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On an inequality for additive arithmetic functions

by

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In memory of Yu. V. Linnik

G. H. Hardy and S. Ramanujan [2] proved that for any fixed $\delta > 0$ and all positive integers $m \leqslant n$, with a possible exception of $o(n)$ of them, the inequality

$$|\omega(m) - \ln\ln n| < (\ln\ln n)^{1/2+\delta}$$

is true. Here $\omega(m)$ denotes the number of different prime divisors of m. This is an analogue of the probabilistic weak law of large numbers. It shows the bounds between which the function $\omega(m)$ oscillates for the great majority of values of the argument.

P. Turán [5], [6] gave a very simple derivation of this statement. He proved the elementary inequality

$$\sum_{m=1}^{n} (\omega(m) - \ln\ln n)^2 \leqslant c_1 n \ln\ln n, \tag{1}$$

where c_1 is a constant, which evidently implies the result of Hardy and Ramanujan.

Naturally there arose a question of the generalization of (1) to a larger class of arithmetic functions. P. Turán [7] obtained the following theorem. Let $f(m)$ be a real-valued strongly additive function such that

$$0 \leqslant f(p) \leqslant K$$

for all primes p and a constant K and

$$M_n = \sum_{p \leqslant n} \frac{f(p)}{p} \to \infty$$

as $n \to \infty$. Then the inequality

$$\sum_{m=1}^{n} \big(f(m) - M_n\big)^2 \leqslant c_2 n M_n$$

holds, where c_2 is a constant depending on K.

The following remarks are very suggestive.

Let $f(m)$ be any complex-valued arithmetic function. The sum

$$(2)\qquad \sum_{m=1}^{n} |f(m)-A_n|^2$$

for any n takes the minimal value, which is equal to

$$\sum_{m=1}^{n} |f^2(m)| - \frac{1}{n}\Big|\sum_{m=1}^{n} f(m)\Big|^2,$$

in case

$$A_n = \frac{1}{n}\sum_{m=1}^{n} f(m).$$

When additive functions are considered this result may be expressed in the form

$$(3)\qquad \sum_{m=1}^{n}\Big|f(m)-\frac{1}{n}\sum_{p^\alpha\leqslant n}\Big(\Big[\frac{n}{p^\alpha}\Big]-\Big[\frac{n}{p^{\alpha+1}}\Big]\Big)f(p^\alpha)\Big|^2$$
$$=\sum_{p^\alpha\leqslant n}\Big(\Big[\frac{n}{p^\alpha}\Big]-\Big[\frac{n}{p^{\alpha+1}}\Big]\Big)|f^2(p^\alpha)|+$$
$$+\sum_{\substack{p^\alpha q^\beta\leqslant n\\ p\neq q}}\Big(\Big[\frac{n}{p^\alpha q^\beta}\Big]-\Big[\frac{n}{p^\alpha q^{\beta+1}}\Big]-\Big[\frac{n}{p^{\alpha+1} q^\beta}\Big]+\Big[\frac{n}{p^{\alpha+1} q^{\beta+1}}\Big]\Big)f(p^\alpha)\bar f(q^\beta)-$$
$$-\frac{1}{n}\Big|\sum_{p^\alpha\leqslant n}\Big(\Big[\frac{n}{p^\alpha}\Big]-\Big[\frac{n}{p^{\alpha+1}}\Big]\Big)f(p^\alpha)\Big|^2.$$

Here and in what follows, p^α, q^β denote prime powers, $\alpha\geqslant 1$, $\beta\geqslant 1$, the bar denotes complex conjugation. However (3) is almost useless for applications. Probabilistic interpretation of additive functions (see [4]) suggests the idea that the centering constant A_n in (2) may be approximated by means of the following sums

$$A_1(n,f) = \sum_{p^\alpha\leqslant n}\frac{f(p^\alpha)}{p^\alpha}\Big(1-\frac{1}{p}\Big),$$

$$A_2(n,f) = \sum_{p^\alpha\leqslant n}\frac{f(p^\alpha)}{p^\alpha},$$

$$A_3(n,f) = \sum_{p\leqslant n}\frac{f(p)}{p}.$$

Set

$$S_k(n,f) = \sum_{m=1}^{n}|f(m)-A_k(n,f)|^2 \qquad (k=1,2,3).$$

As the measure of deviation it is natural to choose the quantities

$$D^2(n,f) = \sum_{p^\alpha\leqslant n}\frac{|f^2(p^\alpha)|}{p^\alpha}$$

or

$$B^2(n,f) = \sum_{p\leqslant n}\frac{|f^2(p)|}{p}$$

in case of strongly additive functions.

