ACTA ARITHMETICA
XXVII (1975)

Large values of Dirichlet polynomials II

by

M. N. Huxley (Cardiff)

Dedicated to the memory
of Yu. V. Linnik

1. Introduction. This paper continues [5], where we use a reflection argument together with the Halász–Montgomery method to obtain new bounds for sums of the moduli of Dirichlet polynomials. Forti and Viola [1], [2] divide ordinates of t into 'good' and 'bad' values, an ordinate being good if the appropriate L-function is not too large. By a standard argument (cf. Section 3) an ordinate can be bad only if there is a zero nearby. Since such bounds imply zero-density theorems, they obtain inequalities connecting the numbers $N(\sigma, T, \chi)$ for different abscissae σ. Here $N(\sigma, T, \chi)$ is the number of zeros $\beta+i\gamma$ of $L(s, \chi)$ in $\beta \geqslant \sigma$, $|\gamma| \leqslant T$ and $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the same rectangle. This method was systematized by Jutila [6], who showed (actually the qT-analogue) that

$$\sum_{q\leqslant Q} \sum_{\chi \bmod q}^{*} N(\sigma, T, \chi) \ll (Q^2T)^{\lambda(1-\sigma)+\varepsilon} \tag{1.1}$$

with $\lambda = 2.4605\ldots$ The asterisk indicates the restriction of the sum to primitive characters. Jutila considered the sum over all characters to a fixed modulus, in which qT replaces Q^2T. In fact the method of [5] already suffices to prove (1.1) with $\lambda = 27/11$; we show this in Section 8. The present paper combines the ideas of [5] and [6]. Recasting the iteration step in [5] enables us to treat good ordinates by estimating the L-function, bad ones by reflecting the sum. Section 5 summarizes our bounds for Dirichlet polynomials.

As an application we prove zero-density theorems for $\zeta(s)$.

Theorem. *The 'density hypothesis'*

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon} \tag{1.2}$$

holds for $\sigma \geqslant \sigma_0$ and for any $\varepsilon > 0$, where $\sigma_0 = 0.80118\ldots$ is a certain quadratic surd. Moreover, for $61/74 \leqslant \sigma \leqslant 37/42$

$$N(\sigma, T) \ll T^{48(1-\sigma)/37(2\sigma-1)+\varepsilon}, \tag{1.3}$$

and for $37/42 \leqslant \sigma \leqslant 1$

(1.4) $$N(\sigma, T) \ll T^{3(1-\sigma)/2\sigma+\varepsilon}.$$

The implied constants in (1.2), (1.3) *and* (1.4) *depend on* ε.

We can also obtain bounds for $3/4 \leqslant \sigma \leqslant \sigma_0$, but in these the exponent of T is an implicit function of σ.

Forti and Viola [2] obtained (1.2) for $\sigma > 0.8059\ldots$, improving earlier results of Turán and Halász, Montgomery [7] and the author [4], [5].

2. The reflection argument. We consider Dirichlet polynomials of the form

(2.1) $$F(s, \chi) = \sum_{N+1}^{2N} a(m)\chi(m)m^{-s},$$

where $\chi(m)$ is a Dirichlet character; a general Dirichlet polynomial can be broken into sums of the form (2.1). Our object is to estimate

(2.2) $$E = \sum_{q\leqslant Q;\, q\equiv 0 (\mathrm{mod}\, q_0)} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{r=1}^{R(\chi)} |F(s(r,\chi),\chi)|,$$

where the asterisk denotes propriety. We assume that the complex numbers $s(r, \chi) = \sigma(r, \chi)+it(r, \chi)$ satisfy

(2.3) $$0 \leqslant \sigma(r, \chi) \leqslant (\log NQ^2T)^{-1},$$

that any pair $s(r_1, \chi_1), s(r_2, \chi_2)$ satisfy

(2.4) $$|t(r_1, \chi_1)-t(r_2, \chi_2)| \leqslant T,$$

and if $\chi_1 = \chi_2$ also

(2.5) $$|t(r_1, \chi_1)-t(r_2, \chi_2)| \geqslant 1$$

for $r_1 \neq r_2$. Let

(2.6) $$R = \sum_{q\leqslant Q;\, q\equiv 0 (\mathrm{mod}\, q_0)} \sideset{}{^*}\sum_{\chi \bmod q} R(\chi).$$

