By (4.7) and (4.3) $I_j$ is estimated by a sum of triple integrals with respect to the variables $a$, $t_1$, $t_2$, where $t_1$ and $t_2$ are of the same sign. Integrating first over $a$ using (4.8)–(4.9) and then over $t_1$ and $t_2$ using (1.1), we obtain

$$I_j \ll XYa_j \log^{17} X.$$

Hence in view of (4.2)

$$\sum_{|D|\leqslant X}' \int_{4Y^{-1}}^{1/2} |S(a, \chi_D)|^2 |T(a)|^2 da \ll \sum_j I_j \min(N^2, a_j^{-2}) \ll XYN\log^{17} X.$$

This combined with (4.6) proves (1.2).

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# The moments of partitions, I

by

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**1.** Let $p_m(n)$ denote the number of partitions of $n$ into $m$ parts. D. H. Lehmer [7] has considered calculating $t_k(n)$, the $k$th moments of $p_m(n)$ defined for $k = 0, 1, \ldots$ by

$$t_k(n) = \sum_{m=1}^{n} m^k p_m(n).$$

The purpose of this paper is the determination of the asymptotic behaviour of the $t_k(n)$ for arbitrary fixed $k$ as $n \to \infty$.

$p_m(n)$ is the number of microstates of a Bose–Einstein gas of $m$ particles and of energy $n$ distributed over the energy levels ($\varepsilon = 1, 2, \ldots$) [5]. It is hoped that the following results are of interest in statistical mechanics as well as in number theory. The first moment has been considered by Husimi [5] and there are certain similarities in method between [5] and this paper. However, we shall avoid using the transformation equation for the generating function of the $t_0(n)$.

We require the generating function for the $t_k(n)$ and we give the derivation of D. H. Lehmer [7]. It is known that ([1], p. 193)

$$G(x, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_m(n) x^n z^m = \prod_{r=1}^{\infty} (1 - zx^r)^{-1}. \tag{1.1}$$

If we introduce the operator

$$\theta = z\frac{\partial}{\partial z},$$

then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m^k p_m(n) x^n z^m = \theta^k G. \tag{1.2}$$

Now

$$\theta G = G \sum_{r=1}^{\infty} \frac{zx^r}{1 - zx^r}, \tag{1.3}$$

and from di Bruno's formula for the Bell polynomials we obtain for $k = 1, 2, \ldots$

$$(1.4)\qquad \theta^k G = GS^{(k)} = G\sum \frac{k!}{l_1!\ldots l_k!}\left(\frac{\theta^0 S}{1!}\right)^{l_1}\ldots\left(\frac{\theta^{k-1}S}{k!}\right)^{l_k}$$

where the summation is over all partitions of $k$ ($k = l_1+2l_2+\ldots+kl_k$) and here and throughout this section

$$S = S(x,z) = \sum_{r=1}^{\infty}\frac{zx^r}{1-zx^r}$$

and $S^{(k)}(x,z)$ will be defined as in (1.4).

We see from (1.2) that the generating function for $t_k(n)$ is $\theta^k G$ evaluated at $z=1$, say $\theta^k G(x)$. Thus from Cauchy's theorem

$$(1.5)\qquad t_k(n) = \frac{1}{2\pi i}\int_C \theta^k G(x)\,x^{-n+1}\,dx$$

where $C$ is any circle about the origin inside the unit circle. We choose the radius of the circle to be $e^{-a}$, where $a$ is the solution of

$$(1.6)\qquad n = \sum_{r=1}^{\infty}\frac{r}{e^{ar}-1} - \big(S^{(k)}(e^{-a})\big)\frac{d}{da}S^{(k)}(e^{-a}).$$

This as we shall see is a saddle-point condition. Furthermore we shall see that (1.6) has a unique solution for $n$ sufficiently large and that an asymptotic expression for $a$ in terms of elementary functions of $n$ may be obtained.

It follows from (1.5) that

$$(1.7)\qquad t_k(n) = \frac{e^{an}\,\theta^k G(e^{-a})}{2\pi}\int_{-\pi}^{\pi}\frac{\theta^k G(e^{-a+i\varphi})}{\theta^k G(e^{-a})}\,e^{-in\varphi}\,d\varphi.$$

**2.** Throughout this section $a$ is defined by (1.6). All equations and estimates involving $a$ may hold only for $a$ sufficiently large. $\varepsilon$ and $\delta$ shall refer to arbitrary real constants $> 0$.