The author of this note [3] proved that for any real-valued strongly additive function

$$S_3(n,f)\leqslant c_3 nB^2(n,f)$$

where c_3 is an absolute constant. Later [4] this inequality was generalized for any complex-valued additive functions. It was proved

$$S_3(n,f)\leqslant c_4 nD^2(n,f),$$

c_4 being an absolute constant too. By the same method it is easy to prove analogous estimates for the sums $S_1(n,f)$, $S_2(n,f)$.

It occured that these elementary inequalities are very useful in the probabilistic number theory for the investigation of the distribution of values of additive and multiplicative arithmetic functions. It is essential that the constants c_4 in these inequalities are absolute. Therefore it is interesting to find numerical values of these constants. Recently P.D.T.A. Elliott [1] by means of the method of the large sieve proved that for sufficiently large n

$$S_2(n,f)\leqslant 51nD^2(n,f).$$

Estimating the sums $S_k(n,f)$ it is sufficient to suppose $D(n,f)>0$, or $B(n,f)>0$ in case of strongly additive functions. Otherwise $f(m) = 0$ $(m=1,\ldots,n)$ and $S_k(n,f)=0$ $(k=1,2,3)$. Set

$$\lambda_k(n,f) = \frac{S_k(n,f)}{nD^2(n,f)} \qquad (k=1,2,3)$$

and

$$\lambda_4(n,f) = \frac{S_3(n,f)}{nB^2(n,f)}.$$

The aim of this note is to estimate the quantities

$$\lambda_k(n) = \sup\lambda_k(n,f) \qquad (k=1,2,3,4),$$

where the upper bound is taken over all complex-valued additive (for $k=4$ over all strongly additive) arithmetic functions with $D(n,f)>0$.

In what follows B is a quantity (not always the same) which is bounded in absolute value by an absolute constant.

THEOREM. *There exists an absolute constant n_0 such that*

$$1.47<\lambda_1(n)<2.08,$$

$$1.47<\lambda_2(n)<2.1,$$

$$1.47<\lambda_3(n)<2.29,$$

$$1.47<\lambda_4(n)<2.08$$

for all $n\geqslant n_0$.

These estimates may be slightly improved by the complication of the method. However it is easy to prove better estimates for some classes of additive functions. From the proof of the theorem it will follow that the upper bound of $\lambda_1(n,f)$, $\lambda_4(n,f)$ over all real-valued additive (in the second case over all strongly additive) arithmetic functions with $D(n,f)>0$, preserving a constant sign for all $m\leqslant n$, equals $1+B(\ln\ln n)^{-1}$.

Proof. We shall need the following well-known estimates

$$\sum_{p\leqslant x}1=\frac{Bx}{\ln x},\qquad \sum_{p^\alpha\leqslant x}1=\frac{Bx}{\ln x},$$

$$(4)\qquad \sum_{p\leqslant x}\frac{1}{p}=\ln\ln x+c_5+\frac{B}{\ln x},\qquad \sum_{p^\alpha\leqslant x}\frac{1}{p^\alpha}=\ln\ln x+c_6+\frac{B}{\ln x},$$

$$\sum_{p\leqslant x}\frac{\ln p}{p}=\ln x+B,\qquad \sum_{p^\alpha\leqslant x}\frac{\ln p^\alpha}{p^\alpha}=B\ln x,$$

where c_5, c_6 are constants. Less exact estimates are sufficient too.

Denote

$$\varphi(m)=\frac{f(m)}{D(n,f)},\qquad u(p^\alpha)=\frac{\varphi(p^\alpha)}{p^{\alpha/2}}.$$

Then

$$(5)\qquad \sum_{p^\alpha\leqslant n}|u^2(p^\alpha)|=1$$

and

$$\lambda_k(n,f)=\lambda_k(n,\varphi)=\frac{1}{n}S_k(n,\varphi)\qquad (k=1,2,3).$$

