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Brun's method and the Fundamental Lemma, II*

by

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To the memory of Yu. V. Linnik

1. Introduction. Let $\mathfrak{A}$ be a finite sequence of (not necessarily distinct nor necessarily positive) integers, and let $\mathfrak{P}$ be a set of primes. Let $\overline{\mathfrak{P}}$ denote the complement of $\mathfrak{P}$ with respect to the set $\mathfrak{P}_1$ of all primes, and let $(d, \overline{\mathfrak{P}}) = 1$ signify that d has no prime factors in $\overline{\mathfrak{P}}$. For any real numbers w and z satisfying $2 \leqslant w \leqslant z$ define

$$P(z) := \prod_{\substack{p<z \\ p\in\mathfrak{P}}} p, \qquad P_{w,z} = P(z)/P(w)$$

and

$$S(\mathfrak{A}; \mathfrak{P}, z) := \left|\{a:\ a \in \mathfrak{A}, (a, P(z)) = 1\}\right|,$$

where $|\{\ldots\}|$ denotes the cardinality of the set $\{\ldots\}$.

Let $\omega(d)$ be a non-negative multiplicative arithmetic function on the sequence of square-free integers d which satisfies the following conditions:

$$\omega(p) = 0 \quad \text{if} \quad p \in \overline{\mathfrak{P}};$$

there exists a constant $A_1 \geqslant 1$ such that

$$(\Omega_1) \qquad \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A_1};$$

there exist constants $\varkappa > 0$ and $A_2 \geqslant 1$ such that

$$(\Omega_2(\varkappa)) \qquad \sum_{w \leqslant p < z} \frac{\omega(p)}{p} \log p \leqslant \varkappa \log\frac{z}{w} + A_2, \qquad 2 \leqslant w \leqslant z.$$

* This paper is a sequel to [1]. A brief announcement of its results was contained in [2].

We shall write

$$V(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right).$$

We postulate the existence of a real number $X > 1$ and an arithmetic function ω of the above class such that the 'remainders'

$$R_d: = \sum_{\substack{a\in\mathfrak{A}\\ a\equiv 0 \bmod d}} 1 - \frac{\omega(d)}{d}X$$

are small on average, in a sense to be made precise in the next section. In an earlier paper bearing the same title (see [1]) we established a general form of Brun's sieve, and derived from it a rather sharp Fundamental Lemma (that is, an asymptotic formula for $S(\mathfrak{A};\mathfrak{P},z)/(XV(z))$ valid in an extensive region of the $X-z$ plane) under the hypothesis (Ω_1), $(\Omega_2(\varkappa))$ and

(R) $\qquad |R_d| \leqslant K\omega(d) \quad$ if $\quad \mu(d) \neq 0,\ (d, \overline{\mathfrak{P}}) = 1,$

for some real number $K \geqslant 1$. In this note we shall replace (R) by a much cruder upper bound condition together with an 'average' condition of Bombieri type (cf. condition $(\mathrm{R}(\varkappa, \alpha))$ in [3] or [4]); and we shall indicate how the method of [1] leads, with very little modification, to an even more general form of Brun's sieve. We shall derive from this form a new Fundamental Lemma, and we shall apply this, by way of illustration, to prove the following companion result, for polynomial sequences with *prime* arguments, of Theorem 5 of [1]:

THEOREM 1. *Let $f_1(n), \ldots, f_g(n)$ be distinct irreducible polynomials with integer coefficients, and suppose that*

(1.1) $\qquad f_i(n) \neq \pm n \quad (i = 1, \ldots, g).$

Write $F(n) = f_1(n)\ldots f_g(n)$, let k denote the degree of F, and let $\varrho(p) = \varrho_F(p)$ be the number of solutions of the congruence

$$F(n) \equiv 0 \bmod p, \qquad 0 \leqslant n < p.$$

Assume that

(1.2) $\qquad \varrho(p) < p \quad$ *for all primes* p

and that

(1.3) $\qquad \varrho(p) < p-1 \quad$ *if* $\quad p \nmid F(0),$

and define

(1.4) $$\varrho'(p) = \begin{cases} \varrho(p)-1, & p \mid F(0), \\ \varrho(p), & p \nmid F(0). \end{cases}$$

