Brun's method and the Fundamental Lemma, II*

by

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To the memory of Yu. V. Linnik

1. Introduction. Let $\mathfrak{A}$ be a finite sequence of (not necessarily distinct nor necessarily positive) integers, and let $\mathfrak{P}$ be a set of primes. Let $\overline{\mathfrak{P}}$ denote the complement of $\mathfrak{P}$ with respect to the set $\mathfrak{P}_1$ of all primes, and let $(d, \mathfrak{P}) = 1$ signify that $d$ has no prime factors in $\mathfrak{P}$. For any real numbers $w$ and $z$ satisfying $2 < w < z$ define

$$P(z) = \prod_{p \leq w} p, \quad P_{w,z} = P(z)/P(w)$$

and

$$S(\mathfrak{A}; \mathfrak{P}, z) = |\{a: a \in \mathfrak{A}, (a, P(z)) = 1\}|,$$

where $|\{\ldots\}|$ denotes the cardinality of the set $\ldots$.

Let $\omega(d)$ be a non-negative multiplicative arithmetic function on the sequence of square-free integers $d$ which satisfies the following conditions:

$$\omega(p) = 0 \quad \text{if} \quad p \in \overline{\mathfrak{P}};$$

there exists a constant $A_1 \geq 1$ such that

$$(\Omega_1) \quad \quad \quad \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1};$$

there exist constants $s > 0$ and $A_2 \geq 1$ such that

$$(\Omega_2(s)) \quad \quad \quad \sum_{w < p < z} \frac{\omega(p)}{p} \log p \leq s \log \frac{z}{w} + A_2, \quad 2 < w < z.$$

* This paper is a sequel to [1]. A brief announcement of its results was contained in [3].
We shall write
\[ V(x) = \prod_{p \leq x} \left( 1 - \frac{\omega(p)}{p} \right) . \]

We postulate the existence of a real number \( X > 1 \) and an arithmetic function \( \omega \) of the above class such that the 'remainders' \( R_d := \sum_{d \leq X / \log X} \mu(d) \omega(d) \) are small on average, in a sense to be made precise in the next section. In an earlier paper bearing the same title (see [1]) we established a general form of Brun's sieve, and derived from it a rather sharp Fundamental Lemma (that is, an asymptotic formula for \( S(N; \mathcal{P}, x)/(XV(x)) \) valid in an extensive region of the \( x - \tau \) plane) under the hypothesis \((\Omega_4), (\Omega_4(x))\) and (R) \[ R_d \ll X \omega(d) \quad \text{if} \quad \mu(d) \neq 0, (d, \mathfrak{P}) = 1, \]
for some real number \( K \gg 1 \). In this note we shall replace (R) by a much cruder upper bound condition together with an 'average' condition of Bombieri type (cf. condition \((R(x, \alpha))\) in [3] or [4]); and we shall indicate how the method of [1] leads, with very little modification, to an even more general form of Brun's sieve. We shall derive from this form a new Fundamental Lemma, and we shall apply this, by way of illustration, to prove the following companion result, for polynomial sequences with prime arguments, of Theorem 5 of [1]:

**Theorem 1.** Let \( f_1(n), \ldots, f_g(n) \) be distinct irreducible polynomials with integer coefficients, and suppose that
\[ f_i(n) \neq \pm n \quad (i = 1, \ldots, g). \]
Write \( F(n) = f_1(n) \cdots f_g(n) \), let \( k \) denote the degree of \( F \), and let \( \varphi(p) = \varphi_F(p) \) be the number of solutions of the congruence
\[ F(n) = 0 \mod p, \quad 0 \leq n \lt p. \]
Assume that
\[ \varphi(p) < p \quad \text{for all primes} \quad p \]
and that
\[ \varphi(p) < p - 1 \quad \text{if} \quad p \nmid F(0), \]
and define
\[ \varphi'(p) = \begin{cases} \varphi(p) - 1, & p \nmid F(0), \\ \varphi(p), & p \mid F(0). \end{cases} \]
Let \( \nu \) and \( x \) be real numbers such that \( x \geq 3 \) and \( x^{1/\nu} \geq 2 \); and let \( g = g(x, \nu) \) (with or without suffixes) denote a number (usually referred to as a quasi-prime) having no prime factor less than \( x^{1/\nu} \). Then we have
\[ \{(p: \ p \leq x, f_i(p) = q_i \text{ for } i = 1, \ldots, g)\} \]
\[ = (\log x)^g \prod_{p \leq x^{1/\nu}} \left( 1 - \frac{\varphi'(p)}{p - 1} \right) \left( 1 + O_P \left( \frac{\varphi'(p) \log \log x - \log x}{\log x} \right) + O_P \left( \frac{1}{\log x} \right) \right) \]
moved, the expression on the right is equal to
\[ \prod_{p \leq x^{1/\nu}} \left( 1 - \frac{\varphi'(p) + 1}{p} \right) \left( 1 - \frac{\varphi(p)}{p} \right)^{\varphi'(p)}/(p - 1) \times \]
\[ \prod_{p \leq x^{1/\nu}} \left( 1 + O_P \left( \frac{1}{\log x} \right) \right) \]
\[ \times \left( 1 + O_P \left( \frac{1}{\log x} \right) \right) \]

2. **Brun's sieve.** It will serve us best to begin with a statement of the form of Brun's sieve that is implicit in [1], in which the remains \( R_d \) are still explicit and which is therefore free of the condition (R).