Now we present the sum $S_k(n,\varphi)$ in the form

$$(6)\qquad S_k(n,\varphi)=\sum_{m=1}^{n}\big(\varphi(m)-A_k(n,\varphi)\big)\big(\overline{\varphi}(m)-\bar{A}_k(n,\varphi)\big)$$
$$=T_2-A_k(n,\varphi)\overline{T}_1-\bar{A}_k(n,\varphi)T_1+n|A_k^2(n,\varphi)|$$

where

$$T_1=\sum_{m=1}^{n}\varphi(m),\qquad T_2=\sum_{m=1}^{n}|\varphi^2(m)|.$$

It follows from the additivity of $\varphi(m)$ that

$$T_1=\sum_{m=1}^{n}\sum_{p^\alpha\|m}\varphi(p^\alpha).$$

Changing the order of the summation and noting that the number of pos i tive integers $m\leqslant n$, for which $p^\alpha\|m$, equals

$$\left[\frac{n}{p^\alpha}\right]-\left[\frac{n}{p^{\alpha+1}}\right],$$

we obtain

$$T_1=\sum_{p^\alpha\leqslant n}\left(\left[\frac{n}{p^\alpha}\right]-\left[\frac{n}{p^{\alpha+1}}\right]\right)\varphi(p^\alpha).$$

If we omit the square brackets we make an error which in view of Cauchy's inequality, (4) and (5) is less in absolute value than

$$\sum_{p^\alpha\leqslant n}|\varphi(p^\alpha)|\leqslant\Big(\sum_{p^\alpha\leqslant n}p^\alpha\sum_{q^\beta\leqslant n}|u^2(q^\beta)|\Big)^{1/2}\leqslant\sqrt{n}\Big(\sum_{p^\alpha\leqslant n}1\Big)^{1/2}=\frac{Bn}{\sqrt{\ln n}}.$$

Hence

$$(7)\qquad T_1=nA_1(n,\varphi)+\frac{Bn}{\sqrt{\ln n}}.$$

Changing the order of summation in the equality

$$T_2=\sum_{m=1}^{n}\sum_{p^\alpha\|m}\varphi(p^\alpha)\sum_{q^\beta\|m}\overline{\varphi}(q^\beta)$$

and taking into account that the number of positive integers $m\leqslant n$, for which $p^\alpha\|m$, $q^\beta\|m$, equals

$$\left[\frac{n}{p^\alpha q^\beta}\right]-\left[\frac{n}{p^{\alpha+1}q^\beta}\right]-\left[\frac{n}{p^\alpha q^{\beta+1}}\right]+\left[\frac{n}{p^{\alpha+1}q^{\beta+1}}\right]$$

if $p \neq q$, we obtain

$$(8)\qquad T_2 = \sum_{p^\alpha \leqslant n} \left(\left[\frac{n}{p^\alpha}\right] - \left[\frac{n}{p^{\alpha+1}}\right] \right) |\varphi^2(p^\alpha)| + $$

$$+ \sum_{\substack{p^\alpha q^\beta \leqslant n \\ p \neq q}} \left(\left[\frac{n}{p^\alpha q^\beta}\right] - \left[\frac{n}{p^{\alpha+1} q^\beta}\right] - \left[\frac{n}{p^\alpha q^{\beta+1}}\right] + \left[\frac{n}{p^{\alpha+1} q^{\beta+1}}\right] \right) \varphi(p^\alpha) \bar{\varphi}(q^\beta).$$

Using (4) we have

$$\sum_{p^\alpha q^\beta \leqslant n} 1 \leqslant 2 \sum_{p^\alpha \leqslant \sqrt{n}} \sum_{q^\beta \leqslant n p^{-\alpha}} 1 = \frac{Bn}{\ln n} \sum_{p^\alpha \leqslant \sqrt{n}} \frac{1}{p^\alpha} = \frac{Bn \ln\ln n}{\ln n}.$$

Therefore the error from the omission of the square brackets in the second sum of the second term of (8) in view of Cauchy's inequality and (5) equals

$$B \sum_{p^\alpha q^\beta \leqslant n} |\varphi(p^\alpha) \varphi(q^\beta)| = B \Big(\sum_{p^\alpha q^\beta \leqslant n} p^\alpha q^\beta \Big)^{1/2} \Big(\sum_{p^\alpha q^\beta \leqslant n} |u^2(p^\alpha) u^2(q^\beta)| \Big)^{1/2}$$

$$= B\sqrt{n} \Big(\sum_{p^\alpha q^\beta \leqslant n} 1 \Big)^{1/2} = Bn \sqrt{\frac{\ln\ln n}{\ln n}}.$$

