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On mean values of L-functions and short character sums with real characters

by

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1. Introduction. Let χ_D be the Dirichlet character (mod $|D|$) determined by Kronecker's symbol (D/n). Here D is assumed to be a non-square integer satisfying $D \equiv 0$ or $1 \pmod 4$. A summation over such values of D will be denoted by $\sum'$ and a summation over fundamental discriminants by $\sum^*$. Our aim in this paper is to prove the following two theorems (the constant in the symbol $\ll$ is everywhere absolute):

THEOREM 1. *For* $X \geqslant 3$, $T > 0$ *we have*

$$\sum_{|D| \leqslant X}' \int_{-T}^{T} |L(\tfrac{1}{2}+it, \chi_D)|^2 dt \ll XT \log^{16}\big(X(T+1)\big). \tag{1.1}$$

THEOREM 2. *For* $X \geqslant 3$ *and* $1 \leqslant N \leqslant Y$ *we have*

$$\sum_{|D| \leqslant X}' \sum_{1 \leqslant n \leqslant Y} \Big| \sum_{0 \leqslant \nu \leqslant N} \chi_D(n+\nu) \Big|^2 \ll XYN \log^{17} X. \tag{1.2}$$

The basic tools in the proof of Theorem 1 are a mean value theorem for Dirichlet polynomials with real characters (Lemma 1 below) which we obtained in [2], and an expression for $L(s, \chi)$ due to Ramachandra [4] (Lemma 2 below). A combination of these yields the theorem by a straightforward calculation.

Theorem 2 is deduced from Theorem 1 using a method of Linnik (Lemmas 3 and 4).

In [1] we proved the following mean value estimate for character sums which we applied to Dirichlet polynomials in [2]:

$$\sum_{|D| \leqslant X}' \Big| \sum_{1 \leqslant n \leqslant Y} \chi_D(n) \Big|^2 \ll XY \log^8 X. \tag{1.3}$$

In an earlier version of the paper we modified the method of [1] to prove Theorem 2 directly, without using L-functions, and then deduced Theorem 1 from it. However, the present method is technically simpler though also less elementary, since we need the functional equation for L-functions and complex integration which we dispensed with in the earlier version.

2. Lemmas. The first lemma is Theorem 1 of [2].

LEMMA 1. *Let $X \geqslant 3$, $N \geqslant 2$ be natural numbers, let a_n, $n = 1, \ldots, N$ be any complex numbers, and define*

$$f(s, \chi) = \sum_{1}^{N} a_n \chi(n) n^{-s}.$$

Write

$$Z_k = \sum_{1}^{N} |a_n|^k.$$

Then for any real T_0 and any positive T we have

$$\sum_{|D| \leqslant X}' \int_{T_0}^{T_0+T} |f(it, \chi_D)|^2 dt \ll TX \sum_{\substack{1 \leqslant m,\, n \leqslant N \\ mn = a^2}} |a_m a_n| + (TX)^{1/2} N^{15/8} Z_{16}^{1/8} \log^7 N.$$

LEMMA 2. *Let χ be a primitive non-principal character, $U \geqslant 2$, and let*

$$L(s, \chi) = \psi(s, \chi) L(1-s, \overline{\chi})$$

be the functional equation for $L(s, \chi)$. Then we have for $s = \frac{1}{2} + it$, $0 < \beta < 1$

$$L(s, \chi) = \sum_{1}^{\infty} \chi(n) n^{-s} e^{-n/U} - \frac{1}{2\pi i} \int_{\mathrm{Re}\, w = -\beta - \frac{1}{2}} \psi(s+w, \chi) \Big(\sum_{n > U} \overline{\chi}(n) n^{-1+s+w} \Big) \Gamma(w) U^w dw - \frac{1}{2\pi i} \int_{\mathrm{Re}\, w = -\beta} \psi(s+w, \chi) \Big(\sum_{n \leqslant U} \overline{\chi}(n) n^{-1+s+w} \Big) \Gamma(w) U^w dw.$$

Proof. This is an analogue of Lemma 3 in Ramachandra's paper [4]. We repeat the proof for completeness. Starting with the well-known identity

$$\sum_{1}^{\infty} \chi(n) n^{-s} e^{-n/U} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s+w, \chi) \Gamma(w) U^w dw,$$

we move the line of integration first to $\mathrm{Re}\, w = -\beta - \frac{1}{2}$ (at $w = 0$ we encounter a pole of the integrand with the residue $L(s, \chi)$), use the functional equation, retain the portion $n > U$ of the series for $L(1-s-w, \overline{\chi})$ and move the line of integration in the integral containing the portion $n \leqslant U$ to the line $\mathrm{Re}\, w = -\beta$.

