,k)=\gamma+\log 2+\frac{\pi}{2}\cot\frac{\pi r}{k}-2\sum_{0<j<k/2}\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}.$$

One useful and immediate consequence of (11) is the formula

$$\text{(12)}\qquad \gamma(k-r,k)=\gamma(r,k)-\frac{\pi}{k}\cot\frac{\pi r}{k}\qquad (r\not\equiv 0 \pmod{k}).$$

Of course (11) can be further simplified when k is a specific small integer. This matter will be discussed at the end of the next section.

4. Primitive $\gamma(r,k)$. When r and k are coprime we call $\gamma(r,k)$ *primitive*, otherwise *imprimitive*. It is clear that if $\gamma(r,k)$ is imprimitive it must be related to some $\gamma(r_1,k_1)$ with $k_1<k$. More precisely we have

THEOREM 2. *Let δ be any common divisor of r and k. Then*

$$\gamma(r,k)=\frac{1}{\delta}\gamma(r/\delta,k/\delta)-\frac{1}{k}\log\delta. \tag{13}$$

Remark. In case δ is the greatest common divisor (r,k) of r and k then $\gamma(r/\delta,k/\delta)$ is primitive. In case $r=0$ then (13) becomes (2).

Proof. Let $r_1=r/\delta$ and $k_1=k/\delta$. Then

$$H(x,r,k)=\frac{1}{\delta}H(x/\delta,r_1,k_1)$$

and so

$$H(x,r,k)-\frac{1}{k}\log x=\frac{1}{\delta}\left\{H(x/\delta,r_1,k_1)-\frac{1}{k_1}\log x/\delta\right\}-\frac{1}{k}\log\delta.$$

Letting $x\to\infty$ gives the theorem.

We now consider the sum of all the $\varphi(k)$ primitive $\gamma(r,k)$ where k is fixed. We denote the sum by

$$\Phi(k)=\sum_{(r,k)=1}\gamma(r,k).$$

We derive an explicit formula for $\Phi(k)$, which depends on the prime factors of k, in a succession of three theorems. The first is

THEOREM 3. *There exists a rational number N_k such that*

$$k\Phi(k)=\varphi(k)\gamma+\log N_k.$$

Proof. The theorem is true for $k=1$ with $N_1=1$. Supposing the theorem holds for all the proper divisors of k. We can then write, in view of (3)

$$\Phi(k)=\gamma-\sum_{\substack{\delta\mid k\\ \delta>1}}\sum_{(r,k)=\delta}\gamma(r,k).$$

Applying Theorem 2 we have

$$\begin{aligned}
-k(\Phi(k)-\gamma)&=\sum_{\substack{\delta\mid k\\ \delta>1}}\sum_{(r,k)=\delta}k\left\{\frac{1}{\delta}\gamma\left(\frac{r}{\delta},\frac{k}{\delta}\right)-\frac{1}{k}\log\delta\right\}\\
&=\sum_{\substack{\delta\mid k\\ \delta>1}}\left\{\frac{k}{\delta}\Phi\left(\frac{k}{\delta}\right)-\Phi\left(\frac{k}{\delta}\right)\log\delta\right\}\\
&=\sum_{\substack{d\mid k\\ d<k}}d\Phi(d)-\sum_{d\mid k}\varphi(d)\log(k/d)\\
&=\sum_{\substack{d\mid k\\ d<k}}\{\varphi(d)\gamma+\log N_d\}-\log k\sum_{d\mid k}\varphi(d)+\sum_{d\mid k}\varphi(d)\log d.
\end{aligned}$$

Here we have used our induction hypothesis. Since

$$\sum_{d\mid k}\varphi(d)=k$$

we have

$$k\Phi(k)-\varphi(k)\gamma=k\log k-\sum_{d\mid k}\varphi(d)\log d-\sum_{\substack{d\mid k\\ d<k}}\log N_d. \tag{14}$$

Now the right hand side is evidently the logarithm of some rational number. Call it N_k since it depends only on k. This proves Theorem 3.

