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Large values of Dirichlet polynomials, III

by

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1. Introduction. This paper continues [3] and [4]. Our first object is to estimate the number of times that a Dirichlet polynomial is large. In [3] we showed that the fate of any Dirichlet polynomial of length N was entwined with that of a standard Dirichlet polynomial of the conjugate length D/N. The question of how often two Dirichlet polynomials can be large simultaneously is important in Halász's method. In this case the two Dirichlet polynomials are of different lengths and we multiply them together. Multiplying together two Dirichlet polynomials of the same length merely estimates the number of times the square of either could be large, and so gives trivial bounds.

Our Theorem 1 generalizes the key estimate of [2] to Dirichlet polynomials containing a variable Dirichlet character. Theorem 2 contains further results and Theorem 3 is the appropriate application of Theorems 1 and 2 to our second object, which is to estimate the number of zeros of Dirichlet L-functions. In the usual notation (see [6]) Theorem 3 implies

$$\sum_{\substack{q\leqslant Q\\ q\equiv 0 \ (\mathrm{mod}\ q_0)}} \sum_{\chi \bmod q}{}^{*} N(\sigma, T, \chi) \ll \left(\frac{Q^2 T}{q_0}\right)^{12(1-\sigma)/5+\varepsilon}, \tag{1.1}$$

and for $\sigma > 4/5$ only

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}. \tag{1.2}$$

Theorem 12.1 of [6] contains (1.1) with $5/2$ in place of $12/5$ in the exponent. Forti and Viola [1] improved $5/2$ to $2.463\ldots$, Jutila [5] to $2.460\ldots$ and the author [4] to $2.432\ldots$ (Jutila actually treats all characters to a fixed modulus, which is essentially the special case $Q = q_0$, where q_0 is a divisor of the given modulus.) The estimate (1.2) is the 'density hypothesis' and has a longer history. It is obtained in [1] for $\sigma > 0.8059\ldots$, and in [4] for $\sigma > 0.801\ldots$

This paper is based on [3], but supersedes neither [3] nor [4], as it provides an alternative way of using the reflection argument of [3].

When $\log N/\log D$ is close to a rational number with small denominator it may be better to apply Lemma 2 several times as in [3] and then to form a product polynomial. The results of [3], [4] and the present paper cannot be subsumed under one simple bound. For instance, an improvement in the upper bound for $\zeta(\frac{1}{2}+it)$ would improve the abscissa of validity of (1.2) found in [4] from 0.801 to one smaller than 4/5. The proof of (1.2) for $\sigma > 4/5$ which is completed in the present paper uses only averages of $\zeta(\frac{1}{2}+it)$.

2. Statement of the results. We consider Dirichlet polynomials of the form

$$F(s,\chi) = \sum_{N+1}^{2N} a(m)\chi(m)m^{-s} \tag{2.1}$$

where χ is a Dirichlet character to some modulus q, $s = \sigma+it$ is a complex variable; a general Dirichlet polynomial can be partitioned into sums of the form (2.1). Let q_0 be a positive integer, $Q \geqslant q_0$ and $T \geqslant 1$ be real numbers. Let

$$D = Q^2T/q_0, \qquad l = \log D. \tag{2.2}$$

We shall assume D is large. Let U be an aggregate of pairs (s,χ) satisfying

$$\chi \text{ proper} \bmod q, \qquad q \leqslant Q, \qquad q \equiv 0 \pmod{q_0}, \tag{2.3}$$

$$0 \leqslant \sigma \leqslant l^{-1}, \tag{2.4}$$

$$|t_1-t_2| \leqslant T, \tag{2.5}$$

$$|t_1-t_2| \geqslant 1 \quad \text{when} \quad \chi_1 = \chi_2. \tag{2.6}$$

Let R be the cardinality of U. The first object of this paper is to obtain bounds for R under the assumption

$$|F(s,\chi)| \geqslant V \tag{2.7}$$

for each pair (s,χ) of U. Let $B(M, U)$ be the least positive B for which

$$\sum_{(s,\chi)\in U} |F(s,\chi)| \leqslant \begin{cases} G^{1/2}B & \text{if} \quad N \leqslant M, \\ (N/M)^{3/2}G^{1/2}B & \text{if} \quad M \leqslant N \leqslant D, \end{cases} \tag{2.8}$$