LEMMA 3. *Write for a Dirichlet character χ and for real a, $Y \geqslant 2$*

$$S(a, \chi) = \sum_{1}^{\infty} e(-na) e^{-n/Y} \chi(n).$$

Then with $x = \dfrac{1}{Y} + 2\pi i a$ we have

$$S(a, \chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(w, \chi) \Gamma(w) x^{-w} dw,$$

where x^{-w} means the principal value of the power.

LEMMA 4. *Let*

$$T(a) = \sum_{0}^{N} e(na) e^{n/Y}.$$

Then

$$\sum_{|D| \leqslant X}' \sum_{n=1}^{\infty} \Big| \sum_{0 \leqslant \nu \leqslant N} \chi_D(n+\nu) \Big|^2 e^{-2n/Y} \leqslant \sum_{|D| \leqslant X}' \int_{0}^{1} |S(a, \chi_D) T(a)|^2 da.$$

Lemma 3 is verified by Mellin's transformation and Lemma 4 by interpreting arithmetically the integral on the right.

3. Proof of Theorem 1. If $T < 1$, then (1.1) follows from (1.3) by partial summation. So we assume that $T \geqslant 1$. If d is a fundamental discriminant, then χ_d is primitive, and by Lemma 2

$$\sum_{|d| \leqslant X}^{*} \int_{-T}^{T} |L(\tfrac{1}{2} + it, \chi_d)|^2 dt \ll \sum_{|d| \leqslant X}^{*} \int_{-T}^{T} \Big| \sum_{1}^{\infty} \chi_d(n) n^{-(1/2+it)} e^{-n/U} \Big|^2 dt + \sum_{|d| \leqslant X}^{*} \int_{-T}^{T} \Big\{ \int_{-\infty}^{\infty} |\psi(-\beta + it + i\tau, \chi_d)| \Big| \sum_{n > U} \chi_d(n) n^{-1-\beta+i(t+\tau)} \Big| \times |\Gamma(-\beta - \tfrac{1}{2} + i\tau)| U^{-\beta-1/2} d\tau \Big\}^2 dt + \sum_{|d| \leqslant X}^{*} \int_{-T}^{T} \Big\{ \int_{-\infty}^{\infty} |\psi(\tfrac{1}{2} - \beta + it + i\tau, \chi_d)| \times \Big| \sum_{n \leqslant U} \chi_d(n) n^{-1/2-\beta+i(t+\tau)} \Big| |\Gamma(-\beta + i\tau)| U^{-\beta} d\tau \Big\}^2 dt = A + B + C,$$

say. Choose $U = (XT)^{1/2}$, $\beta = \big(\log(XT)\big)^{-1}$.

Splitting up the series in A into sums over intervals of the type $[2^j, 2^{j+1})$, $j = 0, 1, \ldots$, and using Lemma 1 to each of these sums we have

$$A \ll XT \log(XT) \sum_{j=0}^{\infty} \sum_{\substack{2^j \leqslant m,\, n < 2^{j+1} \\ mn = a^2}} (mn)^{-1/2} e^{-(m+n)/U} + (XT)^{1/2} \log(XT) \sum_{j=0}^{\infty} (2^j)^{15/8} (2^j)^{-7/8} e^{-2^{j+1}/U} \log^7(2^j),$$

so that

$$A \ll XT\log^8(XT). \tag{3.1}$$

In B and C the integral over $|\tau| \geqslant \log^2(XT)$ is negligible. Consider the remaining terms. In C we have