Halász's inequality gives

(2.7) $$|E|^2 \leqslant G \sum_{\chi_1, r_1} \sum_{\chi_2, r_2} |H(\sigma(r_1, \chi_1)+\sigma(r_2, \chi_2)+it(r_1, \chi_1)-it(r_2, \chi_2), \chi_1\bar{\chi}_2)|$$

where

(2.8) $$G = \sum_{N+1}^{2N} |a(m)|^2$$

and

(2.9) $$H(s, \chi) = \sum_{m=1}^{\infty} b(m)\chi(m)m^{-s},$$

with coefficients $b(m) = b(m, U)$ that satisfy

(2.10) $$\begin{cases} b(m, U) = 0 & \text{if} \quad m \leqslant U \text{ or } m \geqslant e^4 U, \\ b(m, U) = 1 & \text{if} \quad eU \leqslant m \leqslant e^3 U, \\ 0 \leqslant b(m, U) \leqslant 1 & \text{if} \quad U < m < eU \text{ or } e^3 U < m < e^4 U, \end{cases}$$

and

(2.11) $$eU \leqslant N+1 \leqslant 2N \leqslant e^3 U.$$

We generalize the construction of $b(m, U)$ in [5] as follows. Let

(2.12) $$J(\omega) = \frac{2^{1-2h}\pi^{2h}((2h-1)!)^2}{((h-1)!)^2\omega} \prod_{n=1-h}^{h} (\omega-(2n-1)\pi i)^{-1},$$

with a pole of residue $\frac{1}{2}$ at $\omega = 0$, and

(2.13) $$K(\omega) = (e^{4\omega}+e^{3\omega}-e^{\omega}-1)J(\omega).$$

The kernel used in [5] is the case $h = 1$. Then

(2.14) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} K(\omega)L(s+\omega, \chi)U^{\omega}d\omega = \sum_{1}^{\infty} b(m, U)\chi(m)m^{-s}.$$

The assertions (2.10) follow from the lemma below.

Lemma 1. *We have*

(2.15) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} J(\omega)x^{\omega}d\omega = c(x),$$

where

(2.16) $$c(x) = 0 \quad \textit{for} \quad x \leqslant 1,$$

(2.17) $$0 \leqslant c(x) \leqslant 1 \quad \textit{for} \quad x \geqslant 1,$$

and for $x \geqslant 1$

(2.18) $$c(x)+c(ex) = 1.$$

Proof. In partial fractions

(2.19) $$\prod_{1-h}^{h} (\omega-(2n-1)\pi i)^{-1} = \sum_{1-h}^{h} \frac{(-1)^{h-n}(2\pi i)^{1-2h}}{(h-n)!(h+n-1)!}\cdot\frac{1}{\omega-(2n-1)\pi i}.$$

The integral

(2.20) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{\omega}}{\omega(\omega-(2n-1)\pi i)}d\omega$$

is zero for $x \leqslant 1$ and for $x \geqslant 1$ it is

$$\frac{x^{(2n-1)\pi i}-1}{(2n-1)\pi i} = \int_0^{\log x} e^{(2n-1)\pi i\theta}\,d\theta. \tag{2.21}$$

For $x > 1$

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \pi^{2h-1}(2h-1)!\prod_{1-h}^{h}\big(\omega-(2n-1)\pi i\big)^{-1}\omega^{-1}x^{\omega}\,d\omega \tag{2.22}$$
$$= \int_0^{\log x} (2i)^{1-2h}(e^{\pi i\theta}-e^{-\pi i\theta})^{2h-1}\,d\theta = \int_0^{\log x} \sin^{2h-1}\pi\theta\,d\theta.$$

By the periodicity of $\sin\pi\theta$, the right hand side of (2.22) is positive and

$$\Big(\int_0^{\log x} + \int_0^{1+\log x}\Big)\sin^{2h-1}\pi\theta\,d\theta = \int_0^1 \sin^{2h-1}\pi\theta\,d\theta = \frac{2^{2h-1}\big((h-1)!\big)^2}{\pi\big((2h-1)!\big)^2}, \tag{2.23}$$

which verifies (2.18) and (2.17).

