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On mean values of Dirichlet polynomials with real characters

by

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To the memory of Yu. V. Linnik

1. Introduction. The problem concerning the mean square of the Dirichlet polynomials

(1) $$f(s, \chi) = \sum_{1}^{N} a_n \chi(n) n^{-s},$$

where χ runs over a set K of Dirichlet characters and s for each χ runs over a set of "well-spaced" points in the complex plane, can be reduced to the problem of estimating expressions of the type

(2) $$\sum_{\chi \in K} \int_{T_0}^{T_0+T} |f(\alpha+it, \chi)|^2 dt.$$

It is desirable to have "hybrid" estimates, i.e. estimates in which neither the summation nor the integration is carried out trivially.

In the case when K consists of the primitive characters having modulus $\leqslant X$, such "hybrid" results were obtained independently and by different methods by Montgomery [6] and the author [2]. Later Gallagher [1] found a device (Lemma 1 below) which effected a considerable technical simplification. Gallagher's lemma, combined with the large sieve inequality of Linnik–Bombieri, yields the following result (see [1], or [7], Th. 7.1):

(3) $$\sum_{q \leqslant X} \sum_{\chi \bmod q}^{*} \int_{T_0}^{T_0+T} |f(it, \chi)|^2 dt \ll (X^2 T + N) \sum_{1}^{N} |a_n|^2,$$

where the star denotes the primitivity of the character.

If K is the set of the *real* primitive characters having modulus $\leqslant X$, then the inequality (3) of course gives an estimate for (2), but owing to the extra non-real characters in (3), such an estimate may be crude or trivial. Our aim is to show that, using in Gallagher's lemma a mean value theorem for real character sums, obtained by the author in [3] (see Lemma 2 below), one can in some cases prove something better.

To formulate our mean value theorem, we introduce some notation. Any real character arises from Kronecker's symbol $\left(\frac{D}{n}\right)$, and this character will be denoted by χ_D. The modulus of χ_D is $|D|$, and if we write $D = da^2$, where d is a fundamental discriminant, then χ_d is the primitive character, equivalent to χ_D. All real non-principal characters are of the form χ_D with D not a square and with $D \equiv 0$ or $1 \pmod 4$. A sum over such values of D will be denoted by $\sum_D{}'$. The constant implied by the symbol $\ll$ or $O(\ )$ is absolute unless otherwise indicated.

THEOREM 1. *Let $X \geqslant 3$, $N \geqslant 2$ be natural numbers, let a_n, $n = 1, 2, \ldots, N$, be any complex numbers, and define $f(s, \chi)$ by* (1). *Write*

$$Z_k = \sum_1^N |a_n|^k.$$

Then for any real T_0 and any positive T we have

$$\sum_{|D|\leqslant X}{}' \int_{T_0}^{T_0+T} |f(it, \chi_D)|^2 dt \ll TX \sum_{\substack{1\leqslant m,n\leqslant N\\ mn=a^2}} |a_m a_n| + (TX)^{1/2} N^{15/8} Z_{16}^{1/8} \log^7 N. \tag{4}$$

From this we obtain the following analogous result for values of Dirichlet polynomials (for details, see Montgomery [7], Ch. 7).

COROLLARY. *Let a finite set $A(D)$ of complex numbers $s = \sigma + it$ be given for each D with $|D| \leqslant X$. Let T_0, T, α, δ be real numbers such that*

$$T_0 + \delta/2 \leqslant t \leqslant T_0 + T - \delta/2, \qquad \sigma \geqslant \alpha, \qquad |t - t'| \geqslant \delta$$

for each $s \in A(D)$ and for any two different $s, s' \in A(D)$. Write

$$Z_k(\alpha) = \sum_1^N |a_n|^k n^{-k\alpha}.$$

Then we have

$$\sum_{|D|\leqslant X}{}' \sum_{s\in A(D)} |f(s, \chi_D)|^2 \ll (\delta^{-1} + \log N)(\log\log N) \times \Big\{TX \sum_{\substack{1\leqslant m,n\leqslant N\\ mn=a^2}} |a_m a_n| (mn)^{-\alpha} + (TX)^{1/2} N^{15/8} Z_{16}(\alpha)^{1/8} \log^6 N\Big\}. \tag{5}$$

As an application of the corollary we prove the following density theorem for L-functions.

