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On the distribution of values of additive functions*

by

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In 1939 P. Erdös and A. Wintner [2] found the necessary and sufficient conditions for the existence of a limit distribution of additive functions centred about 0 with normalization 1, and proved the following:

THEOREM A. *The necessary and sufficient condition for*

$$\frac{1}{N}\sum_{\substack{n\leqslant N\\ g(n)\leqslant x}} 1 \to F(x)$$

in all points of continuity of $F(x)$, *is that the series*

$$\sum_p \frac{\|g(p)\|^2}{p} \quad \text{and} \quad \sum_p \frac{\|g(p)\|}{p},$$

where

$$\|u\| = \begin{cases} -1 & \text{for} \quad u < -1,\\ u & \text{for} \quad -1\leqslant u\leqslant 1,\\ 1 & \text{for} \quad u>1,\end{cases}$$

are convergent.

In 1971, working independently, Elliott and Ryavec [1], Delange, and the authors of this work [5], proved the following:

THEOREM B. *The necessary and sufficient condition for*

$$\frac{1}{N}\sum_{\substack{n\leqslant N\\ g(n)-A_N\leqslant x}} 1 \to F(x),$$

A_N *arbitrary, is that there exists a real* b *such that*

$$\sum_p \frac{\|g(p)-b\log p\|^2}{p}$$

* The brief summary was published in Успехи матем. наук 28 (1973), No. 1 (169), pp. 243–244.

is convergent and

$$A_N = b\log N + \sum_{p\leqslant N} \frac{\|g(p)-b\log p\|}{p} + C + o(1).$$

In [5] it is proved that in the case of existence of a limit distribution of $(g(n)-A_N)/B_N$, normalization B_N should have either a finite or an infinite limit. The case of finite limit can be finally solved applying the above-mentioned result obtained by Elliott, Ryavec, Delange, and the authors of the present work, and the case $B_N\to\infty$ has not yet been definitely examined. In [5] the case $A_N = 0$ and B_N "regularly" increasing slower than any positive power of $\log N$, is considered. In particular, the necessary and sufficient conditions were found for the existence of a limit distribution for $g(n)/B_N$ if $B_{N^u}/B_N\to 1$ uniformly in u in any closed subinterval of $(0, 1]$, so that $B_N = L(\log N)$, where $L(u)$ is slowly oscillating.

In the present work the necessary and sufficient conditions have been found for the existence of a limit distribution of $g(n)/B_N$ in the case where B_N "regularly" changing, increases not quicker than some power of $\log N$. More precisely, $B_{N^u}/B_N\to\varphi(u)$ uniformly in u in any closed subinterval of $(0, 1]$. The last condition implies that $B_N = \log^m N\times L(\log N)$, where $L(u)$ is a slowly oscillating function. It is interesting to note that these conditions are analogous to those of Erdös and Wintner, and they become identical if we put $B_N = 1$.

If a limit distribution of $(g(n)-A_N)/B_N$ exists then, as shown in Corollary of Theorem 1′, there exists a constant m such that

$$\Delta_N \overset{\text{def}}{=} \max(|A_N|, |B_N|) \leqslant \log^m N.$$

In particular, if a limit distribution of $g(n)/B_N$ exists, then $B_N = O(\log^m N)$. Thus for "regularly" changing and "rapidly" increasing normalization, the question of the existence of a limit distribution of $g(n)/B_N$ is fully solved in this work. Exactly the same results are obtained for the quantities $(g(n)-A_N)/B_N$, assuming that $A_N = b\log N + A_N^*$ and $A_{N^u}^* - A_N^* = \psi(u)B_N + o(B_N)$.

It should be noted that the case of slowly oscillating B_N and the case $B_N = \log^m N\cdot L(\log N)$ (for $m>0$), are different. In the first case, if there exists the limit distribution, then

$$\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}} \frac{\log p}{p} \sim \log N,$$

which means that $g(p) = o(B_N)$ for "almost all" $p\leqslant N$, or, in probability terminology, that each component $g(p)/B_N$ is asymptotically negligible. In the second case,

$$\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}} \frac{\log p}{p} \sim \tau(\xi)\log N,$$

where $\tau(\xi)$, generally speaking, is not identically equal to 1, which means that $g(p)/B_N$ are not asymptotically negligible. To illustrate this case one can take as an example the additive function $g_m(n) = \sum_{p^a\| n}\log^m p^a$, with $m\neq 1$, $m>0$.