First, let us determine $a$ from (1.6). We define the integers $a_j^{(s)}$ for $s = 1, 2, \ldots$, $j = 0, 1, 2, \ldots, s$ recursively by $a_0^{(s)} = 0$, $a_1^{(1)} = 1$, $a_1^{(2)} = 1$, $a_2^{(2)} = 1$, and for $s \geqslant 2$;

$$(2.1)\qquad a_j^{(s+1)} = \begin{cases} ja_j^{(s)} + (s-j+1)\,a_{j-1}^{(s)}, & 1 \leqslant j \leqslant s,\\ 0, & j = s+1.\end{cases}$$



Then it is easily seen that

$$(2.2)\qquad \theta^{s-1}S(x,z) = \sum_{r=1}^{\infty}\frac{\sum_{j=1}^{s} a_j^{(s)} z^j x^{jr}}{(1-zx^r)^s}.$$

Hence we are led to consider sums of the form

$$(2.3)\qquad S_k^j(e^{-a}) = \sum_{r=1}^{\infty}\frac{e^{-ajr}}{(1-e^{-ar})^k}.$$

Now

$$\frac{e^{-ajr}}{(1-e^{-ar})^k} = \sum_{n=k-1}^{\infty} n(n-1)\ldots(n-k+2)\cdot e^{-ar(n+j-k)}.$$

Using the identity

$$e^{-a} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t}\Gamma(t)\,dt,\qquad \sigma > 0,\ |\arg a| < \frac{\pi}{2}-\delta,$$

(throughout this paper $\Gamma(t) = \int_0^\infty u^{t-1}e^{-u}\,du$) we obtain that

$$(2.4)\qquad S_k^j(e^{-a}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t}\Gamma(t)\zeta(t)\zeta_{j,k}(t)\,dt,\qquad \sigma > k,\ |\arg a| \leqslant \frac{\pi}{2}-\delta$$

where $\zeta(t)$ is the Riemann zeta-function and

$$\zeta_{j,k}(t) = \sum_{n=k-1}^{\infty}\frac{n(n-1)\ldots(n-k+2)}{(n+j-k+1)^t}.$$

Note that $\zeta_{1,1}(t) = \zeta(t)$.

LEMMA 2.1. *The function $\zeta_{j,k}(t)$ is regular for all values of $t$ except at $t = i$ ($i \in \{1, 2, \ldots, k\}$). At $t = i$ the function has a simple pole with residue $C_i$ where*

$$(2.5)\qquad (x+k-1-j)(x+k-2-j)\ldots(x+1-j) = \sum_{i=0}^{k-1} C_i x^i.$$

**Proof.** The lemma follows in a manner similar to that sometimes used to prove the case $\zeta_{1,1}(t) = \zeta(t)$. That is, with

$$\varphi(x) = \frac{(x+k-1)(x+k-2)\ldots(x+1)}{(x+j)^s}$$

and $t$ any fixed positive integer ([2], p. 526)

$$(2.6)\qquad \zeta_{j,k}(s) = \int_0^\infty \varphi(x)\,dx + \frac{\varphi(0)}{2} - \frac{B_2}{2!}\varphi'(0) - \frac{B_4}{4!}\varphi'''(0) - \ldots - \frac{B_{2t}}{(2t)!}\varphi^{(2t-1)}(0) + \int_0^\infty P_{2t+1}(x)\varphi^{(2t+1)}(x)\,dx$$

where $B_r$ is the $r$th Bernoulli number and $P_r(x)$ is the $r$th Bernoulli polynomial. The second integral converges for all $s$ with real part $> k-2t-1$. Also $\varphi^{(j)}(0)$ is regular for all $j \geq 0$. Since

$$\int_0^\infty \varphi(x)\,dx = \sum_{l=0}^{k-1} C_l \frac{j^{l-s+1}}{(l-s+1)}$$

for $s \neq 1, 2, \ldots, k$ we have the lemma.

Using (2.4) and Lemma 2.1 we may now determine the asymptotic behaviour of the sums $S_k^j$ as $\alpha \to 0$.

LEMMA 2.2. *Let $k \geq 2$. Then*

$$S_k^j(e^{-\alpha}) = \frac{\Gamma(k)\zeta(k)}{\alpha^k} + O\{\alpha^{-k+1-\varepsilon}\}$$

*and for $s \geq 1$*

$$\theta^s(e^{-\alpha}) = \frac{2(s!)^2\zeta(s+1)}{\alpha^{s+1}} + O\{\alpha^{-s+\varepsilon}\}.$$

Proof. Let $\sigma = \operatorname{Re} t$ and let $k-1+\varepsilon < \sigma < k$. It is known ([2], p. 224) that

$$(2.7)\qquad \Gamma(\sigma+iy) = O\{|y|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|y|}\}$$

as $|y| \to \infty$; and clearly

$$(2.8)\qquad \zeta(\sigma+iy) = O\{1\} \quad \text{as} \quad |y| \to \infty.$$

Finally, it follows readily from (2.6) that

$$(2.9)\qquad \zeta_{j,k}(\sigma+iy) = O\{|y|\} \quad \text{as} \quad |y| \to \infty.$$

With these order relations we may transform the contour in (2.4) from $\sigma > k$ to $\sigma > k-1+\varepsilon$ and the difference is the residue at $t = k$. This gives the first part of the lemma.