THEOREM 4. *The N_k of Theorem 3 is given by*

$$N_k=\prod_{\delta\mid k}\delta^{-k\mu(\delta)/\delta}$$

where μ is Möbius' function.

Proof. Let the numerical function f be defined by

$$f(k)=\sum_{\delta\mid k}\log N_\delta.$$

Then (14) can be written in view of Theorem 3,

$$k\log k-f(k)=\sum_{\delta\mid k}\varphi(\delta)\log\delta.$$

By Möbius inversion

$$\varphi(k)\log k=\sum_{\delta\mid k}\delta\log\delta\mu(k/\delta)-\sum_{\delta\mid k}f(\delta)\mu(k/\delta).$$

Again by Möbius inversion the second sum is simply $\log N_k$. Thus we have

$$\begin{aligned}
\log N_k&=\sum_{\delta\mid k}\delta\log\delta\mu(k/\delta)-\log k\sum_{\delta\mid k}(k/\delta)\mu(\delta)\\
&=\sum_{\delta\mid k}\mu(\delta)\frac{k}{\delta}(\log k-\log\delta)-k\log k\sum_{\delta\mid k}\mu(\delta)/\delta\\
&=-k\sum_{\delta\mid k}(\mu(\delta)/\delta)\log\delta.
\end{aligned}$$

Taking exponentials gives the theorem.

THEOREM 5. *N_k is, in fact, the integer*

$$N_k=\prod_{p\mid k}p^{\varphi(k)/(p-1)}$$

where p ranges over the prime factors of k.

Proof. By Theorem 4 it is clear that N_k is the product of powers (positive, zero or negative) of the prime factors of k. Let p be any one of these primes and let

$$k = p^a m \qquad (p\nmid m),$$
$$N_k = p^\beta n \qquad (p\nmid n).$$

We must show that

$$\beta = \varphi(k)/(p-1). \tag{15}$$

Any δ dividing k is of the form $\delta = p^j d$ where $d\,|\,m$. By Theorem 4

$$\beta = -\sum_{d|m}\sum_{j=0}^{a} jmp^{a-j}\mu(p^j)\mu(d)/d.$$

It is clear that we must take $j=1$ to get any contribution to β. Hence

$$\beta = p^{a-1}\sum_{d|m}\frac{m}{d}\mu(d) = p^{a-1}\varphi(m) = \varphi(p^a)\varphi(m)/(p-1) = \varphi(k)/(p-1)$$

which is (15).

Theorems 3 and 5 now yield

$$\Phi(k) = \frac{\varphi(k)}{k}\left(\gamma+\sum_{p|k}\frac{\log p}{p-1}\right). \tag{16}$$

As one consequence of (16), $\Phi(k)$ depends only on the prime factors of k. In fact if δ is a divisor of k then $\Phi(\delta k) = \Phi(k)$. As another application of (16) we have

THEOREM 6. *If* $k>1$

$$2\sum_{0<j<k/2}\frac{\mu\big(k/(k,j)\big)}{\varphi\big(k/(k,j)\big)}\log\sin\frac{\pi j}{k} = \log 2-\sum_{p|k}\frac{\log p}{p-1}.$$

Proof. If we sum both sides of (11) over the $\varphi(k)$ numbers r which are $<k$ and prime to k we obtain

$$k\Phi(k) = (\gamma+\log 2)\varphi(k)-\sum_{0<j<k/2}\log\sin\frac{\pi j}{k}\sum_{(r,k)=1}2\cos\frac{2\pi rj}{k}$$

since the cotangent terms for r and $k-r$ destroy themselves.

Now the inner sum is

$$2\sum_{(r,k)=1}e^{2\pi ijr/k} = 2c_k(j) = 2\varphi(k)\frac{\mu\big(k/(k,j)\big)}{\varphi\big(k/(k,j)\big)}$$

which is a well-known formula of Hölder for the Ramanujan sum $c_k(j)$ [5]. Substituting for $\Phi(k)$ from (16) and simplifying gives the theorem.

In case $k=p$, a prime, Theorem 6 becomes the familiar result

$$\prod_{r=1}^{p-1}2\sin\frac{\pi r}{p} = p.$$

Theorem 6 is an instance of the catalytic effect of the Euler constants $\gamma(r,k)$. Further examples occur in § 6.