Thus

$$(9)\qquad T_2 = nd + nL + Bn \sqrt{\frac{\ln\ln n}{\ln n}},$$

where

$$(10)\qquad d = \frac{1}{n} \sum_{p^\alpha \leqslant n} \left(\left[\frac{n}{p^\alpha}\right] - \left[\frac{n}{p^{\alpha+1}}\right] \right) p^\alpha |u^2(p^\alpha)| \leqslant 1,$$

$$L = \sum_{\substack{p^\alpha q^\beta \leqslant n \\ p \neq q}} \frac{u(p^\alpha) \bar{u}(q^\beta)}{p^{\alpha/2} q^{\beta/2}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right).$$

One more estimate is necessary:

$$(11)\qquad |A_k(n, \varphi)| \leqslant \sum_{p^\alpha \leqslant n} \frac{|u(p^\alpha)|}{p^{\alpha/2}} \leqslant \Big(\sum_{p^\alpha \leqslant n} \frac{1}{p^\alpha} \Big)^{1/2} \Big(\sum_{p^\alpha \leqslant n} |u^2(p^\alpha)| \Big)^{1/2}$$

$$= B\sqrt{\ln\ln n}.$$

The relation (6) together with (7), (9), (11) implies that

$$(12)\qquad \lambda_k(n, \varphi) = d + R_k + B \sqrt{\frac{\ln\ln n}{\ln n}},$$

where

$$R_k = L - A_1(n, \varphi) \bar{A}_k(n, \varphi) - \bar{A}_1(n, \varphi) A_k(n, \varphi) + |A_k^2(n, \varphi)|.$$

Each of these sums is calculated separately for $k = 1, 2, 3$. We have

$$(13)\qquad R_1 = L - |A_1^2(n, \varphi)| = - \sum_{p \leqslant \sqrt{n}} \sum_{\alpha+\beta \leqslant \gamma_p} \frac{u(p^\alpha) \bar{u}(p^\beta)}{p^{\alpha/2+\beta/2}} \left(1 - \frac{1}{p}\right)^2 -$$

$$- \sum_{\substack{p^\alpha \leqslant n, q^\beta \leqslant n \\ p^\alpha q^\beta > n}} \frac{u(p^\alpha) \bar{u}(q^\beta)}{p^{\alpha/2} q^{\beta/2}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right),$$

where $\gamma_p = [\ln n / \ln p]$. In the first sum we separate the summands with $\alpha = \beta = \gamma_p/2$ and estimate the rest. Using Cauchy's inequality and partial summation and referring to (4), (5) we obtain

$$(14)\qquad \Big(2 \sum_{p \leqslant \sqrt{n}} \sum_{\alpha < \gamma_p/2} \sum_{\gamma_p/2 < \beta \leqslant \gamma_p - \alpha} \frac{\operatorname{Re} u(p^\alpha) \bar{u}(p^\beta)}{p^{\alpha/2+\beta/2}} \left(1 - \frac{1}{p}\right)^2 \Big)^2$$

$$\leqslant 2 \sum_{p \leqslant \sqrt{n}} \sum_{\alpha > \gamma_p/2} \sum_{\gamma_p/2 < \beta \leqslant \gamma_p - \alpha} \frac{1}{p^{\alpha+\beta}} \left(1 - \frac{1}{p}\right)^4$$

$$< 2 \sum_{p \leqslant \sqrt{n}} p^{-2-[\gamma_p/2]} < \frac{2}{\sqrt{n}} \sum_{p \leqslant \sqrt{n}} \frac{1}{\sqrt{p}} = \frac{B}{\ln n}.$$

Thus

$$(15)\qquad R_1 = - \sum_{p \leqslant \sqrt{n}} \left(1 - \frac{1}{p}\right)^2 \Big| \sum_{\alpha \leqslant \gamma_p/2} \frac{u(p^\alpha)}{p^{\alpha/2}} \Big|^2 - \Big| \sum_{\sqrt{n} < p^\alpha \leqslant n} \frac{u(p^\alpha)}{p^{\alpha/2}} \left(1 - \frac{1}{p}\right) \Big|^2 +$$

$$+ V_1 + \frac{B}{\ln n},$$

where

$$V_1 = -2 \sum_{p^\alpha \leqslant \sqrt{n}} \sum_{n p^{-\alpha} < q^\beta \leqslant n} \frac{\operatorname{Re} u(p^\alpha) \bar{u}(q^\beta)}{p^{\alpha/2} q^{\beta/2}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right).$$

