On mean values of Dirichlet polynomials with real characters

by

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To the memory of Yu. V. Linnik

1. Introduction. The problem concerning the mean square of the Dirichlet polynomials

\[ f(s, \chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s}, \]

where \( \chi \) runs over a set \( K \) of Dirichlet characters and \( s \) for each \( \chi \) runs over a set of "well-spaced" points in the complex plane, can be reduced to the problem of estimating expressions of the type

\[ \sum_{\chi \in K} \int_{H_0}^{H} |f(a + it, \chi)|^2 dt. \]

It is desirable to have "hybrid" estimates, i.e. estimates in which neither the summation nor the integration is carried out trivially.

In the case when \( K \) consists of the primitive characters having modulus \( \leq X \), such "hybrid" results were obtained independently and by different methods by Montgomery [6] and the author [2]. Later Gallagher [1] found a device (Lemma 1 below) which effected a considerable technical simplification. Gallagher's lemma, combined with the large sieve inequality of Linnik–Bombieri, yields the following result (see [1], or [7, Th. 7.1]):

\[ \sum_{\chi \in K} \sum_{n \equiv 0 \pmod{\chi}} \int_{H_0}^{H} |f(it, \chi)|^2 dt \ll (X + T + N) \sum_{\chi} |a_\chi|^2, \]

where the star denotes the primitivity of the character.

If \( K \) is the set of the real primitive characters having modulus \( \leq X \), then the inequality (3) of course gives an estimate for (2), but owing to the extra non-real characters in (3), such an estimate may be crude or
trivial. Our aim is to show that, using Gallagher’s lemma a mean value theorem for real character sums, obtained by the author in [3] (see Lemma 2 below), one can in some cases prove something better.

To formulate our mean value theorem, we introduce some notation. Any real character arises from Kronecker’s symbol \( \left( \frac{D}{\nu} \right) \), and this character will be denoted by \( \chi_D \). The modulus of \( \chi_D \) is \( |D| \), and if we write \( D = d \omega \), where \( d \) is a fundamental discriminant, then \( \chi_D \) is the primitive character, equivalent to \( \chi_D \). All real non-principal characters are of the form \( \chi_D \) with \( D \) not a square and with \( D = 0 \) or \( 1 \) (mod 4). A sum over such values of \( D \) will be denoted by \( \chi_D \). The constant implied by the symbol \( \ll \) or \( O(\cdot) \) is absolute unless otherwise indicated.

**Theorem 1.** Let \( X > 3 \), \( N > 2 \) be natural numbers, let \( a_n \) for \( n = 1, 2, \ldots, N \), be any complex numbers, and define \( f(s, \chi) \) by (1). Write

\[
Z_\chi = \sum_{t=1}^N |a_n|^2.
\]

Then for any real \( T \), and any positive \( T \) we have

\[
\sum_{1 \leq n \leq X} \left| f(t, \chi_D) \right|^2 dt \ll TX \sum_{1 \leq m, n \leq N} \left| a_m a_n \right| + (TX)^{1/2} N^{1/2} Z_\chi \log N.
\]

From this we obtain the following analogous result for values of Dirichlet polynomials (for details, see Montgomery [7], Ch. 7).

**Corollary.** Let a finite set \( A(D) \) of complex numbers \( s = \sigma + it \) be given for each \( D \) with \( |D| < X \). Let \( T \), \( T \), \( \alpha \), \( \delta \) be real numbers such that

\[
T + \delta/2 \leq t \leq T + T - \delta/2, \quad \alpha \geq \alpha, \quad |t - s'| \geq \delta
\]

for each \( s \in A(D) \) and for any two different \( s, s' \in A(D) \). Write

\[
Z_\chi(a) = \sum_{n=1}^N |a_n|^2 n^{-\sigma a}.
\]

Then we have

\[
\sum_{1 \leq n \leq X} \left| f(s, \chi_D) \right|^2 \ll \left( \delta^{-1} + \log X \right) (\log \log X) \times \left[ TX \sum_{1 \leq m, n \leq N} |a_m a_n| (mn)^{-\alpha} + (TX)^{1/2} N^{1/2} Z_\chi(a)^{1/2} \log^6 X \right].
\]

As an application of the corollary we prove the following density theorem for \( L \)-functions.