It is well-known [8] that the values of the trigonometric functions in formula (11) are algebraic numbers. In case k is a power of 2 times a (possibly empty) product of distinct primes of the form $2^{2^\lambda}+1$ these algebraic numbers are expressible in terms of successive square roots of positive integers, that is they are constructable with ruler and compass.

We give below a condensed list of $\gamma(r,k)$ expressed in this way. To save printing costs we can omit the imprimitive cases by Theorem 2. We can also dispense with the cases in which $k\equiv 2 \pmod 4$ since by (5) and Theorem 2

$$\gamma(2s+1,4m+2) = \gamma(2s+1,2m+1)-\tfrac{1}{2}\gamma(s+m+1,2m+1)+\frac{\log 2}{4m+2}.$$

Finally we can suppose that $r<k/2$ in view of (12).

5. $\gamma(r,k)$ and $\Gamma'(z)/\Gamma(z)$. From the three familiar basic formulas

$$\Gamma(1+z) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1-z) = \pi\csc\pi z,$$
$$1/\Gamma(1+z) = e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-z/n}\right\}$$

for the Gamma function the following well-known properties of its logarithmic derivative $\psi(z) = \Gamma'(z)/\Gamma(z)$ are immediately derived.

$$\psi(1+z) = \psi(z)+1/z, \tag{17}$$

$$\psi(1-z) = \psi(z)+\pi\cot\pi z, \tag{18}$$

$$\psi(1+z) = -\gamma+z\sum_{n=1}^{\infty}\frac{1}{n(n+z)} = -\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1}. \tag{19}$$

The following theorem indicates a connection between $\gamma(r,k)$ and $\psi(z)$ at the rational point r/k.

THEOREM 7.

$$\gamma(r,k) = -\{\psi(r/k)+\log k\}/k \qquad (0<r\leqslant k).$$

LIST OF CERTAIN CONSTRUCTABLE $\gamma(r,k)$

$$\gamma(1,2)=\frac{1}{2}\gamma+\frac{1}{2}\log 2,$$

$$\gamma(1,3)=\frac{1}{3}\gamma+\frac{\pi}{18}\sqrt{3}+\frac{1}{6}\log 3,$$

$$\gamma(1,4)=\frac{1}{4}\gamma+\frac{\pi}{8}+\frac{1}{4}\log 2,$$

$$\gamma(1,5)=\frac{1}{5}\gamma+\frac{\pi}{10}\sqrt{1+2/\sqrt{5}}+\frac{1}{20}\log 5+\frac{\sqrt{5}}{10}\log(1+\sqrt{5})/2,$$

$$\gamma(2,5)=\frac{1}{5}\gamma+\frac{\pi}{10}\sqrt{1-2/\sqrt{5}}+\frac{1}{20}\log 5-\frac{\sqrt{5}}{10}\log(1+\sqrt{5})/2,$$

$$\gamma(1,8)=\frac{1}{8}\gamma+\frac{1}{8}\left\{\frac{\pi}{2}(\sqrt{2}+1)+\log 2+\sqrt{2}\log(\sqrt{2}+1)\right\},$$

$$\gamma(3,8)=\frac{1}{8}\gamma+\frac{1}{8}\left\{\frac{\pi}{2}(\sqrt{2}-1)+\log 2-\sqrt{2}\log(\sqrt{2}+1)\right\},$$

$$\gamma(1,12)=\frac{1}{12}\gamma+\frac{1}{24}\{\pi(2+\sqrt{3})-2(\sqrt{3}-1)\log 2+\log 3+4\sqrt{3}\log(\sqrt{3}+1)\},$$

$$\gamma(5,12)=\frac{1}{12}\gamma+\frac{1}{24}\{\pi(2-\sqrt{3})+2(\sqrt{3}+1)\log 2+\log 3-4\sqrt{3}\log(\sqrt{3}+1)\}.$$

The small Table 1 giving $\gamma(r,k)$ to $6D$ for all r and for $k<10$ may be helpful in checking formulas and in applications.