THEOREM 2. *Let $N(\alpha, T, \chi)$ be the number of zeros of the function $L(s, \chi)$ in the rectangle*

$$\alpha \leqslant \sigma \leqslant 1, \qquad |t| \leqslant T.$$

Then for $\frac{1}{2} \leqslant \alpha < 1$, $T \geqslant 1$, $\varepsilon > 0$ we have

$$\sum_{|D|\leqslant X}{}' N(\alpha, T, \chi_D) \ll_\varepsilon (XT)^{(7-6\alpha)/(6-4\alpha)+\varepsilon}. \tag{6}$$

In [4] we proved a density theorem of the same type but weaker for increasing T. Similarly as in [4], one might conjecture that

$$\sum_{|D|\leqslant X}{}' N(\alpha, T, \chi_D) \ll_\varepsilon (XT)^{3/2-\alpha+\varepsilon}. \tag{7}$$

As an arithmetical application of Theorem 2 we mention the following mean value estimate for character sums over primes.

THEOREM 3. *For $X \geqslant 3$, $\gamma > 0$, $\varepsilon > 0$ we have*

$$\sum_{|D|\leqslant X}{}' \Big|\sum_{p\leqslant X^\gamma} \chi_D(p)\Big| \ll_\varepsilon X^{h(\gamma)+\varepsilon}, \tag{8}$$

where

$$h(\gamma) = \begin{cases} 1+\gamma/2 & \text{for} \quad 0 < \gamma \leqslant \frac{1}{2}, \\ \frac{3}{2}(1+\gamma) - \sqrt{2\gamma} & \text{for} \quad \frac{1}{2} \leqslant \gamma \leqslant 2, \\ \frac{1}{2}+\gamma & \text{for} \quad \gamma > 2. \end{cases} \tag{9}$$

If the conjecture (7) *is true, then we have in* (8)

$$h(\gamma) = \begin{cases} 1+\gamma/2 & \text{for} \quad 0 < \gamma \leqslant 1, \\ \frac{1}{2}+\gamma & \text{for} \quad \gamma > 1. \end{cases} \tag{10}$$

For example, from (9) we have $h(1) = 3 - \sqrt{2} = 1.585\ldots$, whereas (10) would give $h(1) = 1.5$. For comparison we note that for *any* sequence M of natural numbers we have

$$\sum_{|D|\leqslant X}{}' \Big|\sum_{\substack{n\leqslant X^\gamma\\ n\in M}} \chi_D(n)\Big| \ll (X^{3/4+\gamma} + X^{1+\gamma/2})(\log X^\gamma)^3 \tag{11}$$

if $X^\gamma \geqslant 2$. This is an easy corollary of Lemma 3 of the next section. In the above example this gives the exponent 1.75. If $\gamma \leqslant \frac{1}{2}$, then the assertion of Theorem 3 follows from (11). Hence (8) is "non-trivial", i.e. reflects special properties of the prime number sequence, only in the case $\gamma > \frac{1}{2}$.

2. Lemmas. First we quote the two basic lemmas.

LEMMA 1 (Gallagher [1]). *If the series*

$$S(s) = \sum_{1}^{\infty} a_n n^{-s}$$

is absolutely convergent for $\operatorname{Re} s \geqslant 0$ (*hence in particular if* $a_n = 0$ *for* $n > N$), *then*

$$\int_{-T}^{T} |S(it)|^2 dt \ll T^2 \int_{0}^{\infty} \Big|\sum_{y}^{\tau y} a_n\Big|^2 y^{-1} dy,$$

where $\tau = \exp(T^{-1})$ *and* $T > 0$.