All the machinery of [5] goes through with these more general coefficients $b(m, U)$. The implied constants in the various estimates will depend on h.

3. The effect of a zero-free region. Next we show that $H(s,\chi)$ is small inside a zero-free region. The argument is standard, cf. Forti and Viola [1], [2] or Titchmarsh [8]. Let

$$D = Q^2T/q_0, \tag{3.1}$$

$$\lambda_1 = \log D, \tag{3.2}$$

$$\lambda_2 = \log\lambda_1, \tag{3.3}$$

$$\lambda_3 = \log\lambda_2. \tag{3.4}$$

Suppose that $L(s,\chi)$ has no zeros $\beta+i\gamma$ with $\beta \geqslant \alpha$, $|\gamma-\tau| \leqslant \lambda_1\lambda_2$. Let t_0 be any number with $|t_0-\tau| \leqslant 3\lambda_1\lambda_2/4$. On the circle C_1, centre $2+it_0$, radius $2-\alpha-\frac{1}{2}\lambda_2^{-1}$ we have

$$\max\big(0, \operatorname{Re}\log L(s,\chi)\big) \ll \lambda_1. \tag{3.5}$$

Hence on the circle C_2, centre $2+it_0$, radius $2-\alpha-\lambda_2^{-1}$ the Borel–Carathéodory theorem gives

$$\log L(s,\chi) \ll \lambda_1\lambda_2. \tag{3.6}$$

Now consider the circles C_3 radius $\lambda_2-\alpha-\lambda_2^{-1}$, C_4 radius $\lambda_2-\sigma$, C_5 radius $\lambda_2-1-\lambda_2^{-1}$, each with centre λ_2+it_0, where now $|t_0-\tau| \leqslant 2\lambda_1\lambda_2/3$. On C_3 the inequality (3.6) is valid, whilst on C_5

$$\log L(s,\chi) \ll \lambda_2 \tag{3.7}$$

by direct computation. Hadamard's three circles theorem gives

$$\log L(s,\chi) \ll \lambda_1^{(1-\sigma)/(1-\alpha)}\lambda_2 \tag{3.8}$$

or $\alpha+\lambda_2^{-1} < \sigma < 1$. On $\sigma = \alpha_1$ given by

$$\alpha_1 = \alpha+\lambda_2^{-1}\lambda_3 \tag{3.9}$$

we have

$$L(s,\chi) \ll D^{1/2h} \tag{3.10}$$

if D is large enough. The functional equation gives

$$L(1-\alpha_1+it,\chi) \ll D^{\alpha_1-1/2}|L(\alpha_1+it,\chi)| \ll D^{\alpha-1/2+1/h} \tag{3.11}$$

whether or not χ is proper.

We now define a pair (τ,χ) to be good as far as α if $L(s,\chi)$ is non-zero for $\sigma \geqslant \alpha$ and

$$|t-\tau| \leqslant D^{1/h}. \tag{3.12}$$

In the integral (2.14) if the pair (t,χ) is good as far as α, we take the line of integration to $\operatorname{Re}\omega = 1-\alpha_1$. Then (2.9) and (2.14) give

$$H(s,\chi) \ll N^{1-\alpha}D^{\alpha-1/2+1/h}. \tag{3.13}$$

4. Jutila's convexity argument. Jutila's convexity argument takes its simplest form when we consider zeros of $\zeta(s)$ alone. Let $I(\sigma, T)$ be the total length of those subintervals of $[-T, T]$ on which $\zeta(s)$ is not good as far as σ, and let $f(\sigma)$ be such that