Table 1

r	k	$\gamma(r,k)$	r	k	$\gamma(r,k)$	r	k	$\gamma(r,k)$
1	1	.577216	1	6	.756728	3	8	.084320
1	2	.635182	2	6	.223379	4	8	−.014491
2	2	−.057966	3	6	.028625	5	8	−.078342
1	3	.677807	4	6	−.078921	6	8	−.124198
2	3	.073207	5	6	−.150172	7	8	−.159428
3	3	−.173799	6	6	−.202424	8	8	−.187778
1	4	.710290	1	7	.774010	1	9	.801191
2	4	.144304	2	7	.248515	2	9	.284890
3	4	−.075108	3	7	.059987	3	9	.103868
4	4	−.202270	4	7	−.042448	4	9	.007727
1	5	.735920	5	7	−.109390	5	9	−.053823
2	5	.190389	6	7	−.157931	6	9	−.097666
3	5	−.013764	7	7	−.195528	7	9	−.131110
4	5	−.128886	1	8	.788631	8	9	−.157860
5	5	−.206444	2	8	.268501	9	9	−.180001

Proof. By definition and by (17) and (19)

$$\gamma(r,k)=\lim_{x\to\infty}\left\{\frac{1}{r}+\frac{1}{k+r}+\frac{1}{2k+r}+\ldots+\frac{1}{\left[\frac{x}{k}\right]k+r}-\frac{1}{k}\log x\right\}$$

$$=\frac{1}{r}+\lim_{x\to\infty}\left\{\frac{1}{k}-\left(\frac{1}{k}-\frac{1}{k+r}\right)+\ldots+\frac{1}{\left[\frac{x}{k}\right]k}-\left(\frac{1}{\left[\frac{x}{k}\right]k}-\frac{1}{\left[\frac{x}{k}\right]k+r}\right)-\right.$$

$$\left.-\frac{1}{k}\left(\log\frac{x}{k}+\log k\right)\right\}$$

$$=\frac{1}{r}+\frac{1}{k}\gamma-\frac{\log k}{k}-\sum_{n=1}^{\infty}\frac{r}{nk(nk+r)}=\frac{1}{k}\left\{\frac{k}{r}-\log k+\gamma-\frac{r}{k}\sum_{n=1}^{\infty}\frac{1}{n(n+r/k)}\right\}$$

$$=\frac{1}{k}\left\{-\log k+\frac{k}{r}-\psi\left(1+\frac{r}{k}\right)\right\}=\frac{1}{k}\{-\log k-\psi(r/k)\}$$

which is the theorem.

Solving for $\psi(r/k)$ we find

$$\psi(r/k)=-\{k\gamma(r,k)+\log k\}\qquad(0<r\leqslant k) \tag{20}$$

which, as we note, holds also for $r=k$.

The results we have already obtained for $\gamma(r,k)$ can now be applied to give information about $\psi(z)$. For example (1) gives us at once

$$\psi(r/k)=-\gamma-\log(k/2)-\frac{\pi}{2}\cot\frac{\pi r}{k}+2\sum_{0<j<k/2}\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}.$$

This was discovered by Gauss in 1813 [4]. A simplification of Gauss' proof has been given by Jensen [6] using Abel's theorem on the continuity of convergent power series on the circle of convergence. Our proof via finite Fourier series indicates that Gauss' remarkable result has a completely elementary basis. Gauss used this result to produce the first table of $\psi(z)$ for $z=0(.01)1$. He also pointed out that because of (17) and (18) we can evaluate $\psi(z)$ at any rational point $\neq 0$. For example we have

$$\psi(3/4)=-\gamma+\frac{\pi}{2}-\log 8,$$

$$\psi(7/3)=\frac{15}{4}-\frac{3}{2}\log 3-\gamma-\frac{\pi\sqrt{3}}{6},$$

$$\psi(-3/2)=\frac{8}{3}-\gamma-\log 4.$$

The relation (3) gives us

$$\sum_{r=1}^{k-1} \psi(r/k) = -\{k\log k + (k-1)\gamma\}$$

while from (18) we obtain

$$\sum_{\substack{r=1\\(r,k)=1}}^{k-1} \psi(r/k) = -\varphi(k)\left\{\gamma + \log k + \sum_{p|k} \frac{\log p}{p-1}\right\}.$$