According to (2.2)

$$\theta^s S(e^{-\alpha}) = \left(\sum_{j=1}^{s+1} a_j^{(s+1)}\right)\frac{\Gamma(s+1)\zeta(s+1)}{\alpha^{s+1}} + O\{\alpha^{-s+\varepsilon}\}$$



and from (2.1) it is easily seen that

$$\sum_{j=1}^{s+1} a_j^{(s+1)} = s\sum_{j=1}^{s} a_j^{(s)} = 2s!.$$

Hence we have the second part of the lemma.

Define $\Sigma(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ by

$$(2.10)\qquad \sum \frac{k!}{l_1!\ldots l_k!}\left(\frac{\lambda_0}{1!}\right)^{l_1}\ldots\left(\frac{\lambda_{k-1}}{k!}\right)^{l_k} = \sum(\lambda_0, \lambda_1, \ldots, \lambda_{k-1}),$$

where the summation extends over all partitions of $k$ $(l_1+2l_2+\ldots+kl_k = k)$.

LEMMA 2.3. *Let $\gamma$ denote Euler's constant. Then*

$$\theta^k G(e^{-\alpha}) = \exp\left(\frac{\pi^2}{6\alpha}\right)\cdot\alpha^{\frac{1}{2}-k}\cdot\sum\left(\gamma+\log\frac{1}{\alpha}, 2\zeta(2), \ldots, 2\left((k-1)!\right)^2\zeta(k)\right)\times[1+O\{\alpha^{1-\varepsilon}\}].$$

Proof. In [5] it is shown that

$$(2.11)\qquad G(e^{-\alpha}) = \exp\left(\frac{\pi^2}{6\alpha}\right)\cdot\left(\frac{\alpha}{2\pi}\right)^{1/2}[1+O(\alpha)].$$

The case $k = 1$ has also been considered before in [5] and

$$(2.12)\qquad \theta G = G(e^{-\alpha})\sum\left(\log\frac{1}{\alpha}+\gamma\right)[1+O(\alpha)].$$

We note for future reference that in this case $\zeta_{1,1}(t) = \zeta(t)$ and there is a double pole at $t = 1$. The residue therefore is

$$\left(\frac{d\{\alpha^{-t}\Gamma(t)\}}{dt} + 2\alpha^{-t}\Gamma(t)\left(\zeta(t)-\frac{1}{t-1}\right)\right)_{t=1}$$

and ([4], p. 127) $\left(\zeta(t)-\dfrac{1}{t-1}\right)_{t=1} = \gamma$, $\Gamma'(1) = -\gamma$ ([2], p. 228).

LEMMA 2.4.

$$\frac{dS^{(k)}(e^{-\alpha})}{d\alpha} = -\alpha^{-k-1}\left[\log^k\frac{1}{\alpha} + O\left\{\log^{k-1}\frac{1}{\alpha}\right\}\right].$$

Proof. From (1.4)

$$(2.13)\qquad \frac{dS^{(k)}(e^{-\alpha})}{d\alpha} = \sum\frac{k!}{l_1!\ldots l_k!}\left[l_1\left(\frac{\theta^0 S}{1!}\right)^{l_1-1}\frac{d\theta^0 S}{d\alpha}\cdot\left(\frac{\theta^1 S}{2!}\right)^{l_2}\ldots\left(\frac{\theta^{k-1}S}{k!}\right)^{l_k}+\ldots+\frac{l_k}{k!}\left(\frac{\theta^0 S}{1!}\right)^{l_1}\left(\frac{\theta^1 S}{2!}\right)^{l_2}\ldots\left(\frac{\theta^{k-1}S}{k!}\right)^{l_k-1}\frac{d\theta^{k-1}S}{d\alpha}\right],$$

where the outside summation is over all partitions of $k$. Now

$$\frac{d\theta^u S}{d\alpha} = \sum_{j=1}^{s} a_j^{(u+1)} \frac{dS_j^{u+1}(e^{-\alpha})}{d\alpha}$$