**Theorem 2** \((\Omega_4), (\Omega_4(x))\): Let \( b \) be a positive integer, and \( \lambda \) be a real number satisfying
\[ 0 < \lambda^{d+1} < 1, \]
and let
\[ B = \frac{1}{2} A_3 \left( 1 + A_1 \left( \frac{\lambda + A_2}{\log 2} \right) \right). \]
Define
\[ A = \frac{2 \lambda}{\lambda + 1}, \quad \epsilon = \frac{1}{200 e^B}, \]
and let the sequence
\[ \lambda = \lambda_1 < \lambda_2 < \ldots < \lambda_s < s \]
be given by
\[ \frac{\log \lambda_n}{\log x} = e^{-\lambda_n} \log x \quad (n = 1, \ldots, r - 1). \]
For \( \nu \geq 1 \) or 2, for each \( n = 1, \ldots, r \) and for each positive divisor \( d \) of \( P(x) \) put
\[ \chi_n(d) = \begin{cases} 1 & \text{if } \nu(d, P_{a,n}) \leq 2b - \tau \sum_{n 
(1) Throughout, \( \nu(n) \) denotes the number of prime factors of \( n \).
and

\[
S_{\mathfrak{A}}(\mathfrak{P}, \mathfrak{B}, x) \geq XV(x) \left\{ 1 - \frac{\zeta^{2b} \delta^{3}}{1 - \frac{2}{B} \delta^{2} \log(\pi)} \exp \left( (2b + 2) \frac{B}{\lambda \log \pi} \right) \right\} - \sum_{d \mathfrak{P}(d)} \chi(d) |R_d|;
\]

moreover, for any constant \( A \geq 1 \), we have

\[
\sum_{d \mathfrak{P}(d)} \chi(d) A^{\epsilon_{0}(d)} = O \left( x^{2b+1-\frac{2}{B} \frac{\log(x)}{1^{2}} - \frac{2}{B} \frac{\log(x)}{1^{2}}} \right),
\]

where the implied \( O \)-constant, while it may depend on \( A_{1}, A_{2}, x \) and \( A \), does not depend on \( x \) and \( \lambda \).

We now introduce in place of (B) a pair of new conditions on the remainders \( R_d \). We shall suppose first that there exist a real number \( K \geq 1 \) and a constant \( A_{4} \geq 1 \) such that

\[
|R_{d}| \leq K \left( \frac{X \log X}{d} + 1 \right) A_{4}^{\epsilon_{0}(d)} \quad \text{for} \quad \mu(d) \neq 0, (d, \mathfrak{P}) = 1;
\]

and we shall suppose also that for some constant \( a \) (\( 0 < a \leq 1 \)) there exist constants \( C_{6} \geq 1 \) and \( C_{7} \geq 1 \) such that

\[
|R_{4}(s, a)| d \sum_{d < X \log^{-a} \mathfrak{p}^{-a}} \mu(d)^{2} |R_{d}| \leq C_{1} \frac{X}{\log^{a+\frac{1}{2}} X}.
\]

It is clear that (R_{6}) is, in general, much weaker than (B) (take, for example, the common case when \( \omega(p) \leq A_{4} \) for all \( p \)), and that (B) implies a condition of type \((R_{4}(s, 1))\). We shall see in Section 4 that both conditions are satisfied in the case of Theorem 1.

We shall now apply the new conditions \((R_{4})\) and \((R_{4}(s, a))\) in conjunction with (2.8) to the remainder terms in (2.6) and (2.7); we have, for \( \nu = 1 \) and \( \nu = 2 \) that

\[
\sum_{d \mathfrak{P}(d)} \chi(d) |R_d| \leq \sum_{d < x \log^{-a} \mathfrak{p}^{-a}} |R_d| + K \sum_{d \mathfrak{P}(d)} \left( \frac{X \log X}{d} + 1 \right) A_{4}^{\epsilon_{0}(d)} \chi(d)
\]

\[
\leq C_{1} \frac{X}{\log^{a+\frac{1}{2}} X} + 2KX^{1-\frac{1}{2} \log \log(\pi)} \sum_{d \mathfrak{P}(d)} A_{4}^{\epsilon_{0}(d)} \chi(d)
\]

\[
= O \left( \frac{X}{\log^{a+\frac{1}{2}} X} + KX^{1-\frac{1}{2} \log \log(\pi)} \right).
\]