6. Applications of $\gamma(r, k)$. As a first application we give

THEOREM 8. *Let $g(n)$ be a numerical function which is purely periodic of period k. Then*

$$S(g) \equiv \sum_{n=1}^{\infty} g(n)/n = \sum_{r=1}^{k} g(r)\gamma(r, k)$$

provided $\sum_{r=1}^{k} g(r) = 0$; which is a necessary and sufficient condition for convergence of $S(g)$.

Proof. We have

$$\sum_{1\leqslant n\leqslant x} g(n)/n = \sum_{r=1}^{k} g(r)H(x, r, k)$$
$$= \sum_{r=1}^{k} g(r)\left\{H(x, r, k) - \frac{1}{k}\log x\right\} + \frac{\log x}{k}\sum_{r=1}^{k} g(r).$$

As $x\to\infty$ the first sum tends to $\sum g(r)\gamma(r, k)$. The second sum must vanish for convergence.

Theorem 8 can be used to evaluate $S(g)$ in terms of the $\gamma(r, k)$ or inversely, when $S(g)$ is known, to obtain interesting linear combinations of the $\gamma(r, k)$. To illustrate the latter use we can choose $g(n) = \varepsilon^{nj}$ where $j \not\equiv 0 \pmod k$ and $\varepsilon = e^{2\pi i/k}$. Then $S(g)$ becomes $-\log(1-\varepsilon^j)$ and we obtain (9). To illustrate the former use, consider the example

$$g(1) = g(2) = \ldots = g(k-1) = 1, \qquad g(k) = 1-k.$$

This gives us the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k-1} + \frac{1-k}{k} + \frac{1}{k+1} + \ldots + \frac{1}{2k-1} + \frac{1-k}{2k} + \ldots$$
$$= \log k$$

using (3) and (2). This is a generalization of

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots = \log 2.$$

A more interesting and conspicuous class of examples of $S(g)$ are the Dirichlet L series $L(s, \chi)$ at $s = 1$ namely

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

where $\chi(n)$ is a non-principle character modulo k, that is, a non-constant purely multiplicative periodic function of period k which vanishes whenever n and k have a prime factor in common. For n prime to k, $\chi(n)$ is a $\varphi(k)$-th root of unity, not necessarily primitive. Of particular interest to the theory of quadratic fields are the cases in which k or $-k$ is the discriminant of a monic irreducible quadratic equation, because of the connection between $L(1, \chi)$ and the class number $h(\sqrt{\pm k})$ of the field of that equation. For us this means that the two linear combinations

$$\sum_{r=1}^{k-1}\left(\frac{k}{r}\right)\gamma(r, k) \quad \text{and} \quad \sum_{r=1}^{k-1}\left(\frac{-k}{r}\right)\gamma(r, k),$$

where the symbols are those of Kronecker, can be expressed in terms of $h(\sqrt{\pm k})$. For $k > 1$ the precise formulas are

$$\sum_{r=1}^{k-1}\left(\frac{k}{r}\right)\gamma(r, k) = 2\log\varepsilon_k k^{-1/2} h(\sqrt{k}), \tag{21}$$

$$\sum_{r=1}^{k-1}\left(\frac{-k}{r}\right)\gamma(r, k) = \frac{2\pi}{w} k^{-1/2} h(\sqrt{-k}) \tag{22}$$

where $w = 6$ for $k = 3$, $w = 4$ for $k = 4$ and $w = 2$ for all other k and ε_k is the fundamental unit in the real field $Q(\sqrt{k})$.

Thus for $k = 5$, (21) becomes

$$\gamma(1, 5) - \gamma(2, 5) - \gamma(3, 5) + \gamma(4, 5) = 2\log\left(\frac{1+\sqrt{+5}}{2}\right)/\sqrt{5}$$

and for $k = 12$, (22) becomes

$$\gamma(1, 12) - \gamma(5, 12) + \gamma(7, 12) - \gamma(11, 12) = \pi/\sqrt{12}$$

since in both cases $h = 1$. These relations are easily verified from our list of $\gamma(r, k)$.