and we may apply (2.4) to obtain for $k \geqslant 2$

$$\frac{dS_k^j(e^{-\alpha})}{d\alpha} = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} t\alpha^{-t-1}\Gamma(t)\zeta(t)\zeta_{j,k}(t)\,dt \tag{2.14}$$

$$= -2\alpha^{-k-1}\big((k-1)!\big)^2\zeta(k)$$

and obtain with the argument noted after (2.12)

$$\frac{d\theta^0 S}{d\alpha} = -\left(\frac{d\{t\alpha^{-t-1}\Gamma(t)\}}{dt} + 2t\alpha^{-t-1}\Gamma(t)\left(\zeta(t)-\frac{1}{t-1}\right)\right)_{t=1} \tag{2.15}$$

$$= -\frac{\log\frac{1}{\alpha}+1+\gamma}{\alpha^2}.$$

The lemma follows from (2.13), (2.14) and (2.15) upon noting that the term of (2.13) with $l_1 = k$ is the largest by a factor of $\log\frac{1}{\alpha}$ and that $a_j^{(u+1)} \equiv 1$.

LEMMA 2.5. *Equation* (1.6) *has a unique solution for sufficiently large* $n$ *and*

$$\alpha = \frac{\pi}{\sqrt{6}}n^{-1/2}\{1+f(n)\}$$

*where*

$$f(n) = O\{n^{-1/2}\}.$$

*Furthermore*

$$\exp\{\alpha n\}\cdot\theta^k G(e^{-\alpha}) = \exp\left(\frac{2\pi}{\sqrt{6}}n^{1/2}\right)2^{-1/2}\cdot 6^{-1/4}\cdot(6^{1/2}/\pi)^k n^{k/2}\times$$

$$\times\sum\left(\log\frac{(6n)^{1/2}}{\pi}+\gamma,\ 2\zeta(2),\ \ldots,\ 2\big((k-1)!\big)^2\zeta(k)\right)[1+O\{n^{-1/2+\varepsilon}\}].$$

Proof. From Lemmas 2.3 and 2.4 and (1.6) it follows that

$$n = \frac{\pi^2}{6\alpha^2}+\frac{1}{2\alpha}+O\left\{\frac{1}{-\alpha\log\alpha}\right\}.$$

The first part of the lemma follows routinely from this. The second part follows at once from the first part and Lemma 2.3.



LEMMA 2.6.

$$\int_{-\pi}^{\pi}\frac{\theta^k G(e^{-\alpha+i\psi})}{\theta^k G(e^{-\alpha})}e^{-in\psi}\,d\psi = \pi 6^{-1/4}n^{-3/4}[1+O\{n^{-1/2}\}].$$

Proof. We break the range of integration up as follows:

$$I = \int_{-\pi}^{\pi} = \int_{-\pi}^{-\psi_0} + \int_{-\psi_0}^{\psi_0} + \int_{\psi_0}^{\pi}$$

where $\psi_0 = \alpha^{7/5}$. With this choice of $\psi_0$ it has been shown by Roth and Szekeres [8] that for arbitrary fixed $m > 0$, if $\psi_0 < |\psi| < \pi$ then

$$G(e^{-\alpha+i\psi})/G(e^{-\alpha}) = O\{\alpha^m\}.$$

Clearly

$$|S^{(k)}(e^{-\alpha+i\psi})/S^{(k)}(e^{-\alpha})| \leqslant 1.$$

Thus

$$I = \int_{-\psi_0}^{\psi_0} + O\{\alpha^m\}. \tag{2.16}$$

We shall now see that the factor $S(\theta) = S^{(k)}(e^{-\alpha+i\psi})/S^{(k)}(e^{-\alpha})$ does not significantly change the value of the integral.

We obtain from Taylor's theorem

$$S(\psi) = id_1(\alpha)\psi - d_2(\alpha)\psi^2 + O\{d_3(\alpha)\psi^3\}. \tag{2.17}$$

Here we use the fact that

$$\left|\frac{d^3 S^{(k)}(e^{-\alpha+i\psi})}{d\psi^3}\right| = O\left\{\left|\left(\frac{d^3 S^{(k)}(e^{-\alpha+i\psi})}{d\psi^3}\right)_{\psi=0}\right|\right\}$$

which in turn follows from (2.2) and (2.4) [note that $|\psi| \leqslant \alpha^{7/5}$].