$$I(\sigma,T) \ll T^{f(\sigma)}. \tag{4.1}$$

Let $A(\alpha)$ be the convexity property

$$(1-\alpha)f(\sigma) \leqslant (1-\sigma)f(\alpha)+(\sigma-\alpha)f(1) \tag{4.2}$$

for $\alpha \leqslant \sigma \leqslant 1$. By (2.7) and (3.13) we have (dropping χ from the notation)

$$\sum_{r_2}\big|H\big(\sigma(r_1)+\sigma(r_2)+it(r_1)-it(r_2)\big)\big| \tag{4.3}$$
$$\ll \int_\alpha^1 N^{1-\sigma}T^{\sigma-1/2+1/h}\log(T/N)T^{f(\sigma)}\,d\sigma + N^{1-\alpha}T^{\alpha-1/2+f(\alpha)+1/h}$$
$$\ll \int_\alpha^1 N^{1-\sigma}T^{\sigma-1/2+(1-\sigma)f(\alpha)/(1-\alpha)+(\sigma-\alpha)f(1)/(1-\alpha)+1/h}\log(T/N)\,d\sigma + N^{1-\alpha}T^{\alpha-1/2+f(\alpha)+1/h},$$

where α is chosen so that

$$T^{f(\alpha)} = RT^{1/h}, \tag{4.4}$$

and we have assumed $A(\alpha)$. If

$$f(\alpha) \geqslant f(1)+1-\alpha \tag{4.5}$$

and

$$N \geqslant T^{1/3} \tag{4.6}$$

the expression in (4.3) is

$$\ll N^{1-\alpha}T^{\alpha-1/2+f(\alpha)+1/h} \ll RN^{1-\alpha}T^{\alpha-1/2+2/h}. \tag{4.7}$$

In the general case we let $I(\sigma, T, \chi)$ be the total length of subintervals of $[-T, T]$ on which $L(s, \chi)$ is not good as far as σ, and ask that

$$\sum_{q\leqslant Q;\, q\equiv 0 \,(\mathrm{mod}\, q_0)} \;\sum_{\chi \bmod q}^{*} I(\sigma, T, \chi\chi_1) \ll D^{f(\sigma)} \tag{4.8}$$

where χ_1 is a fixed character to some modulus $q_1 \leqslant Q$ which is a multiple of q_0, the implied constant being uniform in χ_1, and in triples q_0, Q, T of real numbers $\geqslant 1$ with $D = Q^2T/q_0$. Since $(\chi_1\chi_2)\overline{(\chi_1\chi_3)} = \chi_2\overline{\chi}_3|\chi_1|$, and T times the modulus of $\chi\chi_1$ is at most D, any bound obtained using the Halász–Montgomery method for the number of zeros of the functions $L(s, \chi)$ also applies to the number of zeros of the corresponding functions $L(s, \chi\chi_1)$ with χ_1 fixed as above.

5. The iteration step. As in [5] we introduce the number $B(R, M, D)$ defined for each triple R, M, D of real numbers $\geqslant 1$ as the greatest lower bound of numbers B for which

$$E \ll \begin{cases} G^{1/2}B & \text{if} \quad N \leqslant M, \\ (N/M)^{3/2}G^{1/2}B & \text{if} \quad M \leqslant N \leqslant D \end{cases} \tag{5.1}$$

holds for each choice of q_0, Q, T satisfying (3.1), of coefficients $a(N+1), \ldots, a(2N)$ and for each set of points $s(r, \chi)$ satisfying (2.3), (2.4) and (2.5). We quote the following bounds for B.

LEMMA 2. *If $D \geqslant N$ we have*

$$B(R, N, D) \ll D^{1/h}(R^{1/2}N^{1/2}+RD^{1/4}), \tag{5.2}$$

and if k be any positive integer

$$B(R, N, D) \ll D^{1/h}\big(R^{1/2}N^{1/2}+RN^{1/4}+R^{1-1/2k}N^{1/4}\big(B(R, D^k/N^k, D)\big)^{1/2k}\big). \tag{5.3}$$

If $N \geqslant D$ we have

$$B(R, N, D) \ll R^{1/2}N^{1/2+1/h}. \tag{5.4}$$

The implied constants depend only on h and k.