Let v and x be real numbers such that $v \geqslant 3$ and $x^{1/v} \geqslant 2$; and let $q = q(x, v)$ (with or without suffices) *denote a number* (usually referred to as a *quasi-prime*) *having no prime factor less than $x^{1/v}$. Then we have*

(1.5) $$|\{p:\ p \leqslant x, f_i(p) = q_i \text{ for } i = 1, \ldots, g\}|$$
$$= (\mathrm{li}\, x) \prod_{p<x^{1/v}}\left(1-\frac{\varrho'(p)}{p-1}\right)\left\{1+O_F\left(e^{-\frac{1}{2}v(\log v-\log\log 2v-\log 3g-2)}\right)+O_F\left(\frac{1}{\log x}\right)\right\};$$

moreover, the expression on the right is equal to

$$(ve^{-\gamma})^g \prod_p \left(1-\frac{\varrho'(p)+1}{p}\right)\left(1-\frac{1}{p}\right)^{-g-1}\left(\frac{x}{\log^{g+1}x}\right) \times$$
$$\times\left\{1+O_F\left(e^{-\frac{1}{2}v(\log v-\log\log 2v-\log 3g-2)}\right)+O_F\left(\frac{v}{\log x}\right)\right\}.$$

2. Brun's sieve. It will serve us best to begin with a statement of the form of Brun's sieve that is implicit in [1], in which the remainders R_d are still explicit and which is therefore free of the condition (R).

THEOREM 2 (Ω_1), $(\Omega_2(\varkappa))$: *Let b be a positive integer, let λ be a real number satisfying*

(2.1) $\qquad 0 < \lambda e^{1+\lambda} < 1,$

and let

(2.2) $$B = \tfrac{1}{2}A_2\left\{1+A_1\left(\varkappa+\frac{A_2}{\log 2}\right)\right\}.$$

Define

(2.3) $$\Lambda = \frac{2\lambda}{\varkappa}\cdot\frac{1}{1+\varepsilon}, \qquad \varepsilon = \frac{1}{200e^{1/\varkappa}},$$

and let the sequence

$$2 = z_r < z_{r-1} < \ldots < z_1 < z$$

be given by

(2.4) $\qquad \log z_n = e^{-n\Lambda}\log z \quad (n = 1, \ldots, r-1).$

For $\nu = 1$ or 2, for each $n = 1, \ldots, r$ and for each positive divisor d of $P(z)$ put

(2.5) $$\chi_\nu(d) = \begin{cases} 1 & \text{if } \nu\big((d, P_{z_n,z})\big) \leqslant 2b-\nu+2n-1 \text{ for } n = 1, \ldots, r,^{(1)} \\ 0 & \text{if } d \mid P(z) \text{ otherwise.} \end{cases}$$

Then

(2.6) $$S(\mathfrak{A};\mathfrak{P},z) \leqslant XV(z)\left\{1+2\frac{\lambda^{2b+1}e^{2\lambda}}{1-\lambda^2e^{2+2\lambda}}\exp\left((2b+3)\frac{B}{\lambda\log z}\right)\right\} +$$
$$+ \sum_{d\mid P(z)} \chi_1(d)\,|R_d|$$

(1) Throughout, $\nu(n)$ denotes the number of prime factors of n.

and

$$(2.7)\qquad S(\mathfrak{A};\mathfrak{P},z) \geqslant XV(z)\left\{1-2\frac{\lambda^{2b}e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}}\exp\left((2b+2)\frac{B}{\lambda\log z}\right)\right\} - \sum_{d|P(z)}\chi_2(d)|R_d|;$$

moreover, for any constant $A \geqslant 1$, *we have*

$$(2.8)\qquad \sum_{d|P(z)}\chi_\nu(d)A^{\nu(d)} = O\left(z^{2b+1-\nu+\frac{2.01}{e^{2\lambda/\varkappa}-1}}\right),$$

where the implied O-constant, while it may depend on $A_1, A_2, \varkappa$ *and* A, *does not depend on* b *and* λ.