Using again the estimates (4), (5) and Cauchy's inequality we get

$$V_1^2 < 2 \sum_{p^\alpha \leqslant \sqrt{n}} \frac{1}{p^\alpha} \sum_{n p^{-\alpha} < q^\beta \leqslant n} \frac{1}{q^\beta} = 2 \sum_{p^\alpha \leqslant \sqrt{n}} \frac{1}{p^\alpha} \left(\ln \frac{\ln n}{\ln (n/p^\alpha)} + \frac{B}{\ln n} \right)$$

$$= -2 \sum_{p^\alpha \leqslant \sqrt{n}} \frac{1}{p^\alpha} \ln \left(1 - \frac{\ln p^\alpha}{\ln n}\right) + \frac{B \ln\ln n}{\ln n}.$$

Partial summation results in

$$V_1^2 < -2\sum_{p^\alpha\leqslant\sqrt{n}}\frac{1}{p^\alpha}\ln\left(1-\frac{\ln\sqrt{n}}{\ln n}\right)+2\int_2^{\sqrt{n}}\sum_{p^\alpha\leqslant u}\frac{1}{p^\alpha}\,d\ln\left(1-\frac{\ln u}{\ln n}\right) \tag{16}$$

$$= -2(\ln\ln\sqrt{n}+c_6)\ln\tfrac{1}{2}+2\int_2^{\sqrt{n}}(\ln\ln u+c_6)\,d\ln\left(1-\frac{\ln u}{\ln n}\right)+$$

$$+\frac{B}{\ln n}+\frac{B}{\ln n}\int_2^{\sqrt{n}}\frac{du}{u\ln u}$$

$$= -2\int_2^{\sqrt{n}}\ln\left(1-\frac{\ln u}{\ln n}\right)d\ln\ln u+\frac{B\ln\ln n}{\ln n} = \varkappa_1+\frac{B\ln\ln n}{\ln n},$$

where

$$\varkappa_1 = -2\int_0^{1/2}\frac{\ln(1-y)}{y}\,dy.$$

The estimates (10), (12), (13) imply

$$\lambda_1(n,\varphi) < 1+\sqrt{\varkappa_1}+B\sqrt{\frac{\ln\ln n}{\ln n}},$$

where

$$1+\sqrt{\varkappa_1} = 2.0791\ldots$$

We present R_2 in the form

$$R_2 = -\sum_{p\leqslant\sqrt{n}}\sum_{\alpha+\beta\leqslant\gamma_p}\frac{u(p^\alpha)\overline{u}(p^\beta)}{p^{\alpha/2+\beta/2}}\left(1-\frac{2}{p}\right)+\sum_{\substack{p^\alpha q^\beta\leqslant n\\ p\neq q}}\frac{u(p^\alpha)\overline{u}(q^\beta)}{p^{\alpha/2+1}q^{\beta/2+1}}- \tag{17}$$

$$-\sum_{\substack{p^\alpha\leqslant n,\, q^\beta\leqslant n\\ p^\alpha q^\beta>n}}\frac{u(p^\alpha)\overline{u}(q^\beta)}{p^{\alpha/2}q^{\beta/2}}\left(1-\frac{1}{p}-\frac{1}{q}\right).$$

In the first sum, analogously to the sum R_1, we separate the summands with $\alpha=\beta\leqslant\gamma_p/2$. The rest, analogously to (14), equals $B(\ln n)^{-1/2}$. Moreover we note that

$$\left|\sum_{\sqrt{n}<p^\alpha\leqslant n}\frac{u(p^\alpha)}{p^{\alpha/2+1}}\right|^2 \leqslant \sum_{\sqrt{n}<p^\alpha\leqslant n}\frac{1}{p^{\alpha+2}} < \sum_p\sum_{\alpha>\ln n/(2\ln p)}\frac{1}{p^{\alpha+2}}$$

$$= B\sum_p p^{-3-[\ln n/(2\ln p)]} = \frac{B}{\sqrt{n}}\sum_p\frac{1}{p^2} = \frac{B}{\sqrt{n}}.$$

Therefore

$$R_2 = -\sum_{p\leqslant\sqrt{n}}\left(1-\frac{2}{p}\right)\left|\sum_{\alpha\leqslant\gamma_p/2}\frac{u(p^\alpha)}{p^{\alpha/2}}\right|^2-\left|\sum_{\sqrt{n}<p\leqslant n}\frac{u(p^\alpha)}{p^{\alpha/2}}\left(1-\frac{1}{p}\right)\right|^2+ \tag{18}$$

$$+V_2+\frac{B}{\sqrt{\ln n}},$$

where

$$V_2 = \sum_{\substack{p^\alpha q^\beta\leqslant n\\ p\neq q}}\frac{u(p^\alpha)\overline{u}(q^\beta)}{p^{\alpha/2+1}q^{\beta/2+1}}-2\sum_{p^\alpha\leqslant\sqrt{n}}\sum_{np^{-\alpha}<q^\beta\leqslant n}\frac{\operatorname{Re}u(p^\alpha)\overline{u}(q^\beta)}{p^{\alpha/2}q^{\beta/2}}\left(1-\frac{1}{p}-\frac{1}{q}\right).$$