LEMMA 2 (Jutila [2]). *For* $X \geqslant 3$, $Y \geqslant 1$ *we have*

$$\sum_{|D|\leqslant X}' \Big| \sum_{1\leqslant n\leqslant Y} \chi_D(n)\Big|^2 \ll XY\log^3 X.$$

From Lemma 2 we deduce the following

LEMMA 3. *For* $X \geqslant 3$, $N \geqslant 2$ *we have*

$$\sum_{|D|\leqslant X}' \Big|\sum_{1}^{N} a_n \chi_D(n)\Big|^2 \ll X \sum_{\substack{1\leqslant m,n\leqslant N\\ mn=a^2}} |a_m a_n| + X^{1/2} N^{7/4} Z_8^{1/4} \log^6 N. \tag{12}$$

Proof. Put $p(n) = 0$ or 1 according to whether n is a square or not. Then the expression on the left of (12) is

$$\ll X \sum_{1\leqslant m,n\leqslant N} \big(1-p(mn)\big)|a_m a_n| + \sum_{1\leqslant m,n\leqslant N} p(mn)|a_m a_n| \Big|\sum_{|D|\leqslant X}' \Big(\frac{D}{mn}\Big)\Big|. \tag{13}$$

We shall estimate the second sum here by Lemma 2. Write

$$S(r) = \Big|\sum_{|D|\leqslant X}' \Big(\frac{D}{r}\Big)\Big|.$$

Then the sum under consideration is

$$\sum_{1}^{N^2} p(r) b_r S(r) \ll \Big\{\sum_{1}^{N^2} b_r^2 \sum_{1}^{N^2} p(r) S^2(r)\Big\}^{1/2}, \tag{14}$$

where

$$b_r = \sum_{\substack{1\leqslant m,n\leqslant N\\ mn=r}} |a_m a_n|. \tag{15}$$

By Lemma 2 and the quadratic reciprocity law it is easily seen that

$$\sum_{1}^{N^2} p(r) S^2(r) \ll N^2 X \log^8 N. \tag{16}$$

By (15),

$$\sum_{1}^{N^2} b_r^2 = \sum_{\substack{1\leqslant h,k,m,n\leqslant N\\ hk=mn}} |a_h a_k a_m a_n| \leqslant \sum_{\substack{1\leqslant h,k,m,n\leqslant N\\ hk=mn}} (|a_h|^4 + |a_k|^4 + |a_m|^4 + |a_n|^4).$$

Hence, writing $d(n)$ for the number of divisors of n, we have

$$\sum_{1}^{N^2} b_r^2 \ll \sum_{h=1}^{N} |a_h|^4 \sum_{k=1}^{N} d(hk) \ll \sum_{h=1}^{N} |a_h|^4 d(h) \sum_{k=1}^{N} d(k) \tag{17}$$
$$\ll Z_8^{1/2} (N\log^3 N)^{1/2} N \log N,$$

the last step by Schwarz's inequality and by the well-known formulas for the sum functions of $d(n)$ and $d^2(n)$.

The assertion (12) now follows from (13), (14), (16), and (17).

3. Proof of Theorem 1. If $T \leqslant 1$, then we may apply Lemma 3 pointwise and integrate trivially to obtain (4). Hence only the case $T > 1$ remains.

By Lemma 1 we have

$$\sum_{|D|\leqslant X}' \int_{T_0}^{T_0+T} |f(it, \chi_D)|^2 dt \tag{18}$$
$$\ll T^2 \sum_{|D|\leqslant X}' \int_{0}^{\infty} \Big|\sum_{y}^{\tau y} a_n \chi_D(n) n^{-iT_0}\Big|^2 y^{-1} dy$$
$$\ll T^2 \int_{0}^{\infty} \sum_{y\leqslant m,n\leqslant y\tau} |a_m a_n| S(mn) y^{-1} dy = \sum_{1}^{N^2} c_r S(r),$$

where

$$c_r = T^2 \sum_{\substack{1\leqslant m,n\leqslant N\\ mn=r}} |a_m a_n| \int_{y_1}^{y_2} y^{-1} dy,$$

$$y_1 = \tau^{-1}\max(m,n), \quad y_2 = \max\big(y_1, \min(m,n)\big).$$

Obviously

$$c_r \ll T \sum_{\substack{1\leqslant m,n\leqslant N\\ m\leqslant n\leqslant m\tau\\ mn=r}} |a_m a_n|. \tag{19}$$