where

$$G = \sum_{N+1}^{2N} |a(m)|^2 \tag{2.9}$$

for every choice of coefficients $a(m)$, and let

$$B(R, M, D) = \max B(M, U) \tag{2.10}$$

taken over sets U of cardinality R with fixed D. We also define $B^*(M, U)$ as the least positive B for which

$$\sum_{(s,\chi)\in U} |F(s,\chi)| \leqslant \begin{cases} M^{1/2}B & \text{if} \quad N \leqslant M, \\ M^{-1}N^{3/2}B & \text{if} \quad M \leqslant N \leqslant D \end{cases} \tag{2.11}$$

for every choice of coefficients $a(m)$ with $|a(m)| \leqslant 1$.

Clearly

$$B^*(M, U) \ll B(M, U). \tag{2.12}$$

Let

$$\Delta_k = \max_{q\leqslant D} d_k(q) \tag{2.13}$$

be the maximum of the kth divisor function. We shall assume D to be large and

$$\Delta_k \ll D^{\varepsilon} \tag{2.14}$$

for any $\varepsilon > 0$.

Since (2.7) implies that the left hand side of (2.8) is at least RV, a bound for B gives a bound for R in terms of V.

The main result of [3] can be stated as follows

LEMMA 1. *For $D > N$ we have*

$$B(R, N, D) \ll R^{1/2}N^{1/2}\log l + RD^{1/4}l \tag{2.15}$$

and for any positive integer k

$$B(R, N, D) \ll R^{1/2}N^{1/2}\log l + RN^{1/4}l^2 + k^{k/2}R^{1-1/2k}N^{1/4}\Delta_3^{1/2}\{B(R, D^k/N^k, D)\}^{1/2k}l^{2+k/4}. \tag{2.16}$$

In any case

$$B(R, N, D) \ll R^{1/2}(D^{1/2}+N^{1/2})l. \tag{2.17}$$

The implied constants are absolute.

An examination of the proof of Lemma 1 leads us to the following restatement of it. Let $U(h,\omega)$ be any aggregate of ordered pairs (s,ψ) with ψ proper mod f, $f \leqslant Q/h$, $f \equiv 0 \pmod{q_0/(h, q_0)}$ for which $\{(s,\omega\psi)\}$ is a subset of U.

In the notation of [3] our $q, \chi, f, \psi, h, \omega$ are $q_1, \chi_1, f_1, \psi_1, h_1$, and $\chi_2\overline{\psi}_2$. We can sharpen (2.16) to

LEMMA 2. *Let k be any positive integer and $D \geqslant N$. Then*

$$B(N, U) \ll R^{1/2}N^{1/2}\log l + RN^{1/4}l^2 + k^{k/2}R^{1-1/2k}N^{1/4}\Delta_3^{1/2}l^{2+k/4}\max_{(h,\omega)}\{B(D^k/N^k, U(h,\omega))\}^{1/2k}. \tag{2.18}$$

Moreover $B(D^k/N^k, U(h,\omega))$ *may be replaced by*

$$\Delta_k B^*(D^k/N^k, U(h,\omega)). \tag{2.19}$$

The simplest result we obtain from Lemma 2 is as follows.

THEOREM 1. *In the notation above*

$$R \ll GV^{-2}N\log^2 l + G^3V^{-6}ND\Delta_2^{1/2}\Delta_3 l^{11} \tag{2.20}$$

provided that

$$V > c_1 G^{1/2}N^{1/4}l^2, \tag{2.21}$$

where c_1 *is an absolute constant. Moreover for* $N \leqslant D$

$$B(N, U) \ll R^{1/2}N^{1/2}l\log l + RN^{1/4}l^2 + R^{5/6}N^{1/6}\Delta_2^{1/12}\Delta_3^{1/6}D^{1/6}l^3. \tag{2.22}$$

Apart from divisor functions and logarithms, (2.20) contains (2.9) of [2] as a special case and enables a similar zero density theorem

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll \max\{D^{3(1-\sigma)/(3\sigma-1)+\varepsilon}, D^{2(1-\sigma)+\varepsilon}\} \tag{2.23}$$

to be deduced by the methods of [2]. This establishes (1.1) for $\sigma > 3/4$, and the range $1/2 \leqslant \sigma \leqslant 3/4$ follows from Theorem 12.1 of [6].