**Theorem 2.** Let \( N(a, T, \chi) \) be the number of zeros of the function \( L(s, \chi) \) in the rectangle

\[
a \leq \sigma \leq 1, \quad |t| \leq T.
\]

Then for \( 1/2 \leq a < 1, \ T \geq 1, \ c > 0 \) we have

\[
N(a, T, \chi_D) \ll_c (XT)^{(c/4)\varepsilon/4} + (XT)^{1/2} (\log^c X).
\]

In [4] we proved a density theorem of the same type but weaker for increasing \( T \). Similarly as in [4], one might conjecture that

\[
N(a, T, \chi_D) \ll_c (XT)^{(c/2)\varepsilon/2}.
\]

As an arithmetical application of Theorem 2 we mention the following mean value estimate for character sums over primes.

**Theorem 3.** For \( X > \gamma, \ x > 0 \) we have

\[
\sum_{1 \leq n \leq X} \left| \sum_{p \leq x} \chi_D(p) \right| \ll X^{(c/2)\varepsilon/2},
\]

where

\[
h(\gamma) = \begin{cases}
1 + \gamma/2 & \text{for } 0 < \gamma < \frac{1}{2}, \\
\frac{1}{2}(1 + \gamma)^{-1} - \gamma^2 & \text{for } \frac{1}{2} \leq \gamma < 2, \\
\frac{1}{2} + \gamma & \text{for } \gamma \geq 2.
\end{cases}
\]

If the conjecture (7) is true, then we have in (8)

\[
h(\gamma) = \begin{cases}
1 + \gamma/2 & \text{for } 0 < \gamma < 1, \\
\frac{1}{2} + \gamma & \text{for } \gamma > 1.
\end{cases}
\]

For example, from (9) we have \( h(1) = 3 - \sqrt{2} = 1.585 \ldots \), whereas (10) would give \( h(1) = 1.5 \). For comparison we note that for any sequence \( M \) of natural numbers we have

\[
\sum_{1 \leq n \leq X} \left| \sum_{p \leq x} \chi_D(p) \right| \ll (X^{1/4} + x^{1+\gamma}) (\log X)^\gamma
\]

if \( x > 2 \). This is an easy corollary of Lemma 3 of the next section. In the above example this gives the exponent 1.75. If \( \gamma < \frac{1}{4} \), then the assertion of Theorem 3 follows from (11). Hence (8) is "non-trivial", i.e. reflects special properties of the prime number sequence, only in the case \( \gamma > \frac{1}{4} \).
2. Lemmas. First we quote the two basic lemmas.

Lemma 1 (Gallagher [1]). If the series

$$S(x) = \sum_{n=1}^{\infty} a_n x^{-s}$$

is absolutely convergent for $\Re s > 0$ (hence in particular if $a_n = 0$ for $n > N$), then

$$\int_{-T}^{T} |S(it)|^2 dt \ll T^2 \int_{-Y}^{Y} \left| \sum_{n=1}^{\infty} a_n \right|^2 y^{-1} dy,$$

where $T = \exp(T^{-1})$ and $T > 0$.

Lemma 2 (Jutila [2]). For $X \geq 3$, $Y \geq 1$ we have

$$\sum_{|D| \leq X} \left| \sum_{|D| \leq X} \chi_D(n) \right|^2 \ll X Y \log^4 X.$$

From Lemma 2 we deduce the following

Lemma 3. For $X \geq 3$, $N \geq 2$ we have

$$\sum_{|D| \leq X} \left| \sum_{|D| \leq X} \chi_D(n) \right|^2 \ll X \sum_{1 \leq m \leq N} |a_m a_n| + X^{1/2} N^{1/4} Z_{1/4} \log^6 N.$$

Proof. Put $p(n) = 0$ or 1 according to whether $n$ is a square or not. Then the expression on the left of (12) is

$$\ll X \sum_{1 \leq m \leq N} (1 - p(mn)) |a_m a_n| + \sum_{1 \leq m \leq N} p(mn) |a_m a_n| \sum_{|D| \leq X} \left| \sum_{|D| \leq X} \frac{D}{mn} \right|.$$