The bounds (5.2) and (5.4) are due to Montgomery, cf. [7], and (5.3) is the result of [5]. The ideas of Forti, Viola and Jutila lead to the following result.

LEMMA 3. *For $D \geqslant N \geqslant D^{1/3}$ we have*

$$B(R, N, D) \ll D^{2/h}(R^{1/2}N^{1/2}+RN^{\frac{1}{2}(1-\alpha)}D^{\frac{1}{2}(\alpha-\frac{1}{2})}), \tag{5.5}$$

provided that $f(\sigma)$ defined in (4.8) has the property $A(\alpha)$ and satisfies (4.5), and that

$$D^{f(\alpha)} \leqslant RD^{1/h}. \tag{5.6}$$

In any case we have for $N \leqslant D$ and for any α in $1/2 \leqslant \alpha \leqslant 1$

$$B(R, N, D) \ll D^{2/h}(R^{1/2}N^{1/2}+RN^{\frac{1}{2}(1-\alpha)}D^{\frac{1}{2}(\alpha-\frac{1}{2})}+R^{1/2}R_1^{1/2}D^{1/4}), \tag{5.7}$$

and for any positive integer k

$$B(R, N, D) \ll D^{2/h}\big(R^{1/2}N^{1/2}+RN^{\frac{1}{2}(1-\alpha)}D^{\frac{1}{2}(\alpha-\frac{1}{2})}+R^{1/2}R_1^{1/2-1/2k}N^{1/4}\big(B(R_1, D^k/N^k, D)\big)^{1/2k}\big), \tag{5.8}$$

where the implied constants depend on h and k and where

$$R_1 = D^{f(\alpha)}. \tag{5.9}$$

Proof. The term in R to the first power within the brackets comes from ordinates which are good as far as α. In (5.7) bad ordinates are treated as in [7] by estimating the corresponding L-function: they are good as far as 1. In (5.8) the iteration is applied to bad ordinates, whilst in (5.5) bad ordinates are classified according to which σ they are good for, and Jutila's convexity argument of Section 4 is used.

We now turn to zero-density theorems. In Sections 6 and 7 we make repeated use of Lemma 3 to obtain results for $\zeta(s)$, whilst in Section 8, which really belongs to [5], we improve Jutila's value of λ in the more general zero-density theorem (1.1).

6. The range $\sigma > 61/74$ for $\zeta(s)$. We now consider zeros of $\zeta(s)$ only, so that $q_0 = Q = 1$ and $D = T$. In [5] we showed that

$$N(\sigma, T) \ll T^{48(1-\sigma)/37(2\sigma-1)+O(1/h)} \tag{6.1}$$

for $61/74 \leqslant \sigma \leqslant 75/89$. We shall extend the validity of (6.1). The application of Lemmas 2 and 3 to zeros of $\zeta(s)$ has

$$E \gg Rt^{-1/h}, \tag{6.2}$$

$$G \leqslant N^{1-2\sigma+1/h}, \tag{6.3}$$

where t will be chosen below. Equation (5.3) of Lemma 2 gives, with the definition (5.1) of $B(R, N, t)$,

$$R^2t^{-2/h} \ll E^2 \ll N^{1-2\sigma+1/h}t^{2/h}(RN+R^{2-1/4k}t^{1/2}+R^2N^{1/2}t^{1/4k}). \tag{6.4}$$

With $k=2$ (6.4) gives

$$R \ll N^{2-2\sigma+1/h} t^{4/h} + N^{4-8\sigma+4/h} t^{2+16/h} \tag{6.5}$$

provided that

$$N^{\sigma-3/4-1/2h} > c_1 t^{1/16+2/h} \tag{6.6}$$

for a suitable constant c_1 (depending on h). The relation of t to T can now be explained: we divide the interval $[-T, T]$ into subintervals of length t, where t satisfies (6.6). Let