Using again Cauchy's inequality we obtain

$$V_2^2 \leqslant \sum_{\substack{p^\alpha q^\beta\leqslant n\\ p\neq q}}\frac{1}{p^{\alpha+2}q^{\beta+2}}+2\sum_{p^\alpha\leqslant\sqrt{n}}\sum_{np^{-\alpha}<q^\beta\leqslant n}\frac{1}{p^\alpha q^\beta}.$$

The second sum is estimated analogously to (15). Thus we have

$$V_2^2 < \varkappa_1+\varkappa_2+\frac{B\ln\ln n}{\ln n}, \tag{19}$$

where

$$\varkappa_2 = \left(\sum_p\frac{1}{p^2(p-1)}\right)^2-\sum_p\frac{1}{p^4(p-1)^2}.$$

From (12) referring to (10), (18), (19) we have

$$\lambda_2(n,\varphi) < 1+\sqrt{\varkappa_1+\varkappa_2}+B\sqrt{\frac{\ln\ln n}{\ln n}},$$

where

$$1+\sqrt{\varkappa_1+\varkappa_2} = 2.0962\ldots$$

In order to estimate R_3 some preliminary estimates are to be proved. We have

$$\left|\sum_{\sqrt{n}<p\leqslant n}\frac{u(p)}{p^{3/2}}\right|^2 < \sum_{p>\sqrt{n}}\frac{1}{p^3} = \frac{B}{n}.$$

Further

$$\sum_{\substack{p^\alpha>x\\ \alpha\geqslant 2}}\frac{1}{p^\alpha} < \sum_{p\leqslant\sqrt{x}}\sum_{\alpha>\ln x/\ln p}\frac{1}{p^\alpha}+\sum_{p>\sqrt{x}}\sum_{\alpha=2}^{\infty}\frac{1}{p^\alpha}$$

$$= \frac{B}{x}\sum_{p\leqslant\sqrt{x}}1+B\sum_{p>\sqrt{x}}\frac{1}{p^2} = \frac{B}{\sqrt{x}\ln x}.$$

Therefore

$$\Big(-2\sum_{\substack{p^\alpha\leqslant n,\,q\leqslant n\\ p^\alpha q>n,\,\alpha\geqslant 2}}\frac{\operatorname{Re}u(p^\alpha)\bar u(q)}{p^{\alpha/2}q^{1/2}}\Big)^2\leqslant 2\sum_{\substack{p^\alpha\leqslant n,\,q\leqslant n\\ p^\alpha q>n,\,\alpha\geqslant 2}}\frac{1}{p^\alpha q}$$

$$<2\sum_{\substack{p^\alpha\leqslant\sqrt n\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}\sum_{np^{-\alpha}<q\leqslant n}\frac{1}{q}+2\sum_{q\leqslant\sqrt n}\frac{1}{q}\sum_{\substack{nq^{-1}<p^\alpha\leqslant n\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}+2\sum_{\substack{\sqrt n<p^\alpha\leqslant n\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}\sum_{\sqrt n<q\leqslant n}\frac{1}{q}$$

$$=2\sum_{\substack{p^\alpha\leqslant\sqrt n\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}\Big(\ln\frac{\ln n}{\ln(n/p^\alpha)}+\frac{B}{\ln n}\Big)+\frac{B}{\sqrt n\ln n}\sum_{q\leqslant\sqrt n}\frac{1}{\sqrt q}+\frac{B}{\sqrt[4]{n}\ln n}$$

$$=-2\sum_{\substack{p^\alpha\leqslant\sqrt n\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}\ln\Big(1-\frac{\ln p^\alpha}{\ln n}\Big)+\frac{B\ln\ln n}{\ln n}$$

$$=\frac{B}{\ln n}\sum_{p^\alpha\leqslant\ln^2 n}\frac{\ln p^\alpha}{p^\alpha}+\frac{B}{\ln n}\sum_{\substack{\ln^2 n<p^\alpha\leqslant\sqrt n\\ \alpha\geqslant 2}}\frac{\ln p^\alpha}{p^\alpha}+\frac{B\ln\ln n}{\ln n}=\frac{B\ln\ln n}{\ln n}.$$