We now introduce in place of (R) a pair of new conditions on the remainders R_d. We shall suppose first that there exist a real number $K \geqslant 1$ and a constant $A_0 \geqslant 1$ such that

$$(\mathrm{R}_0)\qquad |R_d| \leqslant K\left(\frac{X\log X}{d}+1\right)A_0^{\nu(d)} \quad \text{for} \quad \mu(d)\neq 0,\ (d,\overline{\mathfrak{P}})=1;$$

and we shall suppose also that for some constant α $(0<\alpha\leqslant 1)$ there exist corresponding to any given constant $C \geqslant 1$ constants $C_0 \geqslant 1$ and $C_1 \geqslant 1$ such that

$$(\mathrm{R}_1(\varkappa,\alpha))\qquad \sum_{\substack{d<X^\alpha \log^{-C_0}X\\ (d,\overline{\mathfrak{P}})=1}}\mu^2(d)|R_d| \leqslant C_1\frac{X}{\log^{\varkappa+C}X}.$$

It is clear that (R_0) is, in general, much weaker than (R) (take, for example, the common case when $\omega(p)\leqslant A_0$ for all p), and that (R) implies a condition of type $(\mathrm{R}_1(\varkappa,1))$. We shall see in Section 4 that both conditions are satisfied in the case of Theorem 1.

We shall now apply the new conditions (R_0) and $(\mathrm{R}_1(\varkappa,\alpha))$ in conjunction with (2.8) to the remainder terms in (2.6) and (2.7): we have, for $\nu=1$ and 2 that

$$\sum_{d|P(z)}\chi_\nu(d)|R_d| \leqslant \sum_{\substack{d<X^\alpha\log^{-C_0}X\\ (d,\overline{\mathfrak{P}})=1}}|R_d| + K\sum_{\substack{d|P(z)\\ d\geqslant X^\alpha\log^{-C_0}X}}\left(\frac{X\log X}{d}+1\right)A_0^{\nu(d)}\chi_\nu(d)$$
$$\leqslant C_1\frac{X}{\log^{\varkappa+C}X} + 2KX^{1-\alpha}\log^{C_0+1}X\sum_{d|P(z)}A_0^{\nu(d)}\chi_\nu(d)$$
$$= O\left(\frac{X}{\log^{\varkappa+C}X} + KX^{1-\alpha}z^{2b+1-\nu+\frac{2.01}{e^{2\lambda/\varkappa}-1}}\log^{C_0+1}X\right).$$

If now we adopt the convenient notation

$$u = \frac{\log X}{\log z},$$

and if also we recall from [1] (inequality (2.5)) that

$$(2.9)\qquad 1/V(z) = O(\log^\varkappa z),$$

we arrive at the estimates

$$(2.10)\qquad \sum_{d|P(z)}\chi_\nu(d)|R_d| \ll XV(z)\left\{\frac{u^{-\varkappa}}{\log^C X} + Kz^{-\alpha u+2b+1-\nu+\frac{2.01}{e^{2\lambda/\varkappa}-1}}u^{C_0+1}\log^{C_0+\varkappa+1}z\right\};$$

$$(\nu = 1, 2).$$

From Theorem 2 and (2.10) we now obtain

Theorem 3 (Ω_1), $(\Omega_2(\varkappa))$, (R_0), $(\mathrm{R}_1(\varkappa,\alpha))$: *Let* b *be a positive integer, let* λ *be a real number satisfying* (2.1), *let* B *be as defined in* (2.2) *and write*

$$(2.11)\qquad u = \log X/\log z.$$

Then

$$(2.12)\qquad \frac{S(\mathfrak{A};\mathfrak{P},z)}{XV(z)} \leqslant 1+2\frac{\lambda^{2b+1}e^{2\lambda}}{1-\lambda^2e^{2+2\lambda}}\exp\left((2b+3)\frac{B}{\lambda\log z}\right) + O\left(Kz^{-\alpha u+2b+\frac{2.01}{e^{2\lambda/\varkappa}-1}}u^{C_0+1}\log^{C_0+\varkappa+1}z\right)+O(u^{-\varkappa}\log^{-C}X)$$