$$N = t^{c/(2\sigma-1)}; \tag{6.7}$$

with this choice we can dispose what are called in [5] class (ii) zeros by using Haneke's theorem [3], provided $c > 16/37 + 1/h$ and T is sufficiently large. Jutila's method of breaking up a long Dirichlet polynomial requires N to be large enough for the expression on the right of (6.5) to be

$$\ll N^{3-3\sigma+3/2h} t T^{-1}, \tag{6.8}$$

and then gives

$$N(\sigma, T) \ll N^{3-3\sigma+2/h}, \tag{6.9}$$

so that we require

$$R \ll N^{3-3\sigma+3/2h} t^{1-37c/16+O(1/h)}. \tag{6.10}$$

The first term on the right of (6.5) satisfies (6.10) for

$$c \leqslant 16(2\sigma-1)/(90\sigma-53) + O(1/h), \tag{6.11}$$

which agrees with $c > 16/37 + 1/h$ for $\sigma > 2/3 + O(1/h)$. This choice makes

$$N = t^{16/(90\sigma-53)+O(1/h)} \tag{6.12}$$

and (6.6) is satisfied for $\sigma > 139/166 + O(1/h)$ ($< 75/89$). The second term on the right of (6.5) satisfies (6.10) for $\sigma < 37/42 + O(1/h)$ ($= 0.8809\ldots$), and we have extended (6.1) to the range $61/74 \leqslant \sigma \leqslant 37/42$.

A convenient bound for the range $\sigma \geqslant 37/42$ can be found by putting

$$N = t^{1/(3\sigma-1)} \tag{6.13}$$

so that the two terms in (6.5) are of the same order of magnitude. The condition (6.6) holds for $\sigma > 11/13 + O(1/h)$ ($= 0.846\ldots$). Substituting in (6.8) we have

$$N(\sigma, T) \ll N^{3-3\sigma+O(1/h)} T^{O(1/h)} \ll T^{3(1-\sigma)/2\sigma+O(1/h)}, \tag{6.14}$$

Haneke's theorem again disposing of class (ii) zeros.

7. The density hypothesis for $\zeta(s)$. To prove the density hypothesis for $\zeta(s)$ in the form

$$N(\sigma, T) \ll T^{2(1-\sigma)+O(1/h)} \tag{7.1}$$

we need in place of (6.10)

$$R \ll N^{3-3\sigma} t^{1-3c/2+O(1/h)} \tag{7.2}$$

with

$$N = t^c \tag{7.3}$$

for some $c > 2/3 + O(1/h)$. We use (5.5) of Lemma 3 in place of Lemma 2, which gives

$$R^2 t^{-2/h} \ll N^{1-2\sigma+1/h} t^{4/h} (RN + R^2 N^{1-\alpha} t^{\alpha-1/2}) \tag{7.4}$$

provided that $A(\alpha)$ holds and

$$R > N(\alpha, T) t^{1/h}. \tag{7.5}$$

To obtain a non-trivial upper bound from (7.4) we require

$$N^{\alpha+2\sigma-2-1/h} > c_2 t^{\alpha-1/2+4/h} \tag{7.6}$$

for some c_2 depending only on h, that is

$$\alpha \leqslant \left(\tfrac{1}{2} - 2c(1-\sigma)\right)/(1-c) + O(1/h), \tag{7.7}$$

whence

$$R \ll N^{2-2\sigma+1/h} t^{6/h}. \tag{7.8}$$

This ensures (7.2) when

$$c \leqslant 1/(\sigma + \tfrac{1}{2}) + O(1/h), \tag{7.9}$$

which is consistent with $c > 2/3$ for $\sigma < 1 - O(1/h)$, and gives

$$\alpha \leqslant (10\sigma-7)/(4\sigma-2) + O(1/h). \tag{7.10}$$

Using the bound established in [4]

$$N(\alpha, T) \ll T^{3(1-\alpha)/(3\alpha-1)+1/h}, \tag{7.11}$$

which has the convexity property $A(\alpha)$ for $\alpha > 1/3$, we find that (7.5) is satisfied if

$$\frac{4(1-\sigma)}{2\sigma+1} > \frac{3(5-6\sigma)}{26\sigma-19} + O\left(\frac{1}{h}\right), \tag{7.12}$$

which is true for $5/6 \geqslant \sigma > 0.8020\ldots$ ($< 55/68$).