To establish zero density theorems by the Halász–Montgomery–Jutila method we need information about short sums ($N \leqslant D^{1/r}$). Applying Theorem 1 to $F^r(s,\chi)$ raises N and V to the rth power and replaces G by $\Delta_r G^r$ (see Lemma 3). If V is somewhat larger than the bound (2.21) we can do better.

THEOREM 2. *Let* p, q, r *be positive integers such that* $p \leqslant r-1$ *and*

$$N^{rq-p} \geqslant D^{q-1}. \tag{2.24}$$

Then

$$\begin{aligned} B(N, U) \ll{}& R^{1-1/2r}N^{1/2}\Delta_2 + R\Delta_2(R^{-1}N^pD^q)^{1/(4qr+2p)} + \\ &+ RN^{1/6}\Delta_2(R^{-1}D^{q+1})^{1/(12qr+6p)} + R\Delta_2(N^{qr+p}D^q)^{1/(8qr+4p)}. \end{aligned} \tag{2.25}$$

Moreover for

$$V > c_2G^{1/2}N^{1/4}(DN^{-r})^{q/(8qr+4p)}(\Delta_3^{2q}\Delta_{r+1}^{2q}\Delta_{p+q}^{9q+4})^{1/(4qr+2p)}, \tag{2.26}$$

where c_2 *depends only on* p, q, *and* r, *we have*

$$R \ll (GNV^{-2})^rD^\varepsilon + (GV^{-2})^{2qr+p}N^pD^{q+\varepsilon} + (G^3NV^{-6})^{2qr+p}D^{q+1+\varepsilon}. \tag{2.27}$$

The implied constants depend only on p, q, *and in* (2.27) *also on* ε.

We summarize the application of Theorems 1 and 2 to zero-density theorems as follows.

THEOREM 3. *Suppose* $|F(s,\chi)| \geqslant V$ *at each pair* (s,χ) *of* U *and for some integer* $r \geqslant 2$

$$N^r \leqslant D < N^{r+1}. \tag{2.28}$$

Let

$$VG^{-1/2} = N^{\alpha-1/2} \tag{2.29}$$

and $\alpha > 3/4$. *Then for any* $\varepsilon > 0$ *and* D *sufficiently large*

$$R \ll D^{12(1-\alpha)/5+\varepsilon}, \tag{2.30}$$

the constant depending on r, α *and* ε. *Moreover*

$$R \ll D^{2(1-\alpha)+\varepsilon} \tag{2.31}$$

for

$$\alpha > (3r-1)/(4r-2), \tag{2.32}$$

the constant depending on r, α *and* ε. *If the character* χ *is the same in all pairs* (s,χ) *and if for some integer* $r \geqslant 2$

$$N^r \leqslant T < N^{r+1}, \tag{2.33}$$

then

$$R \ll T^{2(1-\alpha)+\varepsilon} \tag{2.34}$$

holds for

$$\alpha > \begin{cases} 4/5 & \text{if} \quad r = 2, \\ 7(3r-1)/(28r-12) & \text{if} \quad r \geqslant 3. \end{cases} \tag{2.35}$$

If in Sections 6 to 8 of [3] we replace the use of Theorem 1 by (2.31) and Theorem 2 by (2.34) in the estimation of class (i, n) zeros, we obtain (1.1) for $\sigma > 3/4$ and (1.2) for $\sigma > 4/5$.

3. Proof of Theorem 1. We begin with a technical lemma.

LEMMA 3. *For* $i = 0, 1, \ldots, k$ *let*

$$F_i(s,\chi) = \sum_{1}^{M_i} a_i(m)\chi(m)m^{-s} \tag{3.1}$$

with

$$F_0(s,\chi) = \prod_{1}^{k} F_i(s,\chi) \tag{3.2}$$

be Dirichlet polynomials. Let

$$G_i = \sum_{1}^{M_i} |a_i(m)|^2. \tag{3.3}$$

Then if $M_0 \leqslant D$ we have

$$G_0 \leqslant \Delta_k \prod_1^k G_i, \tag{3.4}$$

where Δ_k was defined by (2.13). If $M_0 \leqslant D^r$, where $r > 1$, then

$$G_0 \ll \Delta_k^r \prod_1^k G_i, \tag{3.5}$$

where the constant depends on k and r.