For $|\psi| \leqslant \psi_0$ we may use equation (2.4) to represent the derivatives with respect to $\psi$. By the same methods used to estimate $dS^{(k)}(e^{-\alpha})/d\alpha$ we obtain that

$$d_j = O\{\alpha^{-j}\} \quad \text{for} \quad j = 1, 2, 3. \tag{2.18}$$

Thus

$$\log S(\psi) = id_1(\alpha)\psi - e(\alpha)\psi^2 + O\{\alpha^{-3}\psi^3\}. \tag{2.19}$$

It is shown in [8] that for $|\psi| \leqslant \psi_0 = \alpha^{7/5}$,

$$\log G(e^{-\alpha+i\psi}) = iA_1(\alpha)\psi - A_2(\alpha)\psi^2 + O\{A_3(\alpha)\psi^3\}$$

where

$$A_1(\alpha) = \sum_{r=1}^{\infty}\frac{r}{e^{\alpha r}-1},\qquad A_2(\alpha) = \sum_{r=1}^{\infty}\frac{r^2 e^{r\alpha}}{(e^{r\alpha}-1)^2}, \tag{2.20}$$

$$A_3(\alpha) = \sum_{r=1}^{\infty}\frac{r^3(e^{2r\alpha}-e^{r\alpha})}{(e^{r\alpha}-1)^3}.$$

From the Euler–Maclaurin summation formula, one obtains that

$$(2.21)\qquad A_2(a) = \frac{\pi^2}{3a^3} + O\{a^{-2}\}, \qquad A_3(a) = O\{a^{-4}\}.$$

Equation (1.6) shows that $A_1(a) + d_1(a) - n = 0$. Letting $B_2 = A_2 + e_2$

$$I = \int_{-\psi_0}^{\psi_0} d\psi = \int_{-a^{7/5}}^{a^{7/5}} \exp(-B_2\psi^2)[1 + O\{a^{-4}\psi^3\}]d\psi$$

$$= \frac{1}{\sqrt{B_2}} \int_{-a^{-1/10}}^{a^{-1/10}} \exp(-x^2)dx[1 + O(a)] = \left(\frac{2\pi}{B_2}\right)^{1/2}[1 + O\{a\}].$$

Now $B_2 = A_2[1 + O\{a\}]$ by (2.18) and thus using (2.16) and (2.21) we obtain the lemma.

Now from Lemmas 2.5 and 2.6 we obtain our main result:

THEOREM 2.1.

$$t_k(n) = \frac{1}{4\sqrt{3}} n^{-1} e^{\pi\sqrt{2n/3}} \left(\frac{(6n)^{1/2}}{\pi}\right)^k \times$$

$$\times \sum \left(\log\frac{(6n)^{1/2}}{\pi} + \gamma, 2\zeta(2), \ldots, 2\left((k-1)!\right)^2\zeta(k)\right)[1 + O\{n^{-1/2+\varepsilon}\}],$$

as $n \to \infty$, where the sum is defined by (2.10).

COROLLARY 2.1.

$$t_0(n) = \frac{1}{4\sqrt{3}} n^{-1} e^{\pi\sqrt{2n/3}}[1 + O\{n^{-1/2+\varepsilon}\}];$$

$$t_1(n) = \frac{\sqrt{2}}{4\pi n^{1/2}} e^{\pi\sqrt{2n/3}}\left(\log\frac{\sqrt{6n}}{\pi} + \gamma\right)[1 + O\{n^{-1/2+\varepsilon}\}];$$

$$t_2(n) = \frac{\sqrt{3}}{\pi^2 2} e^{\pi\sqrt{2n/3}}\left(\left(\log\frac{\sqrt{6n}}{\pi} + \gamma\right)^2 + 2\zeta(2)\right)[1 + O\{n^{-1/2+\varepsilon}\}];$$

$$t_3(n) = \frac{3}{\sqrt{2}\pi^3} n^{1/2} e^{\pi\sqrt{2n/3}}\left(\left(\log\frac{\sqrt{6n}}{\pi} + \gamma\right)^3 + \right.$$

$$\left. + 6\left(\log\frac{\sqrt{6n}}{\pi} + \gamma\right)\zeta(2) + 8\zeta(3)\right)[1 + O\{n^{-1/2+\varepsilon}\}].$$

Remark. With a more careful analysis, but along the same lines [5], we may replace the $O\{n^{-1/2+\varepsilon}\}$ terms in the expressions for $t_1(n)$, $t_2(n)$ and $t_3(n)$ by $O\{n^{-1/2}\log n\}$, $O\{n^{-1/2}\log^3 n\}$ and $O\{n^{-1/2}\log^4 n\}$ respectively.