For $99/124+O(1/h) < \sigma \leqslant 55/68+O(1/h)$ we have $61/74 \leqslant \alpha \leqslant 37/42$, and we may use (6.1), which enables us to satisfy (7.5) for

$$\frac{4(1-\sigma)}{2\sigma+1} > \frac{12(5-6\sigma)}{37(4\sigma-3)} + O\left(\frac{1}{h}\right). \tag{7.13}$$

Let x be that root of

$$56x^2-339x+4 = 0 \tag{7.14}$$

which is approximately $0.01182\ldots$, and $\sigma_0 = 4/5+x/10$. Then for $\sigma \geqslant \sigma_0+O(1/h)$ we have the density hypothesis (7.1).

Between σ_0 and $37/42$ the form of the upper bound for $N(\sigma, T)$ is dictated by consideration of class (ii) zeros. To the left of σ_0 we still have the estimate

$$T^{2-2\sigma+1/h} \tag{7.15}$$

for the number of class (ii) zeros, but for class (i) zeros, defined in terms of Dirichlet polynomials, the best estimate we can obtain still exceeds (7.15). We ask (7.5), (7.6) and (7.8) to hold, and now relate t to T by

$$T/t = N^{1-\sigma} \tag{7.16}$$

as in (6.13). The definition of $f(\sigma)$ with (7.5) and (7.8) gives

$$N^{2-2\sigma} = t^{f(\alpha)+O(1/h)}, \tag{7.17}$$

so that the total number of zeros is

$$N(\sigma, T) \ll N^{3-3\sigma}t^{O(1/h)} \ll t^{3f(\alpha)/2+O(1/h)} \ll T^{(3f(\alpha)+O(1/h))/(f(\alpha)+2)}. \tag{7.18}$$

The condition (7.6) now gives

$$(1-\sigma)(2\alpha-1) = (\alpha+2\sigma-2)f(\alpha)+O(1/h), \tag{7.19}$$

or

$$2-2\sigma = \alpha f(\alpha)/\left(\alpha+f(\alpha)-\tfrac{1}{2}\right)+O(1/h), \tag{7.20}$$

which must be read as an implicit equation for α in terms of σ.

8. A 'Q^2T' density theorem. We seek to prove

$$\sum_{q\leqslant Q;\, q\equiv 0 \,(\mathrm{mod}\, q_0)} \ \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll D^{27(1-\sigma)/11+\varepsilon} \tag{8.1}$$

for $\sigma > 7/9$, where as usual $D = Q^2T/q_0$. In Jutila's device of breaking up a long sum (8.1) corresponds to a critical value for N of

$$N = D^{9/11}. \tag{8.2}$$

The work corresponds to that of Section 6 with D replacing both T and t. With $k = 6$ (5.3) becomes

$$R^2D^{-2/h} \ll N^{1-2\sigma+1/h}D^{2/h}\left(RN+R^2N^{1/2}+R^{11/6}N^{1/2}\left(B(R, D^6/N^6, D)\right)^{1/6}\right), \tag{8.3}$$

and since $D^6/N^6 = D^{12/11} > D$, (5.4) gives unconditionally

$$B(R, D^6/N^6, D) \ll D^3N^{-3}R^{1/2}D^{1/h}. \tag{8.4}$$

Thus provided $\sigma > 3/4+O(1/h)$ we have

$$R \ll N^{2-2\sigma+1/h}D^{4/h}+N^{12-24\sigma+12/h}D^{6+50/h}. \tag{8.5}$$

The first term on the right of (8.5) clearly satisfies (8.1), and the second term does so for $\sigma \geqslant 7/9$ when we choose h sufficiently large. For $\sigma < 7/9$ (8.1) follows from the Ingham–Montgomery zero-density theorem ([7], equation (12.9)).