Hence

$$\sum_{1}^{N^2} c_r S(r) \ll XT \sum_{1\leqslant m,n\leqslant N} \big(1-p(mn)\big)|a_m a_n| + \sum_{1}^{N^2} p(r) S(r) c_r.$$

To complete the proof, we need an estimate for the sum

$$\sum_{1}^{N^2} p(r)S(r)c_r \ll \Big\{\sum_{1}^{N^2} p(r)S^2(r)\sum_{1}^{N^2} c_r^2\Big\}^{1/2}. \tag{20}$$

By (19) we have

$$\sum_{1}^{N^2} c_r^2 \ll T^2 \sum_{\substack{1\leqslant h,k,m,n\leqslant N\\ h\leqslant k\leqslant h\tau\\ m\leqslant n\leqslant m\tau\\ hk=mn}} |a_h a_k a_m a_n|$$

$$\ll T^2 \sum_{h=1}^{N} |a_h|^4 \sum_{h\leqslant k\leqslant h\tau} d(hk) + T^2 \sum_{k=1}^{N} |a_k|^4 \sum_{k\tau^{-1}\leqslant h\leqslant k} d(hk).$$

The first double sum is

$$\ll T^2 \sum_{h=1}^{N} |a_h|^4 d(h) \sum_{h\leqslant k\leqslant h+N(\tau-1)} d(k)$$

$$\ll T^2 \Big\{\sum_{1}^{N} |a_h|^8 d^2(h)\Big\}^{1/2} \Big\{\sum_{h=1}^{N}\Big(\sum_{\nu=0}^{N(\tau-1)} d(h+\nu)\Big)^2\Big\}^{1/2}$$

$$\ll T^2 \Big\{\sum_{1}^{N} |a_h|^8 d^2(h)\Big\}^{1/2} \Big\{\sum_{0\leqslant\mu,\nu\leqslant N(\tau-1)} \sum_{h=1}^{N} d(h+\mu)d(h+\nu)\Big\}^{1/2}$$

$$\ll T^2 Z_{16}^{1/4}(N\log^{15}N)^{1/4}\Big(\sum_{0\leqslant\mu,\nu\leqslant N(\tau-1)} N\log^3 N\Big)^{1/2}$$

$$\ll T Z_{16}^{1/4} N^{7/4}(\log N)^{21/4}$$

and the second double sum is estimated similarly.

This and (16), substituted into (20), complete the proof of Theorem 1.

4. Proof of Theorem 2. Put $L = \log XT$. From the zeros under consideration we pick out a subset A (a set of "good" zeros) satisfying the following conditions:

(i) if $\varrho = \beta + i\gamma$ and $\varrho' = \beta' + i\gamma'$ are counted in A as zeros of the same $L(s, \chi_D)$, then $|\gamma - \gamma'| \geqslant 2L^2$;

(ii) the region $\sigma > \beta + L^{-1}$, $|t-\gamma| \leqslant L^2$ does not contain any zero of $L(s, \chi_D)$ if $\varrho = \beta + i\gamma$ is counted in A as a zero of $L(s, \chi_D)$.

It is easily seen that it suffices to find an upper estimate for the cardinality of the set A.

For any zero ϱ of $L(s, \chi_D)$ we can construct a Dirichlet polynomial

$$F(s, \chi_D) = \sum_{x\leqslant n\leqslant y} b_n \chi_D(n) n^{-s} \tag{21}$$

with $|b_n| \leqslant d(n)$ such that $|F(\varrho, \chi_D)| \gg 1$. If $\varepsilon > 0$ is fixed and $XT \gg_\varepsilon 1$, then for $\varrho \in A$ we may choose in (21)

$$x = (XT)^\varepsilon, \quad y = (XT)^{1/2+8\varepsilon}$$

(see [5], Lemma 2).

Split up the polynomial $F(s, \chi_D)$ into a sum of $\ll L$ polynomials of the type

$$\sum_{N}^{N'} b_n \chi_D(n) n^{-s} \tag{22}$$

with $N \ll N' \ll N$. Then for any "good" zero ϱ of $L(s, \chi_D)$ at least one of these polynomials is $\gg L^{-1}$ at ϱ.