COROLLARY 2.2. *Let $\sigma^2(n)$ denote the variance or*

$$\sigma^2(n) = \frac{t_2(n)}{t_0(n)} - \frac{t_1^2(n)}{t_0^2(n)}.$$

*Then*

$$\sigma^2(n) = 2n[1 + O\{n^{-1/2+\varepsilon}\}].$$

COROLLARY 2.3. *Let $M_3(n)$ denote the third moment about the mean, that is*

$$M_3(n) = \frac{t_3(n)}{t_0(n)} - 3\frac{t_2(n)t_1(n)}{t_0^2(n)} + 2\frac{t_1^3(n)}{t_0^3(n)}.$$

*Then*

$$M_3(n) = \frac{48\sqrt{6}}{\pi^3}\zeta(3)n^{3/2}[1 + O\{n^{-1/2+\varepsilon}\}].$$

Let

$$P_{\bar{m}\pm h\sigma} = \sum_{m \geqslant \bar{m} - h\sigma}^{m \leqslant \bar{m} + h\sigma} p_m(n)/t_0(n)$$

where $\bar{m} = t_1(n)/t_0(n)$. Then from a generalization of Tchebycheff's inequality [6] we obtain

THEOREM 2.2. *Let $h$ be any number $\geqslant 0$. Let $r$ be any positive fixed integer. Then*

$$P_{\bar{m}\pm h}[t_r(n)/t_0(n)]^{1/r} \geqslant 1 - \frac{1}{h^r}.$$

COROLLARY. *Let $f(n)$ be any function tending to infinity, then the number of summands in almost all partitions of $n$ lies between*

$$\frac{1}{\pi}\left(\frac{3n}{2}\right)^{1/2}\log n \pm f(n)n^{1/2}.$$

Of course, this result has been obtained before by Erdös and Lehner [3] as a corollary of the following result (Erdös and Lehner): Let $P_m(n) = \sum_{l=1}^{m} p_l(n)$. For

$$m = c^{-1}n^{1/2}\log n + xn^{1/2}, \qquad c = \left(\frac{2}{3}\right)^{1/2}\pi \lim_{n\to\infty}\frac{P_m(n)}{t_0(n)} = \exp\left(-\frac{2}{c}e^{-c\frac{x}{2}}\right).$$

A further result of interest here is that Szekeres [9] has shown that the distribution $p_m(n)$ is unimodal.

**3.** In this section we shall carry out a quite analogous treatment of $q_m(n)$, the number of partitions of $n$ into exactly $m$ parts. Hence we shall avoid giving the details and shall only discuss the differences when they are significant.

Since $q_m(n)$ is the number of microstates of a Fermi–Dirac gas of $m$ particles and of energy $n$ distributed over the energy levels ($t = 1, 2, \ldots$) [5], it is hoped again that these results will be of some interest in statistical mechanics as well as in number theory.

First of all $q_m(n)$ has the generating function ([1], p. 193)

$$H(x, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_m(n) x^n z^m = \prod_{r=1}^{\infty}(1+zx^r).$$

Letting

$$u_k(n) = \sum_{m=1}^{n} m^k q_m(n),$$

and again letting $\theta$ denote the operator

$$\theta = z\frac{\partial}{\partial z}$$

we obtain that

$$\sum_{n=0}^{\infty} u_k(n) x^n = H(x, 1)\sum(\theta^0 T, \theta^1 T, \ldots, \theta^{k-1} T)$$

where

$$T = \sum_{r=1}^{\infty}\frac{zx^r - z^2x^{2r}}{1-z^2x^{2r}}.$$

Here we define the number $b_j^{(s)}$ for $s = 1, 2, \ldots; j = 1, 2, \ldots, 2s$ by

$$b_1^{(1)} = 1,\ b_2^{(1)} = -1;$$
$$b_1^{(2)} = 1,\ b_2^{(2)} = -1,\ b_3^{(2)} = 2,\ b_4^{(2)} = -2;$$

and for $s \geqslant 2$

$$b_j^{(s+1)} = \begin{cases} jb_j^{(s)} + (2s-j+2)b_{j-2}^{(s)}, & j = 2s; \\ b_{2s-1}^{(s)}, & j = 2s+1; \\ 0, & j = 2s+2. \end{cases}$$

Then one may verify that

$$\theta^k T = \sum_{r=1}^{\infty}\frac{\sum_{j=1}^{2k+2} b_j^{(k+1)} z^j x^{2r}}{(1-z^2x^{2r})^{k+1}},$$



and that for all $k \geqslant 1$

$$\sum_{j=1}^{2k} b_j^{(k)} = 0. \tag{3.1}$$

Corresponding to (1.7) we have

$$u_k(n) = e^{\beta n}\theta^k H(e^{-\beta})\int_{-\pi}^{\pi}\frac{\theta^k H(e^{-\beta+i\psi})}{\theta^k H(e^{-\beta})}e^{-in\psi}d\psi$$

where $\beta$ is determined by

$$n = \sum_{r=1}^{\infty}\frac{r}{e^{\beta r}-1} - \left(T^{(k)}(e^{-\beta})\right)^{-1}\frac{d}{d\beta}T^{(k)}(e^{-\beta})$$

($T^{(k)}$ corresponds to $S^{(k)}$ in (1.4)). Throughout this section $\beta$ shall be defined by this relation.