Proof. We have

$$G_0 = \sum_m \Big| \sum_{m_1 \ldots m_k = m} a_1(m_1) \ldots a_k(m_k) \Big|^2 \tag{3.6}$$

$$\leqslant \sum_m d_k(m) \sum_{m_1 \ldots m_k = m} |a_1(m_1)|^2 \ldots |a_k(m_k)|^2,$$

which proves (3.4) at once, and (3.5) when we observe that

$$d_k(m) \ll \Delta_k^r \tag{3.7}$$

for $m \leqslant D^r$.

We now proceed to Theorem 1. Take the h and ω for which the maximum in (2.18) is attained, and replace U by the corresponding subset $\{(s, \omega\psi)\}$. The bound $B^*(D/N, U(h, \omega))$ is attained by some Dirichlet polynomial $F_1(s, \psi)$ of length M say. We consider the Dirichlet polynomial

$$F_0(s, \psi) = F_1(s, \psi) F(s, \omega\psi). \tag{3.8}$$

In the notation of Lemma 3

$$G_0 \ll \Delta_2 MG \tag{3.9}$$

if $M \leqslant D/4N$ and

$$G_0 \ll \Delta_2 M (MN/D)^{1/4} G \tag{3.10}$$

if $M > D/4N$, by (3.5) and the estimate (2.14). For simplicity we shall suppose $M \leqslant D/4N$. Both cases give the same upper bound apart from the constant.

If (2.7) holds at each (s, χ) we have by Lemma 2,

$$V \sum |F_1(s, \psi)| \ll (\Delta_2 MG)^{1/2} B\big(4MN, U(h, \omega)\big) \tag{3.11}$$

$$\ll G^{1/2} M^{1/2} \Delta_2^{1/2} R^{1/2} D^{1/2} l,$$

where we have used (2.17). We deduce that

$$B^*\big(D/N, U(h, \omega)\big) \ll G^{1/2} V^{-1} R^{1/2} D^{1/2} \Delta_2^{1/2} l^{1/2}. \tag{3.12}$$

Substituting (3.12) into Lemma 2 we have

$$\sum |F(s, \chi)| \tag{3.13}$$

$$\ll G^{1/2}\{R^{1/2} N^{1/2} \log l + R N^{1/4} l^2 + G^{1/4} R^{3/4} V^{-1/2} N^{1/4} D^{1/4} \Delta_2^{1/4} \Delta_3^{1/2} l^{11/4}\}.$$

Hence if (2.21) holds with a sufficiently large c_1, the coefficient of R on the left of (3.13) is at least twice that on the right, and we have

$$R \ll G V^{-2} N \log^2 l + G^3 V^{-6} N D \Delta_2^{1/2} \Delta_3 l^{11}, \tag{3.14}$$

which proves (2.20). The assertion (2.22) follows by dividing the set U up according to

$$V \leqslant |F(s, \chi)| < 2V, \tag{3.15}$$

and summing over a geometric progression of values of V.

4. Proof of Theorem 2. We work with powers of $F(s, \chi)$. Let $F_1(s, \psi)$ of length M be extremal for $B^*\big(D/N^r, U(h, \omega)\big)$. We consider

$$F_0(s, \chi) = F^p(s, \chi) F_1^q(s, \chi), \tag{4.1}$$

where p and q are positive integers, $p \leqslant r-1$, with

$$N^p (D/N^r)^q \leqslant D. \tag{4.2}$$

As in the proof of Theorem 1 there are two cases. We shall treat the case

$$2^{p+q} N^p M^q \leqslant D. \tag{4.3}$$

Then in the notation of Lemma 3 we have

$$G_0 \leqslant \Delta_{p+q} G^p M^q. \tag{4.4}$$

As in the proof of Theorem 1 we have

$$V^p \sum |F_1(s, \psi)|^q \ll (G^p M^q \Delta_{p+q})^{1/2} B\big(2^{p+q} M^q N^p, U(h, \omega)\big) \tag{4.5}$$