and

$$(2.13)\qquad \frac{S(\mathfrak{A};\mathfrak{P},z)}{XV(z)} \geqslant 1-2\frac{\lambda^{2b}e^{2\lambda}}{1-\lambda^2e^{2+2\lambda}}\exp\left((2b+2)\frac{B}{\lambda\log z}\right) + O\left(Kz^{-\alpha u+2b-1+\frac{2.01}{e^{2\lambda/\varkappa}-1}}u^{C_0+1}\log^{C_0+\varkappa+1}z\right)+O(u^{-\varkappa}\log^{-C}X),$$

where the O-constants, while they may depend on $A_0, A_1, A_2, \varkappa, \alpha$ *and* C, *do not depend on* λ *or* b.

To illustrate the effectiveness of Theorem 3, let us apply it to the 'prime twins' problem. We take $\mathfrak{A} = \{p+2\colon p\leqslant x\}$ and $\mathfrak{P} = \{p\colon p>2\}$, so that $\overline{\mathfrak{P}} = \{2\}$. Then if d is square-free and odd,

$$\sum_{\substack{a\in\mathfrak{A}\\ a\equiv 0 \bmod d}}1 = \sum_{\substack{p\leqslant x\\ p\equiv -2 \bmod d}}1 = \pi(x;d,-2) = \frac{\operatorname{li}x}{\varphi(d)} + R_d;$$

accordingly we take $X = \operatorname{li}x$, $\omega(p)=0$ if $p=2$ and $\omega(p)=\dfrac{p}{p-1}$ if p is odd, and we find that (Ω_1) is then satisfied with $A_1 = 2$, $(\Omega_2(\varkappa))$ with

$\varkappa = 1 = A_2$, (R_0) with $K = 2$ and $A_0 = 1$, and $(\mathrm{R}_1(1, a))$ holds with $a = \frac{1}{2}$ (and $C_0 = C+13$) by virtue of Bombieri's theorem (for the version used here, see Montgomery [5]). With this choice of parameters we take $b = 1$ in (2.13) and obtain (note that now $B < 2$ by (2.2))

$$(2.14)\qquad \frac{S(\mathfrak{A}; \mathfrak{P}, z)}{XV(z)} \geqslant 1 - 2\frac{\lambda^2 e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}} \exp\left(\frac{8}{\lambda \log z}\right) + \\ + O\left(z^{-\frac{1}{2}u+1+\frac{2.01}{e^{2\lambda}-1}} u^{C+14} \log^{15+C} z\right) + O(u^{-1}\log^{-C} X).$$

To ensure that the right hand side is positive for all large enough values of x we must choose positive *numerical* values for λ and u so that

$$(2.15)\qquad 1 - \frac{2\lambda^2 e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}} > 0$$

(of course, (2.1) must hold too!) and

$$(2.16)\qquad u > 2 + \frac{4.02}{e^{2\lambda}-1};$$

the quality of our result will depend on choosing u as small as possible, and therefore λ as large as possible. The choice $e^{\lambda} = 1.288$ just satisfies (2.15) and (easily) (2.1); and when substituted in (2.16) permits a choice of $u < 9$. It follows at once that $p+2$ is infinitely often a number having at most *eight* prime factors (counted according to multiplicity).

Actually, it is clear from the proof of Theorem 3 of [1] that our estimates for $S(\mathfrak{A}; \mathfrak{P}, z)$ can be improved by separating the term corresponding to $n = 1$ before embarking upon the various approximations that lead to the relatively simple forms (2.6) and (2.7) above (see the calculations following (3.17) in [1]) — this is the usual procedure in most classical formulations of Brun's sieve. In particular, the first two terms on the right of (2.14) can be replaced by, essentially

$$1 - 2\lambda^2 e^{2\lambda}\left(1 + \frac{16}{3}\frac{\lambda^2 e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}}\right)$$

(we may ignore the exponential term in this application, as we did earlier, because λ is a numerical constant and $z = X^{1/u}$ is therefore large) and this expression exceeds 0.04 with the choice $e^{\lambda} = 1.293$. Now this choice satisfies (2.1) and allows us to select a value of u satisfying (2.16) which is less than 8. Hence we may conclude that $p+2$ *is infinitely often a number having at most seven prime factors* (counted according to multiplicity).