If in (21) and (22) we substitute for $\gamma(r, k)$ from (11) we obtain, after simplification, a pair of class number formulas for positive and negative discriminants.

These are

$$\varepsilon_k^{h(\sqrt{k})} = \frac{\prod\limits_N \sin(\pi N/k)}{\prod\limits_R \sin(\pi R/k)},$$

$$2\sqrt{k}\,h(\sqrt{-k}) = w\sum_r \left(\frac{-k}{r}\right)\cot(\pi r/k).$$

In the first relation, which is due to Dirichlet [3], N and R range over integers $\leqslant k/2$ for which $\left(\frac{k}{N}\right) = -1$ and $\left(\frac{k}{R}\right) = +1$. In the second relation, which is due to V. A. Lebesgue [7], r ranges over the positive integers $\leqslant k/2$.

As a corollary to Theorem 8 we can evaluate a more general sum.

COROLLARY. *Let a and b be relatively prime integers with $0 < a < b$ and let $g(n)$ be any numerical function periodic of period k. Then*

$$S(g, a, b) \equiv \sum_{n=0}^{\infty} \frac{g(n)}{n + a/b} = b\sum_{r=0}^{k-1} g(r)\gamma(br + a, bk)$$

provided $\sum_{r=1}^{k} g(r) = 0$, which is necessary and sufficient for convergence of $S(g, a, b)$.

Proof. Define $g_1(n)$ by

$$g_1(n) = \begin{cases} bg(n) & \text{if } n \equiv a \pmod{b}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can apply Theorem 8 with k replaced by bk.

Another type of sum that can be evaluated by $\gamma(r, k)$ is considered in

THEOREM 9. *Let $m \geqslant 2$ and let*

$$(r_1, k_1), (r_2, k_2), \ldots, (r_m, k_m)$$

be pairs of positive integers for which

$$0 < r_j \leqslant k_j \quad (j = 1(1)m)$$

and the m rational numbers r_j/k_j are distinct. Finally let $p(x)$ be any polynomial of degree $\leqslant m-2$, a necessary condition for convergence.

Then

$$S = \sum_{n=0}^{\infty} \frac{p(n)}{(k_1 n + r_1)(k_2 n + r_2)\ldots(k_m n + r_m)} = \sum_{j=1}^{m} c_j\left\{\gamma(r_j, k_j) + \frac{\log k_j}{k_j}\right\}$$

where the coefficients c_j are defined by the partial fraction decomposition

$$\frac{p(x)}{(k_1 x + r_1)(k_2 x + r_2)\ldots(k_m x + r_m)} = \sum_{j=1}^{m} \frac{c_j}{k_j x + r_j}. \tag{23}$$

Proof. By (23)

$$p(x) = \sum_{j=1}^{m} c_j \frac{(k_1 x + r_1)\ldots(k_m x + r_m)}{k_j x + r_j}.$$

The right hand side is a formal polynomial of degree $m-1$. Hence its leading coefficient must vanish. That is

$$\sum_{j=1}^{m} c_j/k_j = 0. \tag{24}$$

Again by (23)

$$\begin{aligned}
S &= \lim_{x\to\infty} \sum_{n\leqslant x} \frac{p(n)}{(k_1 n + r_1)\ldots(k_m n + r_m)} \\
&= \lim_{x\to\infty} \sum_{j=1}^{m} c_j \sum_{0\leqslant n\leqslant x} \frac{1}{k_j n + r_j} = \lim_{x\to\infty} \sum_{j=1}^{m} c_j H(k_j x, r_j, k_j) \\
&= \sum_{j=1}^{m} c_j \lim_{x\to\infty}\left\{H(k_j x, r_j, k_j) - \frac{1}{k_j}\log k_j x\right\} + \\
&\qquad + \lim_{x\to\infty}\left\{(\log x)\sum_{j=1}^{m} c_j/k_j\right\} + \sum_{j=1}^{m} \frac{c_j}{k_j}\log k_j \\
&= \sum_{j=1}^{m} c_j\left\{\gamma(r_j, k_j) + \frac{1}{k_j}\log k_j\right\}
\end{aligned}$$

in view of (24). This gives the theorem.