We shall estimate the second sum here by Lemma 2. Write

$$S(r) = \left| \sum_{|D| \leq X} \frac{D}{r} \right|.$$

Then the sum under consideration is

$$\sum_{1 \leq r \leq N^2} p(r) S(r) \ll \left( \sum_{1 \leq r \leq N^2} b_r \right)^2,$$

where

$$b_r = \sum_{1 \leq m, n \leq r} |a_m a_n|.$$

By Lemma 2 and the quadratic reciprocity law it is easily seen that

$$\sum_{1 \leq r \leq N^2} p(r) S(r) \ll N^2 X \log^6 N.$$
To complete the proof, we need an estimate for the sum
\[ \sum_{r} p(r) S(r)c_r \leq \left( \sum_{r} p(r) S^2(r) \sum_{1}^{N} c_r^2 \right)^{1/2}. \]

By (19) we have
\[ \sum_{1}^{N} c_r^2 \ll T^3 \sum_{l \leq N} |a_l|^2 \sum_{h \leq N} \hat{d}(h) + T^2 \sum_{h=1}^{N} a_h \sum_{h=1}^{N} \hat{d}(hh). \]

The first double sum is
\[ \ll T^3 \sum_{h=1}^{N} |a_h|^2 \hat{d}(h) \sum_{h \leq N} \hat{d}(h) \]
\[ \ll T^3 \left\{ \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \right\}^{1/2} \left\{ \sum_{h=1}^{N} \left( \sum_{l=1}^{N} \hat{d}(h+l) \right)^2 \right\}^{1/2} \]
\[ \ll T^3 \left\{ \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \right\} \sum_{h=1}^{N} \sum_{h=1}^{N} \hat{d}(h+l) \hat{d}(h+l) \]
\[ \ll T^3 \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \sum_{h=1}^{N} \sum_{h=1}^{N} \hat{d}(h+l) \hat{d}(h+l) \]
\[ = T^3 \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \sum_{h=1}^{N} \hat{d}(h)^2 \]
\[ \ll T^3 \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \sum_{h=1}^{N} \hat{d}(h)^2 \]
\[ = T^3 \sum_{h=1}^{N} |a_h|^2 \hat{d}(h)^2 \sum_{h=1}^{N} \hat{d}(h)^2 \]
and the second double sum is estimated similarly.

This and (20), substituted into (20), complete the proof of Theorem 1.

4. Proof of Theorem 2. Put \( L = \log XT \). From the zeros under consideration we pick out a subset \( A \) (a set of "good" zeros) satisfying the following conditions:

(i) if \( \sigma = \beta + it \) and \( \sigma' = \beta' + it' \) are counted in \( A \) as zeros of the same \( L(s, \chi_D) \), then \( |y-y'| \geq 2L^2; \)

(ii) the region \( \sigma > \beta + L^{-1}, |t| \leq L^2 \) does not contain any zero of \( L(s, \chi_D) \) if \( \sigma = \beta + it \) is counted in \( A \) as a zero of \( L(s, \chi_D) \).

It is easily seen that it suffices to find an upper estimate for the cardinality of the set \( A \).

For any zero \( \sigma \) of \( L(s, \chi_D) \) we can construct a Dirichlet polynomial
\[ F(s, \chi_D) = \sum_{\sigma < \sigma < \sigma'} b_s \chi_D(n)n^{-s} \]
with \( |b_s| \leq d(n) \) such that \( |F(\epsilon, \chi_D)| \geq 1 \). If \( \epsilon > 0 \) is fixed and \( XT \gg 1 \), then for \( \epsilon \leq A \) we may choose in (21)
\[ x = (XT)^{\epsilon}, \quad y = (XT)^{1/2+\epsilon} \]
(see [3], Lemma 2).

Split up the polynomial \( F(s, \chi_D) \) into a sum of \( \ll L \) polynomials of the type
\[ \sum_{n=1}^{N} b_s \chi_D(n)n^{-s} \]
with \( N < N' \ll N \). Then for any "good" zero \( \sigma \) of \( L(s, \chi_D) \) at least one of these polynomials is \( \gg L^{-1} \) at \( \sigma \).