$$\ll G^{p/2} M^{q/2} \Delta_{p+q}^{1/2}\{R^{1/2} M^{q/2} N^{p/2} l^2 + R M^{q/4} N^{p/4} l^2 +$$

$$+ R^{5/6} M^{q/6} N^{p/6} D^{1/6} \Delta_2^{1/2} \Delta_3^{1/6} l^3\},$$

where we have used (2.22). Hence

$$B^*\big(D/N^r, U(h, \omega)\big) \ll R^{(q-1)/q} G^{p/2q} V^{-p/q} \Delta_{p+q}^{1/2q}\{R N^{p-rq} D^q l^4 + \tag{4.6}$$

$$+ R^2 N^{(p-rq)/2} D^{q/2} l^4 + R^{5/3} N^{(p-rq)/3} D^{(q+1)/3} \Delta_2^{1/6} \Delta_3^{1/3} l^6\}^{1/2q}.$$

Lemma 2 applied to $F^r(s, \chi)$ gives

$$R V^r \ll (\Delta_{r+1} G^r)^{1/2}\big(R^{1/2} N^{r/2} \log l + R N^{r/4} l^2 + \tag{4.7}$$

$$+ R^{1/2} N^{r/4} \Delta_3^{1/3} l^{9/4} \max_{h, \omega}\{B^*\big(D/N^r, U(h, \omega)\big)\}^{1/2}\big).$$

Comparing (4.7) with (4.6), we see that if

$$V > c_3 G^{1/2} N^{1/4} (\Delta_{r+1})^{1/2r} l^{2/r}, \tag{4.8}$$

where c_3 is an absolute constant, we have either

$$R \ll G^r V^{-2r} N^r \Delta_{r+1} \log^2 l \tag{4.9}$$

or

$$\begin{aligned} R^2 (V^2/G)^{2qr+p} N^{-qr} \Delta_3^{-2q} \Delta_{r+1}^{-2q} \Delta_{p+q}^{-1} l^{-9q} \\ \ll R N^{p-rq} D^q l^4 + R^2 N^{(p-rq)/2} D^{q/2} l^4 + R^{5/3} N^{(p-rq)/3} D^{(q+1)/3} \Delta_2^{1/6} \Delta_3^{1/3} l^6. \end{aligned} \tag{4.10}$$

The condition (2.26) with a suitable c_2 implies (4.8). Also if (2.26) with a suitable c_2 and (4.10) hold then either

$$R \ll (GV^{-2})^{2qr+p} N^p D^q \Delta_3^{2q} \Delta_{r+1}^{2q} l^{9q+4} \tag{4.11}$$

or

$$R \ll (GV^{-2})^{6qr+3p} N^{2qr+p} D^{q+1} \Delta_2^{1/2} \Delta_3^{6q+1} \Delta_{r+1}^{6q} \Delta_{p+q}^{3} l^{27q+18}. \tag{4.12}$$

This proves (2.27) subject to (2.26). The bound (2.25) follows from (2.27) and the formulae for $d_k(m)$.

5. Proof of Theorem 3. The first two assertions of Theorem 3 are easy consequences of Theorem 1. For (2.34) we use Theorem 2. We can absorb the fixed character χ into the coefficients $a(n)$ without increasing G, and take $Q = 1$, $D = T$. In Theorem 2 we take $q \geq 2$; the case $q = 1$ is different because an exponent of T_0 below becomes zero. We divide the range for T in (2.27) into intervals of length T_0 where $T_0 \leq T$ and

$$V \gg G^{1/2} (N^{qr+p} T_0^{q+\varepsilon})^{1/(8qr+4p)} \tag{5.1}$$

the constant necessary in (5.1) depending on ε. We use ε for any exponent which is $o(1)$, not necessarily the same at each occurrence. Then

$$\begin{aligned} R \ll (GNV^{-2})^r T^{1+\varepsilon} T_0^{-1} + (GV^{-2})^{2qr+p} N^p T^{1+\varepsilon} T_0^{q-1} + \\ + (G^3 N V^{-6})^{2qr+p} T^{1+\varepsilon} T_0^{q+\varepsilon}. \end{aligned} \tag{5.2}$$