To give a simple illustration consider the sum

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)}.$$

Here we have

$$c_1 = \tfrac{1}{3}, \quad c_2 = -2, \quad c_3 = \tfrac{8}{3}$$

and so the sum is

$$S = \tfrac{1}{3}\gamma(1,1) - 2\{\gamma(1,2) + \tfrac{1}{2}\log 2\} + \tfrac{8}{3}\{\gamma(1,4) + \tfrac{1}{4}\log 4\}.$$

This reduces to $\pi/3 = 1.04719755\ldots$

Because of (24), those sums S for which the k_j are all equal will have the value

$$\sum_{j=1}^{m} c_j \gamma(r_j, k).$$

As an example we may cite the sum

$$S_k = \sum_{n=0}^{\infty} \frac{1}{(kn+1)(kn+2)\ldots(kn+k)} = \Big(\sum_{r=1}^{k} (-1)^{r-1}\binom{k-1}{r-1}\gamma(r,k)\Big)/(k-1)!.$$

For example when $k = 6$ we find

$$S_6 = \{192\log 2 - 81\log 3 - 7\pi\sqrt{3}\}/4320 = .01390480727.$$

The reader will have observed that in all these applications Euler's constant γ cancels out.

7. Numerical evaluation of $\gamma(r, k)$**.** Formula (11) which gives the exact value of $\gamma(r,k)$ is somewhat unwieldy and expensive for the numerical calculation of $\gamma(r,k)$ expecially for large k. If one has access to a good table of $\psi(1+z)$ such as [1] or [2] one can use the formula of Theorem 7

$$\gamma(r,k) = \frac{1}{r} - \frac{\log k}{k} - \frac{1}{k}\psi(1+r/k).$$

Alternatively one can use the series (19) instead of a table.

For automatic computing and where greater accuracy is desired one can apply the Euler–Maclaurin summation formula. This method avoids the use of trigonometric functions and allows one to "wholesale" the computation of $\gamma(r,k)$ for k fixed.

If in the Euler–Maclaurin formula

$$f(0)+f(1)+\ldots+f(N) = \int_0^N f(t)\,dt + \frac{1}{2}\{f(N)+f(0)\} + \\ + \frac{1}{12}\{f'(N)-f'(0)\} - \frac{1}{720}\{f'''(N)-f'''(0)\} + \\ + \frac{1}{30240}\{f^{(5)}(N)-f^{(5)}(0)\}\ldots$$

we set

$$f(t) = (kt+r)^{-1}, \quad kN+r = x$$

we obtain the asymptotic formula

$$(25)\quad \gamma(r,k) = H(x,r,k) - \frac{1}{k}\log x - \frac{1}{2x} + \frac{k}{12x^2} - \frac{k^3}{120x^4} + \frac{k^5}{252x^6} - \ldots$$

If we choose x near to 1000 and $k < 100$ these six terms alone give an error less than $5\cdot 10^{-13}$ in absolute value; for small k it is, of course, very much smaller.

For fixed k we can give a polynomial approximation to $\gamma(r,k)$ once the values of $H(1000, r, k)$ have been computed. To this effect coefficients C_i are defined as follows.

$$C_0 = \frac{1}{k}\log 1000 + \frac{1}{2\cdot 10^3} - \frac{k}{12\cdot 10^6} + \frac{k^3}{12\cdot 10^{13}} - \frac{k^5}{252\cdot 10^{18}},$$

$$C_1 = \frac{1}{k\cdot 10^3} - \frac{1}{2\cdot 10^6} + \frac{k}{6\cdot 10^9} - \frac{k^3}{3\cdot 10^6},$$

$$C_2 = \frac{1}{2k\cdot 10^6} - \frac{1}{2\cdot 10^9} + \frac{k}{4\cdot 10^{12}},$$