To apply the corollary of Theorem 1, we first raise each polynomial (22) to a suitable integral power. This power \( w \) is chosen in such way that \( X^w \) is "close" \( (XT)^{1/2} \) since then the corollary gives the best estimate for the number of "large" values of the polynomials. Given a number \( z \) with \( y^{1/2} < z < y, \) an integer \( w \) exists such that \( z < N^w < z' \). Hence by the corollary the cardinality of the set \( A \) is
\[ \ll \left( \int_{0}^{\infty} (TXx^{1/2}+TX)^{1/2}(x^{2}+2\omega) \right)^{1/2}, \]
where \( c(x) \) is a constant depending on \( s \). The optimal choice for \( s \) is
\[ s = \max \{ (XT)^{1/2}, (XT)^{1/4+4\epsilon} \}. \]

Putting this into (23) we obtain the required estimate.

5. Proof of Theorem 3. We use the classical formula
\[ \sum_{n=1}^{N} \chi_D(n) \Lambda(n) = - \sum_{x \leq \epsilon_0} \frac{x^{-1}}{\log x} + O(x \log \log x + x^{1/2} \log \log x), \]
where \( \epsilon > 0 \), \( T \geq 2, \) \( x \) is of the form \( N + \frac{1}{2} \) with integral \( N \), and \( \epsilon \) runs over the zeros of \( L(s, \chi_D) \) except possibly the zero \( 1 - \frac{1}{2} \) if there exists an exceptional zero \( \beta \) of \( L(s, \chi_D) \) (see [8], Satz 4.6). It suffices to prove an estimate of the type (8) for this modified sum, where \( x = X' \).

The \( O \)-term is negligible if \( T \geq X \). The sum over the zeros is estimated by absolute values. Using Theorem 2 and partial summation, we see that the function \( h(y) \) is determined from the condition
\[ h(y) = \max \left\{ ya + (7 - 6a)/(6 - 4a) \right\}, \]
and (9) is obtained by a short calculation. The conditional result (10) is verified similarly.
On sets of integers containing no \( k \) elements in arithmetic progression

by

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Dedicated to the memory of Yu. V. Linnik

1. Introduction. In 1926 van der Waerden [15] proved the following startling theorem: If the set of integers is arbitrarily partitioned into two classes then at least one class contains arbitrarily long arithmetic progressions. It is well known and obvious that neither class must contain an infinite arithmetic progression. In fact, it is easy to see that for any sequence \( a_n \), there is another sequence \( b_n \), with \( b_n > a_n \), which contains no arithmetic progression of three terms, but which intersects every infinite arithmetic progression. The finite form of van der Waerden's theorem goes as follows: For each positive integer \( n \), there exists a least integer \( f(n) \) with the property that if the integers from 1 to \( f(n) \) are arbitrarily partitioned into two \( (2) \) classes, then at least one class contains an arithmetic progression of \( n \) terms. (For a short proof, see the note of Graham and Rothschild [7].) However, the best upper bound on \( f(n) \) known at present is extremely poor. The best lower bound, known to Berlekamp [3], asserts that \( f(n) > n^{2^n} \), which improves previous results of Erdös, Rado and W. Schmidt.

More than 40 years ago, Erdös and Turán [4] considered the quantity \( r_k(n) \), defined to be the greatest integer \( i \) for which there is a sequence of integers \( 0 < a_1 < a_2 < \ldots < a_i \leq n \) which does not contain an arithmetic progression of \( k \) terms. They were led to the investigation of \( r_k(n) \) by several things. First of all the problem of estimating \( r_k(n) \) is clearly interesting in itself. Secondly, \( r_k(n) < n/2 \) would imply \( f(k) < n \), i.e., they hoped to improve the poor upper bound on \( f(k) \) by investigating \( r_k(n) \). Finally, an old question in number theory asks if there are arbitra-

\(^{(2)} \) In fact, van der Waerden proved this for partitions into \( r \) classes for any positive integer \( r \).