We deduce that

$$\theta^k T(e^{-\beta}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t}\gamma(t)\zeta(t)\sum_{j=1}^{2k} b_j^{(k)}\zeta_{j,k}^{(d)}(t)\,dt, \qquad \sigma > k, \tag{3.2}$$

where

$$\zeta_{j,k}^{(d)}(t) = \sum_{n=k-1}^{\infty}\frac{n(n-1)\ldots(n-k+2)}{(2n+2-2k+j)^t}.$$

Hence in particular

$$b_1^{(1)}\zeta_{1,1}^{(d)}(t) + b_2^{(1)}\zeta_{2,1}^{(d)}(t) = (1-2^{-t+1})\zeta(t). \tag{3.3}$$

LEMMA 3.1. *The function $\zeta_{j,k}^{(d)}(t)$ is regular for all values of $t$ except at $t = 1, 2, \ldots, k$. At $t = s$, $s$ some integer with $1 \leqslant s \leqslant k$, the function has a simple pole with residue $2^{-k}C_{k,s-1}$ where*

$$\sum_{s=1}^{k-1} C_{k,s}x^s = (x+4k+4-j)(x+4k+2-j)\ldots(x+2k+4-j).$$

LEMMA 3.2. *Let $k \geqslant 1$. Then*

$$\theta^k T(e^{-\beta}) = O\{\beta^{-k+\varepsilon}\}.$$

Proof. In view of (3.1) the residue of (3.2) at $t = k+1$ is zero, otherwise the proof is the same as that of Lemma 2.2.

LEMMA 3.3. *Let $k \geqslant 0$. Then*

$$\theta^k H(e^{-\beta}) = 2^{-1/2}\exp\left\{\frac{\pi^2}{12\beta}\right\}\left(\frac{\log 2}{\beta}\right)^k [1+O\{\beta^{1-\varepsilon}\}].$$

Proof. If we apply the Euler–Maclaurin formula to $\sum \log(1+e^{-\beta r})$ we obtain

$$H(e^{-\beta}) = 2^{-1/2}\exp\left\{\frac{\pi^2}{12\beta}\right\}[1+O\{\beta\}].$$

In the case $k=1$ we do not have a double pole at $t=1$ since

$$\lim_{t\to 1}(1-2^{-t+1})\zeta(t) = \log 2.$$

The lemma now follows in all cases $k \geqslant 2$ from Lemma 3.2 since

$$\sum(\beta^{-1}\log 2, 0, 0, \ldots, 0) = (\beta^{-1}\log 2)^k.$$

Lemma 3.4. *Let $k \geqslant 1$. Then*

$$\frac{dT^{(k)}(e^{-\beta})}{d\beta} = \beta^{-k-1}[1+O(\beta)].$$

Lemma 3.5.

$$\beta = \frac{\pi}{2\sqrt{3}}n^{-1/2}\{1+f(n)\}$$

*where*

$$f(n) = O\{n^{-1/2}\}.$$

*Furthermore*

$$\exp(\beta n)\,\theta^k H(e^{-\beta}) = 2^{-1/2}\exp\left(\frac{\pi}{\sqrt{3}}n^{-1/2}\right)\cdot\left(n^{1/2}\frac{2\sqrt{3}}{\pi}\log 2\right)^k[1+O(n^{-1/2+\varepsilon})].$$

Lemma 3.6.

$$\int_{-\pi}^{\pi}\frac{\theta^k H(e^{-\beta+i\varphi})}{\theta^k H(e^{-\beta})}e^{-in\varphi}d\varphi = \left(2\sqrt{2}\,3^{-1/4}n^{3/4}\right)^{-1}[1+O\{n^{-1/2+\varepsilon}\}].$$

Thus we obtain

Theorem 3.1. *Let $k \geqslant 0$. Then*

$$u_k(n) = \frac{1}{4\cdot 3^{1/4}\cdot n^{3/4}}e^{\frac{\pi}{\sqrt{3}}n^{1/2}}\left(\frac{n^{1/2}2\sqrt{3}\log 2}{\pi}\right)^k[1+O\{n^{-1/2+\varepsilon}\}].$$