$$C_3 = \frac{1}{3k\cdot 10^9} - \frac{1}{2\cdot 10^{12}},$$

$$C_4 = \frac{1}{4k\cdot 10^{12}}.$$

Then from (25)

$$\gamma(r,k) \cong H(1000, r, k) - C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4$$

where

$$t \equiv 1000 - r \pmod{k} \quad (0 \leqslant t < k).$$

Similar polynomials based on other limits than 1000 can be written down from (25) whose general term is $k^{2n-1}B_{2n}x^{-2n}$, B_{2n} being, of course, the Bernoulli number of index $2n$. Such formulas are sometimes useful in case k is large and only a few values of r are involved. Such an occasion is the following question.

Inspection of our modest table of $\gamma(r,k)$ leads one to guess that about half of the $\gamma(r,k)$ are positive. This is in reality far from the truth, as we see from

THEOREM 10. *For large k the monotone sequence*

$$\gamma(1,k), \gamma(2,k), \ldots, \gamma(k,k)$$

changes sign in the neighborhood of $r = k/\log k$, so that almost all $\gamma(r,k)$ are negative.

Proof. We shall prove somewhat more. Let $L = \log k - \gamma$.
By Theorem 7 to make $\gamma(r, k) = 0$ we must have

$$\psi(z) = -\log k, \quad z = r/k$$

or, by (17),

$$zψ(1+z) = 1 - z\log k. \tag{26}$$

But by (19)

$$z\psi(1+z) = -\gamma z + \sum_{n=2}^{\infty} \zeta(n)(-z)^n.$$

Therefore (26) becomes

$$z = L^{-1}\{1-\zeta(2)z^2+\zeta(3)z^3- \ldots\}. \tag{27}$$

Taking only the first term the theorem follows. Solving (27) by iteration we find

$$z = L^{-1}-\zeta(2)L^{-3}+\zeta(3)L^{-4}+[2\{\zeta(2)\}^2-\zeta(4)]L^{-5}- -[5\zeta(2)\zeta(3)+\zeta(5)]L^{-6}- \ldots \tag{28}$$

As an illustration we take $k = 100$. The terms of (28) become

$$.2483-.0252+.0046+.0041-.0025 = .2292.$$

By actual computation we find

$$\gamma(22, 100) = .00204268,$$
$$\gamma(23, 100) = -.00056747.$$

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Received on 13. 6. 1973 (413)

ACTA ARITHMETICA
XXVII (1975)

On the distribution of additive arithmetic functions

by

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Dedicated to the memory of Yu. V. Linnik

Let $g(n)$ be a real valued additive arithmetic function (i.e. $g(mn) = g(m)+g(n)$ if $(m, n) = 1$). The distribution of values of such functions has been extensively investigated. As a new direction, Erdös, Ruzsa and Sárközi [1] proposed to estimate

$$\max_{-\infty<a<\infty} N(a, x) \overset{\text{def}}{=} \max_{-\infty<a<\infty} \sum_{\substack{n\leqslant x\\ g(n)=a}} 1 \tag{1}$$

for general additive functions. They found bounds cx in various cases, often giving the best possible value of c. If, however, $g(n) = \omega(n)$, the number of prime divisors of n, then this quantity is about const $\frac{x}{\sqrt{\log\log x}}$ and they conjectured (oral communication) that this order of magnitude cannot be exceeded in any case, provided that $g(p) \neq 0$ for each prime p. The aim of this paper is to prove this conjecture in the following more precise form.

THEOREM. *Let $g(n)$ be an arbitrary real valued additive function and put*

$$E(x) = \sum_{\substack{p\leqslant x\\ g(p)\neq 0}} \frac{1}{p}.$$

Then there is a universal constant c_1 such that

$$N(a, x) = \sum_{\substack{n\leqslant x\\ g(n)=a}} 1 \leqslant c_1 \frac{x}{\sqrt{E(x)}}.$$

The result is sharp even in this more general form: The bound is attained if $g(p) = 0$ or 1 and $\sum_{\substack{p\leqslant x\\ g(p)=1}} 1/p = E(x)$ as is seen from [2] and [3] where much more detailed information is given in this special case. (For refer-