If now we adopt the convenient notation

\[
u = \frac{\log X}{\log \pi},
\]

and if also we recall from \((1)\) (inequality (2.3)) that

\[
1/V(x) = O(\log^{a}(x)),
\]

we arrive at the estimates

\[
\sum_{d \mathfrak{P}(d)} \chi(d) |R_d|
\]

\[
\leq XV(x) \left\{ \frac{\nu^{-\epsilon}}{\log X} + \frac{\nu^{b} \log \log(\pi)}{\log X} \right\} \quad \text{for} \quad \nu = 1, 2.
\]

From Theorem 2 and (2.10) we now obtain

**Theorem 3 (\( \Omega_{1} \)), \( \Omega_{2}(x), (R_{4}), (R_{4}(s, a)) \): Let \( b \) be a positive integer, let \( \lambda \) be a real number satisfying (2.1), let \( B \) be as defined in (2.3) and write

\[
u = \log X/\log \pi.
\]

Then

\[
S_{\mathfrak{A}}(\mathfrak{P}, \mathfrak{B}, x) \leq 1 + 2 \sum_{d \mathfrak{P}(d)} \left( \frac{X \log X}{d} + 1 \right) A_{4}^{\epsilon_{0}(d)} \chi(d)
\]

\[
+ O(KX^{1-\frac{1}{2} \log \log(\pi)} \log \log(\pi)) + O(\nu^{b} \log \log(\pi))
\]

and

\[
S_{\mathfrak{A}}(\mathfrak{P}, \mathfrak{B}, x) \leq 1 + 2 \sum_{d \mathfrak{P}(d)} \left( \frac{X \log X}{d} + 1 \right) A_{4}^{\epsilon_{0}(d)} \chi(d)
\]

\[
+ O(KX^{1-\frac{1}{2} \log \log(\pi)} \log \log(\pi)) + O(\nu^{b} \log \log(\pi))
\]

where the \( O \)-constants, while they may depend on \( A_{2}, A_{1}, A_{4}, x \) and \( a \), do not depend on \( \lambda \) or \( b \).

To illustrate the effectiveness of Theorem 3, let us apply it to the 'prime twins' problem. We take \( \mathfrak{A} = (p+1: p < x) \) and \( \mathfrak{B} = (p: p > 2) \), so that \( \mathfrak{P} = (2) \). Then if \( d \) is square-free and odd,

\[
\sum_{d \text{prime}} 1 = \sum_{d = 1 \text{mod} 2} 1 = \pi(x; d, -2) = \frac{\pi(x)}{x} + R_{d};
\]

accordingly we take \( X = \pi(x), \omega(p) = 0 \) if \( p = 2 \) and \( \omega(p) = \frac{p}{p-1} \) if \( p \) is odd, and we find that \((\Omega_{3})\) is then satisfied with \( A_{1} = 2, (\Omega_{2}(x)) \) with
3. A Fundamental Lemma. We may now deduce from Theorem 3
THEOREM 4 \((\Omega_1), (\Omega_4(x)), (B_0), (R_1(x, a))\): \(L \gg x\) and write
\[ u = \frac{\log X}{\log z}. \]
Then
\[ S(\Psi; \mathbb{P}, z) = XV(z) \left( 1 + O(e^{-\log u - \log \log u - \log(\log u)^2}) + O(K \log X) \right), \]
where the \(O\)-constants depend at most on \(A_0, A_1, A_2, z, a, c, C\).

Proof. We follow the argument of the proof of Theorem 4 of [1].

The result is of interest only if \(u \to +\infty\), and we concentrate therefore on the case of \(u\) large (although we can deal also with small \(u\) as in [1]).

For \(u \gg \log z\), that is to say, \(\log z < \log X\), we can easily check that the analogues of Theorems 1 and 2 of [1] are respectively
\[ S(\Psi; \mathbb{P}, z) = XV(z) \left( 1 + O(\log X) + O(KX^{-1} \log X^2 + X (1 + A_0)^{n(2)}) \right) \]
and
\[ S(\Omega_1; \mathbb{P}, z) = XV(z) \left( 1 + O(\log X) + O(KX^{-1} \log X^2) \right); \]
and that both these are better, in their limited ranges of effectiveness, than the stated result.

This allows us to suppose that
\[ u < \log z, \]
and here an application of Theorem 3 with
\[ b = \left[ \frac{a}{2} - \frac{2}{3} \log u \right], \quad \lambda = \frac{e^\delta}{a} \log u \]
leads readily to the result.