Using these estimates we obtain

(20)
$$R_3=-\sum_{p\leqslant\sqrt n}\frac{|u^2(p)|}{p}\Big(1-\frac{2}{p}\Big)-\Big|\sum_{\sqrt n<p\leqslant n}\frac{u(p)}{p^{1/2}}\Big(1-\frac{1}{p}\Big)\Big|^2+V_3+\frac{B\ln\ln n}{\ln n},$$

where

$$V_3=-2\sum_{p\leqslant n^{1/3}}\Big(1-\frac{1}{p}\Big)\sum_{2\leqslant\alpha\leqslant\gamma_p-1}\frac{\operatorname{Re}u(p^\alpha)\bar u(p)}{p^{\alpha/2+1/2}}+$$

$$+\sum_{\substack{pq\leqslant n\\ p\neq q}}\frac{u(p)\bar u(q)}{p^{3/2}q^{3/2}}-2\sum_{\substack{p^\alpha q\leqslant n\\ p\neq q,\,\alpha\geqslant 2}}\frac{\operatorname{Re}u(p^\alpha)\bar u(q)}{p^{\alpha/2}q^{3/2}}+$$

$$+\sum_{\substack{p^\alpha q^\beta\leqslant n\\ \alpha\geqslant 2,\,\beta\geqslant 2\\ p\neq q}}\frac{u(p^\alpha)\bar u(q^\beta)}{p^{\alpha/2}q^{\beta/2}}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)-$$

$$-2\sum_{p\leqslant\sqrt n}\sum_{np^{-1}<q\leqslant n}\frac{\operatorname{Re}u(p)\bar u(q)}{p^{1/2}q^{1/2}}\Big(1-\frac{1}{p}-\frac{1}{q}\Big).$$

Again Cauchy's inequality states that

(21)
$$V_3^2<2\sum_{p^\alpha,\,\alpha\geqslant 2}\frac{1}{p^{\alpha+1}}\Big(1-\frac{1}{p}\Big)^2+\sum_{\substack{p,q\\ p\neq q}}\frac{1}{p^3q^3}+2\sum_{\substack{p^\alpha,q\\ p\neq q,\,\alpha\geqslant 2}}\frac{1}{p^\alpha q^3}\Big(1-\frac{1}{p}\Big)^2+$$

$$+\sum_{\substack{p^\alpha,q^\beta\\ p\neq q\\ \alpha\geqslant 2,\,\beta\geqslant 2}}\frac{1}{p^\alpha q^\beta}\Big(1-\frac{1}{p}\Big)^2\Big(1-\frac{1}{q}\Big)^2+2\sum_{p\leqslant\sqrt n}\sum_{np^{-1}<q\leqslant n}\frac{1}{pq}$$

$$=\varkappa_1+\varkappa_3+\frac{B\ln\ln n}{\ln n},$$

where

$$\varkappa_3=\Big(\sum_p\frac{1}{p^2}\Big)^2+2\sum_p\frac{1}{p^3}-3\sum_p\frac{1}{p^4}.$$

From (9), (12), (20), (21) it follows that

$$\lambda_3(n,\varphi)<1+\sqrt{\varkappa_1+\varkappa_3}+B\sqrt{\frac{\ln\ln n}{\ln n}},$$

where

$$1+\sqrt{\varkappa_1+\varkappa_3}=2.2812\ldots$$

In case of strongly additive functions we put

$$\varphi(m)=\frac{f(m)}{B(n,f)},\qquad u(p)=\frac{\varphi(p)}{p^{1/2}}.$$

Then

$$\sum_{p\leqslant n}|u^2(p)|=1$$

and

$$\lambda_4(n,f)=\lambda_4(n,\varphi)=\frac{1}{n}S_3(n,\varphi).$$

Analogous considerations lead to

(22)
$$\lambda_4(n,\varphi)=b+R_4+B\sqrt{\frac{\ln\ln n}{\ln n}},$$

where

(23)
$$b=\frac{1}{n}\sum_{p\leqslant n}\Big[\frac{n}{p}\Big]p\,|u^2(p)|\leqslant 1,$$