J. P. Kubilius called our attention to the fact that the problem of finding the limit distribution of $(g_m(n)-A_N)/B_N$ was posed by P. Erdös. From Theorem 4 (see also Lemma 4, § 5) it follows that $g_m(n)/\log^m N$ has a limit distribution with a characteristic function equal to

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{e^z}{z}\exp\Big(\int_0^1 \frac{e^{i\xi u^m}-1}{u} e^{-uz}du\Big)dz.$$

Note that $g_m(p)/\log^m N$ are not asymptotically negligible, since for $m>0$ we have

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\left(\frac{\log p}{\log N}\right)^m} \frac{\log p}{p} = \int_0^1 e^{i\xi u^m}du + o(1) \not\equiv 1+o(1).$$

In § 5 a class of additive functions such that for $(g(n)-A_N)/B_N$ the limit distribution does not exist for any A_N and B_N, has been singled out. For this, it is sufficient that

$$\sum_{p\leqslant N}\left\|\frac{g(p)}{\log^m N}\right\|^2\cdot\frac{1}{p} = \Omega(1)$$

for any $m>0$.

Throughout this work, B_N is a sequence of real numbers $B_x = B_{[x]}$, all estimates containing ξ, unless otherwise stated, are uniform in ξ in an arbitrary interval of the form $|\xi|\leqslant C$ where C is an arbitrary constant, $g(n)$ is an additive function, symbols O_K and o_K denote that the constants represented by them depend on K, and

$$E(u) = \begin{cases} 1 & \text{if} \quad u>0, \\ 0 & \text{if} \quad u\leqslant 0. \end{cases}$$

In the following, without any loss of generality, we will assume that $B_N>0$, in the case where the limit law for $(g(n)-A_N)/B_N$ is symmetric. In fact let $\{N^+\}$ and $\{N^-\}$ be two sequences on which $B_{N^+}>0$ and $B_{N^-}<0$, respectively.

Putting

$$F_N(u) = \frac{1}{N}\sum_{\substack{n\leqslant N\\ \frac{g(n)-A_N}{B_N}\leqslant u}} 1 \quad \text{and} \quad \Phi_N(u) = \frac{1}{N}\sum_{\substack{n\leqslant N\\ \frac{g(n)-A_N}{|B_N|}\leqslant u}} 1,$$

we obtain

$$F_{N^+}(u) = \Phi_{N^+}(u) \quad \text{and} \quad F_{N^-}(u) = 1-\Phi_{N^-}(-u)+\frac{1}{N^-}\sum_{\substack{n\leqslant N^- \\ \frac{g(n)-A_{N^-}}{B_{N^-}}=u}} 1.$$

Hence $(g(n)-A_N)/B_N$ and $(g(n)-A_N)/|B_N|$ have or do not have symmetric limit distributions simultaneously.

1. Rapidly increasing normalizations. Since an additive function is defined by its values on the powers of prime numbers, and since, in the question of existence of limit distribution, powers of primes greater than 1 do not play any part, one expects that the existence of a limit distribution should, "on average", be followed by limitations on the values of an additive function in primes. To establish this we apply a simple method analogous to the classical method of finding the asymptotic values for $\sum_{p\leqslant x}\log p/p$.

LEMMA 1. *Let $(g(n)-A_N)/B_N$ have a limit distribution with the characteristic function $\tau(\xi)$, and let B_{N^u}/B_N be uniformly bounded in u, $u\leqslant 1$, then*

$$(1)\qquad \frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}\,\tau\Big(\xi\frac{B_{N/p}}{B_N}\Big)e^{i\xi\frac{A_{N/p}}{B_N}} = \tau(\xi)e^{i\xi\frac{A_N}{B_N}}+o(1).$$