We choose T_0, if possible, to make the first two terms in (5.2) of the same order, that is

$$T_0 = (V^2 G^{-1})^{(2qr+p-r)/q} N^{(r-p)/q}. \tag{5.3}$$

The condition (5.1) requires

$$V \gg G^{1/2} N^{r(q+1+\varepsilon)/(4qr+2p+2r)}, \tag{5.4}$$

and (5.3) already implies $T_0 \geq N^r$. The bound (5.2) becomes

$$RT^{-1-\varepsilon} \ll (GV^{-2})^{(3qr+p-r)/q} N^{(qr+p-r)/q} + (GV^{-2})^{4qr+2p+r} N^{2qr+r}. \tag{5.5}$$

The second term in (5.5) is less than the first for

$$V > G^{1/2} N^{1/4+(r+2q)(r-p)/4(4q^2r-2qr+2pq+p-r)}. \tag{5.6}$$

The condition (5.6) is stronger than (5.4).

We now suppose (2.33) holds, so that $T < N^{r+1}$. Substituting for V from (2.29), we find that the first term in (5.5) is less than $T^{2-2\alpha+\varepsilon}$ for

$$\alpha > \frac{3}{4} + \frac{p+q-r}{4(2qr+p-q-r)}, \tag{5.7}$$

which simplifies to

$$\alpha > \frac{3}{4} + \frac{1}{4(4r-3)} \tag{5.8}$$

when $q = 2$, $p = r-1$. These values of p and q makes (5.6)

$$\alpha > \frac{3}{4} + \frac{r+4}{4(16r-3)}. \tag{5.9}$$

For $r \geq 3$ (5.9) is the stronger condition. We need not check $T_0 \leq T$, for replacing T_0 by T will reduce the second and third terms in (5.2), if $T_0 > T$, and increase the first one to $N^{2r(1-\alpha)+\varepsilon}$, which is still less than $T^{2-2\alpha+\varepsilon}$.

Alternatively we choose T_0 to equalise the first and third terms in (5.2), giving

$$T_0 = (V^2 G^{-1})^{(6qr+3p-r)/(q+1)} N^{-(2qr+p-r)/(q+1)}. \tag{5.10}$$

We have $T_0 \geq N^r$ for

$$V \geq G^{1/2} N^{1/4+(r-p)/4(6qr+3p-r)}. \tag{5.11}$$

The condition (5.1) is satisfied for

$$V < G^{1/2} N^{1/4+q(r-p)/4(2q^2r-5qr+pq-2p)} \tag{5.12}$$

for $q > 2$, or for

$$V > G^{1/2} N^{1/4-q(r-p)/4(5qr-2q^2r+2p-pq)} \tag{5.13}$$

when $q = 1$ or 2. The choice of T_0 makes (5.2)

$$\begin{aligned} RT^{-1-\varepsilon} \ll (GV^{-2})^{(7qr+3p)/(q+1)} N^{(3qr+p)/(q+1)} + \\ + (V^2 G^{-1})^{(4q^2r-9qr+2pq-4p+r)/(q+1)} N^{-(2q^2r-3qr-2p+r)/(q+1)}. \end{aligned} \tag{5.14}$$

The second term in (5.14) is less than the first if (5.6) is false. Substituting for V from (2.29) we find that the first term in (5.14) is less than $T^{2-2\alpha+\varepsilon}$

for

$$a > \frac{3}{4} + \frac{r-p+q+1}{4(6qr+3p-r-q-1)}, \tag{5.15}$$

a condition which implies (5.11). With $q = 2$, $p = r-1$ (5.15) is

$$a > \frac{3}{4} + \frac{1}{14r-6}, \tag{5.16}$$

which is (2.35).

Errata to *Large values of Dirichlet polynomials*

Equation(1.12): Insert factor $N^{1/4}$ in the final term.
Equation(3.20): Right hand side should be $M(1-u)L(u, \bar{\psi})$.
Equation(4.5): should read $f = q_2/h_1$.
Equation(6.9): Replace T_1 by FT_1.

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