(24)
$$R_4=-\sum_{p\leqslant\sqrt n}\frac{|u^2(p)|}{p}-\Big|\sum_{\sqrt n<p\leqslant n}\frac{u(p)}{p^{1/2}}\Big|^2-2\sum_{p\leqslant\sqrt n}\sum_{np^{-1}<q\leqslant n}\frac{\operatorname{Re}u(p)\bar u(q)}{p^{1/2}q^{1/2}}.$$

Hence it follows that

$$\lambda_4(n,\varphi)\leqslant 1+\sqrt{\varkappa_1}+B\sqrt{\frac{\ln\ln n}{\ln n}}.$$

In order to estimate $\lambda_k(n)$ from below we shall calculate $\lambda_k(n,\psi)$ for a special function $\psi(m)$. Let $a=7.579$; $h=0.09599$; $\delta=0.299$; $\varepsilon=7$. We define the function $\psi(m)$ by its values for prime powers

$$\psi(p^a)=\psi(p)=\begin{cases}-\nu\ln^{a-\varepsilon}p & \text{if}\quad p\leqslant n^h,\\ -\mu\ln^{\delta}n\cdot\ln^{a-\delta-\varepsilon}p & \text{if}\quad n^h<p\leqslant\sqrt{n},\\ \ln^{-\varepsilon}n\cdot\ln^{a}p & \text{if}\quad \sqrt{n}<p\leqslant n,\end{cases}$$

where

$$\nu=\frac{1}{2a\tau}\left(\left(1-\frac{1}{2^a}\right)^2+\sqrt{\left(1-\frac{1}{2^a}\right)^4+2^{2a-2\varepsilon+2}\left(1-\frac{1}{2^{2a}}\right)a(a-\varepsilon)\tau^2}\right),$$

$$\mu=\frac{1}{\tau_2}\left(w_1+\sqrt{w_1^2+(a-\delta-\varepsilon)\tau_2^2\frac{w_2}{w_3}}\right),$$

$$w_1=-\nu\tau_1+\frac{1}{2a}\left(1-\frac{1}{2^a}\right)^2,$$

$$w_2=\frac{\nu^2}{a-\varepsilon}h^{2a-2\varepsilon}+\frac{1}{a}\left(1-\frac{1}{2^{2a}}\right),$$

$$w_3=\left(\frac{1}{2}\right)^{2a-2\delta-2\varepsilon}-h^{2a-2\delta-2\varepsilon},$$

$$\tau=\int_0^{1/2}y^{a-\varepsilon-1}\left(1-(1-y)^a\right)dy,$$

$$\tau_1=\int_0^{h}y^{a-\varepsilon-1}\left(1-(1-y)^a\right)dy,$$

$$\tau_2=\int_h^{1/2}y^{a-\delta-\varepsilon-1}\left(1-(1-y)^a\right)dy.$$

Using partial summation we can show that

$$D^2(n,\psi)=\frac{D}{2}\left(1+\frac{B}{\sqrt{\ln n}}\right)\ln^{2a-2\varepsilon}n,$$

where

$$D=w_2+\frac{\mu^2w_3}{a-\delta-\varepsilon}.$$

The same estimate is true for $B^2(n,\psi)$. Moreover

$$\sum_{p^a\leqslant n}\frac{\psi^2(p^a)}{p^{a+1}}=B,\qquad \sum_{p^a\leqslant n}\psi^2(p^a)=B\ln^{2a-2\varepsilon-1}n.$$

Thus

$$d\geqslant 1+\frac{B}{\sqrt{\ln n}},\qquad b\geqslant 1+\frac{B}{\sqrt{\ln n}}.$$

Further, from (15), (18), (20), (24) we deduce that for the function $\psi(m)$

$$R_k=\frac{2}{aw_2}\left(-w_1+\sqrt{w_1^2+(a-\delta-\varepsilon)\tau_2^2\frac{w_2}{w_3}}\right)+\frac{B}{\sqrt{\ln n}}\qquad (k=1,2,3,4).$$

The principal term in the last formula equals 0.47... Therefore for $n\geqslant n_0$

$$\lambda_k(n)>1.47.$$

Finally we shall prove the truth of the remark to the theorem. Referring to (10), (12), (13), (22), (23), (24) we see that

$$\lambda_k(n,f)\leqslant 1+B(\ln\ln n/\ln n)^{1/2}$$

for real-valued additive functions preserving a constant sign. On the other hand, it is easy to show that for the function $\omega(m)$

$$\lambda_k(n,\omega)=1+\frac{B}{\ln\ln n}.$$

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