The same calculation occurs in the proof of

$$\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll D^{27(1-\sigma)/11+\varepsilon}, \tag{8.6}$$

where now $D = qT$; this is almost a special case of (8.1) — put $q_0 = Q = f$ and sum over f which divide q — but (8.6) can be proved directly. The result stated by Jutila in [6] is of the form (8.6) rather than (8.1).

The exponent $27(1-\sigma)/11$ can be improved for $\sigma > 7/9$, but to determine the best exponent given by the present methods appears to be tedious and complicated: — witness the fact that the simple result (8.1) was overlooked by the author when preparing [5]. The exponent $2(1-\sigma)$ for $\sigma \geqslant 5/6$ has been obtained by different methods in Jutila [6] and in [5].

References

[1] M. Forti and C. Viola, *On large sieve type estimates for the Dirichlet series operator*, Proc. of Symposia in Pure Math., vol. XXIV.

[2] — — *Density estimates for the zeros of L-functions*, Acta Arith. 23 (1973), pp. 379–391.

[3] W. Haneke, *Verschärfung der Abschätzung von $\zeta(\frac{1}{2}+it)$*, Acta Arith. 8 (1963), pp. 357–430.

[4] M. N. Huxley, *On the difference between consecutive primes*, Inventiones Math. 15 (1972), pp. 164–170.

[5] — *Large values of Dirichlet polynomials*, Acta Arith. 24 (1973), pp. 329–346.

[6] M. Jutila, *On a density theorem of H. L. Montgomery for L-functions*, Ann. Acad. Sci. Fenn., Ser. A I, 520 (1972), 13 pp.

[7] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Berlin, Heidelberg and New York 1971.

[8] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford 1951.

Received on 20. 5. 1973 (426)

ACTA ARITHMETICA
XXVII (1975)

Transformations of a quadratic form which do not increase the class-number (II)

by

G. L. Watson (London)

In memory of Yu. V. Linnik

1. Introduction and notation. Let f be a quadratic form with integer coefficients, in any number n of variables. Then by $c(f)$, the class-number of f, is meant the number of classes in the genus of f. I showed in [1] that under certain transformations the class-number does not increase. The results of [1], which were used in [2] to show that $c(f) > 1$ for every positive-definite f with $n \geqslant 11$, will here be improved, so as to make possible some further applications explained in §§ 8, 11 below.

The transformations will be defined in a slightly different way, so that we shall have two alternative ways of dealing with the prime number 2. The effect of the transformations on the arithmetic properties of the form, and the cases in which they leave the class-number unaltered, will be investigated more fully than in [1]. The present paper is independent of [1].

Italic letters, with or without accents and subscripts, denote integers, p always prime, except f, g, h, used for quadratic forms (always with integer coefficients). Latin capitals, except F, G, also used for quadratic forms, denote square matrices, I being an identity matrix. Small Latin letter in bold type denote column vectors, with integer elements. An accent is used to denote transposition of a matrix or vector. Λ_n is the standard lattice in n-space, and its points are regarded as column vectors; its origin is $\mathbf{0} = \mathrm{col}\{0, \ldots, 0\}$. $M\Lambda_n$ is the sub-lattice $\{M\boldsymbol{x}\colon \boldsymbol{x} \in \Lambda_n\}$ and $m\Lambda_n$ $(m \neq 0)$ means $(mI)\Lambda_n$.

The matrix $A(f)$, $= A'(f)$, of the quadratic form f is defined so that we have the identities

$$(1.1)\qquad f(\boldsymbol{x}+\boldsymbol{y}) = f(\boldsymbol{x}) + \boldsymbol{x}'A(f)\boldsymbol{y} + f(\boldsymbol{y}), \quad f(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}'A(f)\boldsymbol{x}.$$

The discriminant $d = d(f)$ is defined by

$$(1.2)\qquad d(f) = \begin{cases} (-1)^{\frac{1}{2}n}\det A(f) & \text{if } 2 \mid n, \\ \tfrac{1}{2}(-1)^{\frac{1}{2}n-\frac{1}{2}}\det A(f) & \text{if } 2 \nmid n. \end{cases}$$