Proof. Since

$$\tau_N(\xi) \overset{\text{def}}{=} \frac{1}{N}\sum_{n\leqslant N} e^{i\xi\frac{g(n)-A_N}{B_N}} = \tau(\xi)+o(1),$$

then

$$\frac{1}{N}\sum_{n\leqslant N} e^{i\xi\frac{g(n)}{B_N}}\log n = \big(\tau(\xi)e^{i\xi\frac{A_N}{B_N}}+o(1)\big)\log N + O\Big(\frac{1}{N}\sum_{n\leqslant N}\log\frac{N}{n}\Big)$$
$$= \tau(\xi)e^{i\xi\frac{A_N}{B_N}}\log N+o(\log N).$$

On the other hand, using the additivity of $\log n$, we obtain

$$\frac{1}{N}\sum_{n\leqslant N} e^{i\xi\frac{g(n)}{B_N}}\log n = \sum_{p^\alpha\leqslant n} e^{i\xi\frac{g(p^\alpha)}{B_N}}\frac{\log p^\alpha}{p^\alpha}\,\frac{1}{(N/p^\alpha)}\sum_{\substack{n\leqslant N/p^\alpha\\ (n,p)=1}} e^{i\xi\frac{g(n)}{B_N}}$$
$$= \sum_{p^\alpha\leqslant N} e^{i\xi\frac{g(p^\alpha)}{B_N}}\frac{\log p^\alpha}{p^\alpha}\,\frac{1}{(N/p^\alpha)}\Big(\sum_{n\leqslant N/p^\alpha} e^{i\xi\frac{g(n)}{B_N}} - \sum_{n\leqslant N/p^{\alpha+1}} e^{i\xi\frac{g(n\cdot p)}{B_N}}\Big)$$
$$= \sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}\,\tau_{N/p}\Big(\xi\frac{B_{N/p}}{B_N}\Big)e^{i\xi\frac{A_{N/p}}{B_N}} + O\Big(\sum_{\substack{p^\alpha\leqslant N\\ \alpha\geqslant 2}}\frac{\log p^\alpha}{p^\alpha}\Big)$$
$$= \sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}\,\tau\Big(\xi\frac{B_{N/p}}{B_N}\Big)e^{i\xi\frac{A_{N/p}}{B_N}} + o(\log N).$$

In passing from $\tau_{N/p}(\xi)$ to $\tau(\xi)$ the boundedness of $B_{N/p}/B_N$ and the existence of limit distribution have been used. By equating the above equalities we complete the proof of the lemma.

Lemma 1 is particularly useful when B_{N^u}/B_N tends to 0 or 1, and plays an important part especially in proving the impossibility of existence of a limit distribution with rapidly increasing normalizations.

THEOREM 1. *$(g(n)-A_N)/B_N$ does not have a proper limit distribution for any A_N if B_N is such that $\lim_{N\to\infty} B_{N^a}/B_N = 0$ uniformly in a in any closed subinterval of $(0,1)$.*

Proof. Let $(g(n)-A_N)/B_N$ have a limit distribution. For fixed $M\geqslant 2$ we have

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}\,\tau\Big(\xi\frac{B_{N/p}}{B_N}\Big)e^{i\xi\frac{A_{N/p}}{B_N}}$$
$$= \frac{1}{\log N}\sum_{N^{1/M}\leqslant p\leqslant N^{1-1/M}} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}\big(e^{i\xi\frac{A_{N/p}}{B_N}}+o_M(1)\big)+$$
$$+O\Big(\frac{1}{\log N}\sum_{p\leqslant N^{1/M}}\frac{\log p}{p}\Big)+O\Big(\frac{1}{\log N}\sum_{N^{1-1/M}\leqslant p\leqslant N}\frac{\log p}{p}\Big)$$
$$= \frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}e^{i\xi\frac{A_{N/p}}{B_N}}+O\Big(\frac{1}{M}\Big)+o_M(1).$$

Since $M\geqslant 2$ is arbitrary, then from (1)

$$(2)\qquad \frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p}e^{i\xi\frac{A_{N/p}}{B_N}} = \tau(\xi)e^{i\xi\frac{A_N}{B_N}}+o(1).$$