4. Proof of Theorem 1. We take \(\Psi\) to be the sequence \(\{F(p): p \leq x\}\) and \(\mathbb{P}\) to be the set \(\mathbb{P}_1\) of all primes. Then, if \(\mu(d) \neq 0\),
\[ \sum_{a=\mod d} 1 = \left( \{p: p \leq x, F(p) \equiv 0 \mod d\} \right) = \sum_{d=1}^{\theta} \sum_{F(p) \equiv 0 \mod d} \sum_{a=\mod d} 1 \]
\[ = \sum_{p=\mod d} \pi(x; a) \cdot 1 = \sum_{p=\mod d} \pi(x; a) + O(e(d)), \]
\[ 0 \leq \theta < 1; \]
writing
\[ E(x; d, l) = \pi(x; d, l) - \frac{\log x}{\varphi(d)}, \]
we obtain
\[
\sum_{a=0}^{d-1} 1 = \frac{\log x}{\varphi(d)} \frac{\varphi'(d)}{\varphi(d)} d = \sum_{a=0}^{\varphi(d)} E(x; d, l) + B(x; d, l),
\]
where \( \varphi'(d) \) is the number of solutions of
\[ \mathcal{F}(n) = 0 \mod d, \quad 0 \leq n < d, \quad (n, d) = 1, \]
so that \( \varphi'(d) \leq \varphi(d) \), and where
\[ E(x; d, l) = \max_{\substack{1 \leq d \leq x \mod l \leq d \leq x}} |E(x; d, l)|. \]
It is not hard to prove that \( \varphi' \) is a multiplication function, and therefore an appropriate choice of \( X \) and \( \omega \) here is
\[ X = \frac{\log x}{\varphi(d)}, \quad \omega(d) = \frac{\varphi'(d)}{\varphi(d)}; \]
we may clearly assume that \( X > 1 \). It follows that
\[ E(x; d, l) \leq \frac{\log x}{\varphi(d)} \varphi'(d). \]

In order to apply Theorem 4, we must check that the basic conditions are satisfied. From a well known elementary result we know that
\[ \frac{\varphi'(p)}{p} \leq \frac{1}{k+1} \]
whenever \( \varphi(p) < p \), whence, by (1.4),
\[ \frac{\varrho(p)}{p} = \frac{\varphi'(p)}{p} = \frac{\varphi(p)}{p} - 1 \leq 1 - \frac{1}{k+1} \quad \text{for all } p; \]

hence \( \Omega_1 \) holds with \( A_1 = k+1 \). Next, \( \Omega_2(x) \) is satisfied with \( k = g \) and \( A_2 = O_p(1) \) by virtue of a classical result of Negul [6] (see the proofs of Theorems 4 and 6 in [4]). We come to verify \( R_2 \) and \( R_2(x, a) \), and here we base ourselves on (4.2). Since \( \varphi(d) \leq k^{\varphi(2)} \) for square-free \( d \), the second

of these two conditions follows from (4.2) by Bombieri's theorem (as is demonstrated in full detail in the proof of Theorem 6 of [4]), with \( x = g, \ a = \frac{1}{k} \) and taking (as we may do) \( C = 1 \). As for the first condition, we have
\[ |R_2| \leq |E(x, d, l) + 1| \leq \frac{x}{d} + 2, \quad k^{\varphi(2)} \leq 2 \left\{ \frac{1}{\log x} + 1 \right\} k^{\varphi(2)}, \]
so that \( \Omega_2 \) holds with \( X = 2 \) and \( A_2 = k \).

We may therefore apply Theorem 4. Here we take \( x = \frac{x}{\log x} \), so that by (4.1) and because \( x \geq \frac{21}{2} \) by hypothesis,
\[ u = \frac{\log x}{\log x} = \frac{\log \left( \frac{x}{\log x} \right)}{\log x} \geq \frac{\log x}{\log x} = \frac{1 - \log x}{\log x} \geq \frac{1}{2}; \]

hence, by (4.1) again, (1.5) follows at once from Theorem 4.

As for the last statement in the theorem, we have
\[ 1 - \frac{\varphi'(p)}{p} = \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 - \frac{\varphi(p)}{p} \right), \]
and therefore the product on the right of (1.5) is equal to (see [3], Lemma 2, (2.12), noting that condition \( \Omega_2 \) on p. 24 and \( \varphi(p) = \varphi(p) + 1 \) is satisfied with \( k = g+1 \) and \( A_2 = O_2(1) \), \( L = O_2(1) \))
\[ \prod_{p} \left( 1 - \frac{\varphi'(p)}{p} \right)^{1 - \frac{1}{p}} \leq \exp \left( - \frac{1}{p} \log \varphi x \right) \left( 1 + O_2 \left( \frac{u}{\log x} \right) \right); \]

this completes the proof of Theorem 1.

References


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