To obtain an asymptotic relation for the variance we must obtain sharper estimates for $u_2(n)$ and $u_1(n)$. (See the remark following Corollary 2.1.) From (3.2) and (3.3)

$$\theta^2 H = 2^{-1/2}\exp\left(\frac{\pi^2}{12\beta}\right)\cdot\left[\left(\frac{\log 2}{\beta}\right)^2 + \operatorname*{Res}_{t=1}\left[\beta^{-t}\zeta(t)\sum_{j=1}^{4}\zeta_{j,2}^{(d)}(t)\right] + O\{\alpha^{-1}\}\right]. \tag{3.4}$$



In a sufficiently small neighbourhood of $t=1$

$$\zeta(t) = \frac{1}{1-t} + \gamma + O\{|1-t|\}$$

and

$$\sum_{j=1}^{4}\zeta_{j,2}^{(d)}(t) = \frac{3}{4(1-t)} + b + O\{|1-t|\}$$

where $\gamma$ is again Euler's constant and $b$ is a constant whose value we shall not require.

The residue in (3.4)

$$= \frac{3}{4}\,\frac{\log\frac{1}{\beta}}{\beta} + O\left(\frac{1}{\beta}\right),$$

hence

$$\frac{u_2(n)}{u_1(n)} = \left[\left(\frac{\sqrt{12n}}{\pi}\log 2\right)^2 + n^{1/2}\frac{3\sqrt{3}}{4\pi}\log n + O\{n^{1/2}\}\right].$$

If we examine the proof of Theorem 3.1 more carefully, we see that the pole at zero is a single pole, hence the $O$-term for $k=1$ is $O\{\beta\}$ and not $O\{\beta^{1-\varepsilon}\}$. Thus

$$\left(\frac{u_1(n)}{u_0(n)}\right)^2 = \left(\frac{\sqrt{12n}\log 2}{\pi}\right)^2[1+O\{n^{-1/2}\}]$$

and we obtain

Theorem 3.2. *Let $\sigma^2(n)$ be defined by*

$$\sigma^2(n) = \frac{u_2(n)}{u_0(n)} - \left(\frac{u_1(n)}{u_0(n)}\right)^2.$$

*Then*

$$\sigma^2(n) = \frac{3\sqrt{3}}{4\pi}n^{1/2}\log n[1+O\{\log^{-1}n\}].$$

Corollary. *Let $f(n)$ be any function tending to infinity with $n$, then the number of summands in almost all partitions of $n$ into distinct summands lies between*

$$n^{1/2}\frac{2\sqrt{3}\log 2}{\pi} \pm f(n)n^{1/4}\log^{1/2}n.$$

This corollary is an improvement on the previous result of Erdös and Lehner [3].

Again we point out that Szekeres [9] has proven that the $q_m(n)$ form a unimodal distribution.

**4.** The author is very grateful to Professor D. H. Lehmer for bringing these problems to his attention, for suggesting that the saddle-point method might be applicable and finally for supplying some of the following numerical results for the comparison of the exact values with the asymptotic values. We include some of these results:

| Function | Exact Value | Asymptotic Value |
|---|---|---|
| $t_0(90)$ | 56634173 | 59366780 |
| $t_1(90)$ | 1149288434 | 1132188000 |
| $t_2(90)$ | 26649644186 | 32278070000 |
| $t_3(90)$ | 703921667714 | 1254201000000 |
| $u_0(125)$ | 3207086 | 3258254 |
| $u_1(125)$ | 27314955 | 27842340 |
| $u_2(125)$ | 240371878 | 237917760 |
| $u_3(125)$ | 2178385203 | 2033046000 |

The exact values were computed using algorithms of D. H. Lehmer [7]. The agreement for the $u_k(n)$ seems somewhat better than for the $t_k(n)$. However, all estimates of $t_2$ and $t_3$ differ by at least four in the second significant digit for $n = 90$. The results for $t_3(n)$ differ by more than five in the first significant digit. This does not appear to be due to errors in calculating or deriving the asymptotic expression. We can determine additional terms in the expression, but have not. In any case the method is sound, and we hope to publish soon a generalization of this work to partitions with summands taken from a given set of integers subject to certain conditions.

Added in proof: $\sum_{j=1}^{4} \zeta_{j,2}^{(d)}(t) = (1-2^{-t})\,\zeta(t+1)$, hence the residue in (3.4) is $\beta^{-1}\pi^2/12$ and one obtains in Theorem 3.2

$$\sigma^2(n) = \frac{\pi}{2\sqrt{3}}\, n^{1/2} + O\{1\}.$$

This of course improves the Corollary to Theorem 3.2. We point out that the distribution is Gaussion as stated by Erdös and Lehner.

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