$$\psi(\tfrac{1}{2}-\beta+it+i\tau, \chi_d)\Gamma(-\beta+i\tau)U^{-\beta} \ll \log(XT),$$

and estimating the integral over t and the sum over d as above, we obtain

$$C \ll XT\log^{14}(XT). \tag{3.2}$$

To estimate B note that

$$\{\psi(-\beta+it+i\tau, \chi_d)\Gamma(-\beta-\tfrac{1}{2}+i\tau)U^{-\beta-1/2}\}^2 \ll (XT)^{1/2},$$

and that by Lemma 1

$$\sum_{|d|\leqslant X}{}^{*}\int_{-T}^{T}\Big|\sum_{n>U}\chi_d(n)n^{-1-\beta+i(t+\tau)}\Big|^2 dt \ll (XT)^{1/2}\log^8(XT).$$

Hence $B \ll XT\log^{12}(XT)$. Combining this with (3.1)–(3.2) we obtain

$$\sum_{|d|\leqslant X}{}^{*}\int_{-T}^{T}|L(\tfrac{1}{2}+it, \chi_d)|^2 dt \ll XT\log^{14}(XT). \tag{3.3}$$

For the proof of (1.1) note that $D = da^2$, where d is a (uniquely determined) fundamental discriminant, and that then

$$L(s, \chi_D) = L(s, \chi_d)\prod_{p|a}(1-\chi_d(p)p^{-s}).$$

So

$$|L(\tfrac{1}{2}+it, \chi_D)|^2 \leqslant |L(\tfrac{1}{2}+it, \chi_d)|^2\prod_{p|a}(1+p^{-1/2})^2,$$

where

$$\prod_{p|a}(1+p^{-1/2})^2 \leqslant \Big(\sum_{\delta|a}\delta^{-1/2}\Big)^2 \ll \tau(a)\sum_{\delta|a}\delta^{-1} \ll \tau(a)\log X.$$

Hence (1.1) follows easily from (3.3).

4. Proof of Theorem 2. We may suppose that $Y \leqslant X$, for the general case follows from this by the periodicity of the characters. By Lemma 4 we have to estimate the expression

$$\sum_{|D|\leqslant X}{}'\int_0^1 |S(\alpha, \chi_D)|^2|T(\alpha)|^2 d\alpha. \tag{4.1}$$

The sum $T(\alpha)$ is estimated as follows:

$$T(\alpha) \ll \min(N, \|\alpha\|^{-1}), \tag{4.2}$$

where $\|\alpha\|$ means the distance of α from the nearest integer.

By periodicity we may integrate in (4.1) also over $[-\frac{1}{2}, \frac{1}{2}]$. Consider separately the integrals over $[-4Y^{-1}, 4Y^{-1}]$ and $4Y^{-1} \leqslant |\alpha| \leqslant \frac{1}{2}$. In estimating these we shall make use of the calculations in Linnik's paper [3].

If $|\alpha| \leqslant 4Y^{-1}$, then by Lemma 3 we have, on moving the integration to the line $\operatorname{Re} w = \frac{1}{2}$

$$\sum_{|D|\leqslant X}{}'|S(\alpha, \chi_D)|^2 \ll \sum_{|D|\leqslant X}{}'\Big|\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(w, \chi_D)\Gamma(w)x^{-w}dw\Big|^2. \tag{4.3}$$

Using the estimates

$$x^{-1/2-it} = |x|^{-1/2-it}\exp\Big\{(t-\tfrac{1}{2}i)\Big(\frac{\pi}{2}-\arctan\Big(\frac{1}{2\pi Y\alpha}\Big)\Big)\Big\} \tag{4.4}$$

$$\ll |x|^{-1/2}\exp\Big\{t\Big(\frac{\pi}{2}-\arctan\Big(\frac{1}{2\pi Y\alpha}\Big)\Big)\Big\},$$

$$\Gamma(\tfrac{1}{2}+it) \ll \exp(-\pi|t|/2) \tag{4.5}$$

as well as (1.1), we see that if $|\alpha| \leqslant 4Y^{-1}$, then

$$\sum_{|D|\leqslant X}{}'|S(\alpha, \chi_D)|^2 \ll XY\log^{16}X.$$

Hence

$$\sum_{|D|\leqslant X}{}'\int_{-4Y^{-1}}^{4Y^{-1}}|S(\alpha, \chi_D)|^2|T(\alpha)|^2 d\alpha \ll XN^2\log^{16}X \leqslant XYN\log^{16}X. \tag{4.6}$$