The refinement indicated in the second footnote in Section 3 of [1], possibly combined with a weighting procedure (see [4]), should lead to further improvement.

3. A Fundamental Lemma. We may now deduce from Theorem 3

THEOREM 4 (Ω_1), $(\Omega_2(\varkappa))$, (R_0), $(\mathrm{R}_1(\varkappa, a))$: *Let $X \geqslant z$ and write*

$$u = \frac{\log X}{\log z}.$$

Then

$$S(\mathfrak{A}; \mathfrak{P}, z) \\ = XV(z)\{1 + O(e^{-au(\log u - \log\log 3u - \log(\varkappa/a) - 2)}) + O(K\log^{-C} X)\},$$

where the O-constants depend at most on $A_0, A_1, A_2, \varkappa, a$ and C.

Proof. We follow the argument of the proof of Theorem 4 of [1]. The result is of interest only if $u \to +\infty$, and we concentrate therefore on the case of u large (although we can deal also with small u as in [1]).

For $u \geqslant \log z$, that is to say, for $\log z \leqslant \sqrt{\log X}$, we can easily check that the analogues of Theorems 1 and 2 of [1] are respectively

(Ω_1), $(\Omega_2(\varkappa))$, (R_0), $(\mathrm{R}_1(\varkappa, a))$:

$$S(\mathfrak{A}; \mathfrak{P}, z) = XV(z)\{1 + O(\log^{-C} X) + O(KX^{-a}\log^{C_0+\varkappa} X \cdot (1+A_0)^{\pi(z)})\}$$

and

(Ω_1), $(\Omega_2(\varkappa))$, (R_0), $(\mathrm{R}_1(\varkappa, a))$:

$$S(\mathfrak{A}; \mathfrak{P}, z) = XV(z)\{1 + O(\log^{-C} X)\} + O(KX^{1-\frac{a}{2}}\log^{C_0} X);$$

and that both these are better, in their limited ranges of effectiveness, than the stated result.

This allows us to suppose that

$$u < \log z,$$

and here an application of Theorem 3 with

$$b = \left[\frac{a}{2}u - \frac{a}{2}\frac{u}{\log u}\right], \qquad \lambda = \frac{e\varkappa}{a}\frac{\log u}{u}$$

leads readily to the result.

4. Proof of Theorem 1. We take $\mathfrak{A}$ to be the sequence $\{F(p)\colon p \leqslant x\}$ and $\mathfrak{P}$ to be the set $\mathfrak{P}_1$ of all primes. Then, if $\mu(d) \neq 0$,

$$\sum_{\substack{a \in \mathfrak{A} \\ a \equiv 0 \bmod d}} 1 = |\{p\colon\ p \leqslant x,\ F(p) \equiv 0 \bmod d\}| = \sum_{l=1}^{d} \sum_{\substack{p \leqslant x \\ p \equiv l \bmod d \\ F(p) \equiv 0 \bmod d}} 1$$

$$= \sum_{\substack{l=1 \\ F(l) \equiv 0 \bmod d}}^{d} \pi(x; d, l) = \sum_{\substack{l=1 \\ (l,d)=1 \\ F(l) \equiv 0 \bmod d}}^{d} \pi(x; d, l) + \theta\varrho(d),$$

$$0 \leqslant \theta \leqslant 1;$$

writing

$$E(x;d,l) = \pi(x;d,l) - \frac{\operatorname{li}x}{\varphi(d)},$$

we obtain

$$\sum_{\substack{a\in\mathfrak{A}\\ a\equiv 0 \bmod d}} 1 = \frac{\operatorname{li}x}{\varphi(d)}\varrho'(d) + \sum_{\substack{l=1\\ (l,d)=1\\ F(l)\equiv 0 \bmod d}}^{d} E(x;d,l) + \theta\varrho(d)$$