To apply the corollary of Theorem 1, we first raise each polynomial (22) to a suitable integral power. This power u is chosen in such way that N^u is "near" $(XT)^{1/2}$ since then the corollary gives the best estimate for the number of "large" values of the polynomials. Given a number z with $y^{1/2} \leqslant z \leqslant y$, an integer u exists such that $z \leqslant N^u \leqslant z^2$. Hence by the corollary the cardinality of the set A is

$$\ll_\varepsilon \big(TXz^{1-2\alpha} + (TX)^{1/2}(z^2)^{2-2\alpha}\big)L^{c(\varepsilon)}, \tag{23}$$

where $c(\varepsilon)$ is a constant depending on ε. The optimal choice for z is

$$z = \max\{(XT)^{1/(6-4\alpha)}, (XT)^{1/4+4\varepsilon}\}.$$

Putting this into (23) we obtain the required estimate.

5. Proof of Theorem 3. We use the classical formula

$$\sum_{n\leqslant x} \chi_D(n)\Lambda(n) = -\sum_{|\mathrm{Im}\,\varrho|\leqslant T} \varrho^{-1}x^\varrho + O\big(xT^{-1}\log^2(|D|x) + x^{1/4}\log(|D|x)\big),$$

where $x \geqslant T \geqslant 2$, x is of the form $N + \frac{1}{2}$ with integral N, and ϱ runs over the zeros of $L(s, \chi_D)$, except possibly the zero $1-\beta_1$ if there exists an exceptional zero β_1 of $L(s, \chi_D)$ (see [8], Satz 4.6). It suffices to prove an estimate of the type (8) for this modified sum, where $x = X^\gamma$.

The O-term is negligible if $T = X$. The sum over the zeros is estimated by absolute values. Using Theorem 2 and partial summation, we see that the function $h(\gamma)$ is determined from the condition

$$h(\gamma) = \max_{\frac{1}{2}\leqslant\alpha\leqslant 1}\{\gamma\alpha + (7-6\alpha)/(6-4\alpha)\},$$

and (9) is obtained by a short calculation. The conditional result (10) is verified similarly.

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On sets of integers containing no k elements in arithmetic progression

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Dedicated to the memory of Yu. V. Linnik

1. Introduction. In 1926 van der Waerden [15] proved the following startling theorem: *If the set of integers is arbitrarily partitioned into two classes then at least one class contains arbitrarily long arithmetic progressions.* It is well known and obvious that neither class must contain an infinite arithmetic progression. In fact, it is easy to see that for any sequence a_n there is another sequence b_n, with $b_n > a_n$, which contains no arithmetic progression of three terms, but which intersects every infinite arithmetic progression. The finite form of van der Waerden's theorem goes as follows: *For each positive integer n, there exists a least integer $f(n)$ with the property that if the integers from 1 to $f(n)$ are arbitrarily partitioned into two* (1) *classes, then at least one class contains an arithmetic progression of n terms.* (For a short proof, see the note of Graham and Rothschild [7].) However, the best upper bound on $f(n)$ known at present is extremely poor. The best lower bound known, due to Berlekamp [3], asserts that $f(n) > n2^n$, which improves previous results of Erdös, Rado and W. Schmidt.

More than 40 years ago, Erdös and Turán [4] considered the quantity $r_k(n)$, defined to be the greatest integer l for which there is a sequence of integers $0 < a_1 < a_2 < \ldots < a_l \leqslant n$ which does not contain an arithmetic progression of k terms. They were led to the investigation of $r_k(n)$ by several things. First of all the problem of estimating $r_k(n)$ is clearly interesting in itself. Secondly, $r_k(n) < n/2$ would imply $f(k) < n$, i.e., they hoped to improve the poor upper bound on $f(k)$ by investigating $r_k(n)$. Finally, an old question in number theory asks if there are arbitra-

(1) In fact, van der Waerden proved this for partitions into r classes for any positive integer r.