Now consider the integral over $[4Y^{-1}, \frac{1}{2}]$. Subdivide this interval into subintervals $[2^jY^{-1}, 2^{j+1}Y^{-1}]$, $j = 2, 3, \ldots$ (the last interval is possibly incomplete) and consider the integral

$$I_j = \sum_{|D|\leqslant X}{}'\int_{\alpha_j}^{\alpha_{j+1}}|S(\alpha, \chi_D)|^2 d\alpha, \tag{4.7}$$

where $\alpha_j = \min(2^jY^{-1}, \frac{1}{2})$. Using (4.4)–(4.5) it is seen by calculation that for $s_k = \frac{1}{2}+it_k$, $t_k \geqslant 0$, $k = 1, 2$

$$\int_{\alpha_j}^{\alpha_{j+1}} x^{-s_1}\overline{x^{-s_2}}\Gamma(s_1)\overline{\Gamma(s_2)}\,d\alpha \tag{4.8}$$

$$\ll \min\Big(\frac{1}{|t_1-t_2|}, 1\Big)\exp\Big\{-(t_1+t_2)\arctan\Big(\frac{1}{2\pi Y\alpha_{j+1}}\Big)\Big\}$$

(see [3], § 5). Similarly, if $t_k < 0$, $k = 1, 2$, this integral is

$$\ll \min\Big(\frac{1}{|t_1-t_2|}, 1\Big)\exp\Big\{-(|t_1|+|t_2|)\Big(\pi-\arctan\Big(\frac{1}{2\pi Y\alpha_j}\Big)\Big)\Big\}. \tag{4.9}$$

By (4.7) and (4.3) I_j is estimated by a sum of triple integrals with respect to the variables a, t_1, t_2, where t_1 and t_2 are of the same sign. Integrating first over a using (4.8)–(4.9) and then over t_1 and t_2 using (1.1), we obtain

$$I_j \ll XYa_j \log^{17} X.$$

Hence in view of (4.2)

$$\sum_{|D|\le X}' \int_{4Y^{-1}}^{1/2} |S(a, \chi_D)|^2 |T(a)|^2 da \ll \sum_j I_j \min(N^2, a_j^{-2}) \ll XYN\log^{17} X.$$

This combined with (4.6) proves (1.2).

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The moments of partitions, I

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1. Let $p_m(n)$ denote the number of partitions of n into m parts. D. H. Lehmer [7] has considered calculating $t_k(n)$, the kth moments of $p_m(n)$ defined for $k = 0, 1, \ldots$ by

$$t_k(n) = \sum_{m=1}^{n} m^k p_m(n).$$

The purpose of this paper is the determination of the asymptotic behaviour of the $t_k(n)$ for arbitrary fixed k as $n \to \infty$.

$p_m(n)$ is the number of microstates of a Bose–Einstein gas of m particles and of energy n distributed over the energy levels ($\varepsilon = 1, 2, \ldots$) [5]. It is hoped that the following results are of interest in statistical mechanics as well as in number theory. The first moment has been considered by Husimi [5] and there are certain similarities in method between [5] and this paper. However, we shall avoid using the transformation equation for the generating function of the $t_0(n)$.

We require the generating function for the $t_k(n)$ and we give the derivation of D. H. Lehmer [7]. It is known that ([1], p. 193)

$$G(x, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_m(n) x^n z^m = \prod_{r=1}^{\infty} (1 - zx^r)^{-1}. \tag{1.1}$$

If we introduce the operator

$$\theta = z\frac{\partial}{\partial z},$$

then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m^k p_m(n) x^n z^m = \theta^k G. \tag{1.2}$$

Now

$$\theta G = G \sum_{r=1}^{\infty} \frac{zx^r}{1 - zx^r}, \tag{1.3}$$