$$= \frac{\operatorname{li}x}{d}\cdot\frac{d\varrho'(d)}{\varphi(d)} + \theta\varrho(d)\{E(x,d)+1\}, \qquad |\theta|\leqslant 1,$$

where $\varrho'(d)$ is the number of solutions of

$$F(n)\equiv 0 \bmod d, \quad 0\leqslant n<d, \quad (n,d)=1,$$

so that $\varrho'(d)\leqslant\varrho(d)$, $\varrho'(p)$ satisfies (1.4), and where

$$E(x;d) = \max_{\substack{1\leqslant l\leqslant d\\ (l,d)=1}} |E(x;d,l)|.$$

It is not hard to prove that ϱ' is a multiplication function, and therefore an appropriate choice of X and ω here is

$$X = \operatorname{li}x, \quad \omega(d) = \frac{d\varrho'(d)}{\varphi(d)}; \tag{4.1}$$

we may clearly assume that $X>1$. It follows that

$$|R_d|\leqslant\{E(x,d)+1\}\varrho(d). \tag{4.2}$$

In order to apply Theorem 4, we must check that the basic conditions are satisfied. From a well known elementary result we know that

$$\varrho(p)\leqslant k$$

whenever $\varrho(p)<p$, whence, by (1.4),

$$\frac{\omega(p)}{p} = \frac{\varrho'(p)}{p-1}\leqslant 1-\frac{1}{k+1} \quad \text{for all } p;$$

hence (Ω_1) holds with $A_1 = k+1$. Next, $(\Omega_2(\varkappa))$ is satisfied with $\varkappa = g$ and $A_2 = O_F(1)$ by virtue of a classical result of Nagell [6] (see the proofs of Theorems 4 and 6 in [4]). We come to verify (R_0) and $(\mathrm{R}_1(\varkappa,\alpha))$, and here we base ourselves on (4.2). Since $\varrho(d)\leqslant k^{\nu(d)}$ for square-free d, the second of these two conditions follows from (4.2) by Bombieri's theorem (as is demonstrated in full detail in the proof of Theorem 6 of [4]), with $\varkappa = g$, $\alpha=\frac{1}{2}$ and taking (as we may do) $C=1$. As for the first condition, we have

$$|R_d|\leqslant\{E(x,d)+1\}k^{\nu(d)}\leqslant\left\{\frac{x}{d}+2\right\}k^{\nu(d)}\leqslant 2\left\{\frac{X\log X}{d}+1\right\}k^{\nu(d)},$$

so that (R_0) holds with $K=2$ and $A_0=k$.

We may therefore apply Theorem 4. Here we take $z=x^{1/v}$, so that by (4.1) and because $x\geqslant 2^v\geqslant 8$ by hypothesis,

$$u = \frac{\log X}{\log z} = v\frac{\log(\operatorname{li}x)}{\log x}\geqslant v\frac{\log(x/\log x)}{\log x} = v\left(1-\frac{\log\log x}{\log x}\right)\geqslant\frac{2}{3}v;$$

hence, by (4.1) again, (1.5) follows at once from Theorem 4.

As for the last statement in the theorem, we have

$$1-\frac{\varrho'(p)}{p-1} = \left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{\varrho'(p)+1}{p}\right),$$

and therefore the product on the right of (1.5) is equal to (see [3], Lemma 2, (2.12), noting that condition (Ω_2) on p. 244 with $\omega(p)=\varrho'(p)+1$ is satisfied with $\varkappa = g+1$ and $A_2 = O_F(1)$, $L=O_F(1)$)

$$\prod_p\left(1-\frac{\varrho'(p)+1}{p}\right)\left(1-\frac{1}{p}\right)^{-g-1}\frac{e^{-\gamma g}u^g}{\log^g x}\left\{1+O_F\left(\frac{u}{\log x}\right)\right\};$$

this completes the proof of Theorem 1.

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