We will show that (2) contradicts the assumption of the existence of a proper limit distribution. If $A_N/B_N = O(1)$, then

$$\frac{A_{N^a}}{B_N} = \frac{A_{N^a}}{B_{N^a}}\cdot\frac{B_{N^a}}{B_N}\to 0, \quad \text{for} \quad N\to\infty,$$

uniformly in the above interval. Hence, proceeding in the same way as for the derivation of (2), we obtain

$$(3)\qquad \frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p} = \tau(\xi)e^{i\xi\frac{A_N}{B_N}}+o(1).$$

If, however, $A_N/B_N\neq O(1)$, then we choose such a sequence N_l that $\left|\frac{A_{N_l}}{B_{N_l}}\right|$ increasing monotonously, tends to infinity, and for all $N\leqslant N_l$ the inequality $\left|\frac{A_N}{B_N}\right|\leqslant\left|\frac{A_{N_l}}{B_{N_l}}\right|$ holds. Putting $\xi = \eta\frac{B_{N_l}}{A_{N_l}}$ in (2), we obtain

$$(4)\qquad \frac{1}{\log N_l}\sum_{p\leqslant N_l} e^{i\eta\frac{g(p)}{\Delta_{N_l}}}\frac{\log p}{p} = e^{i\eta}+o(1),$$

since $\tau(0) = 1$ and

$$\frac{A_{N_l^a}}{A_{N_l}} = \frac{A_{N_l^a}}{B_{N_l^a}}\cdot\frac{B_{N_l}}{A_{N_l}}\cdot\frac{B_{N_l^a}}{B_{N_l}} = \frac{B_{N_l^a}}{B_{N_l}}\cdot\frac{A_{N_l^a}}{B_{N_l^a}}\Big/\frac{A_{N_l}}{B_{N_l}}\to 0$$

uniformly in a in the above interval. We will show that the left-hand sides of (3) and (4) tend to 1 for any ξ. In fact, putting $\gamma = 1-1/M$, by (2) we have

$$\frac{1}{\log N_l}\sum_{p\leqslant N_l} e^{i\xi\frac{g(p)}{C_{N_l}}}\frac{\log p}{p} = \frac{\gamma}{\log N_l^\gamma}\sum_{p\leqslant N_l^\gamma} e^{i\xi\frac{g(p)}{C_{N_l^\gamma}}\cdot\frac{C_{N_l^\gamma}}{C_{N_l}}}\frac{\log p}{p}+O\left(\frac{1}{M}\right)$$

$$= \begin{cases} \gamma e^{i\xi\frac{A_{N_l^\gamma}}{B_{N_l}}}+O\left(\frac{1}{M}\right)+o_M(1) & \text{if}\quad C_N = B_N,\\ \gamma e^{i\xi\frac{A_{N_l^\gamma}}{A_{N_l}}}+O\left(\frac{1}{M}\right)+o_M(1) & \text{if}\quad C_N = A_N. \end{cases}$$

Hence in the first case,

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p} = 1+o_M(1)+O\left(\frac{1}{M}\right),$$

which contradicts (3) if there exists the proper limit distribution ($\tau(\xi) \not\equiv e^{i\xi a}$). In the second case we obtain

$$\frac{1}{\log N_l}\sum_{p\leqslant N_l} e^{i\eta\frac{g(p)}{A_{N_l}}}\frac{\log p}{p} = 1+o_M(1)+O\left(\frac{1}{M}\right),$$

which contradicts (4).

Remark 1. Improper limit distribution can exist only in the case when A_N/B_N has a limit; for $A_N = 0$ limit distribution is centred about 0.

Remark 2. If A_N/B_N is bounded and if N_l is a subsequence of the sequence of positive integers, then $(g(n)-A_{N_l})/B_{N_l}$ does not have the proper limit distribution if $B_{N_l^a}/B_{N_l}\to 0$ uniformly in a in any closed subinterval of $(0, 1)$.

Analogous considerations permit us, in the case of existence of a limit distribution of $(g(n)-A_N)/B_N$, to find limitations not only on B_N but also on the behaviour of $\Delta_N = \max(|B_N|, |A_N|)$.

THEOREM 1'. *If on any subsequence of the sequence of positive integers* $\lim_{N_l\to\infty}\Delta_{N_l^a}/\Delta_{N_l} = 0$ *uniformly in* a *in any closed subinterval of* $(0, 1)$, *then* $(g(n)-A_N)/B_N$ *does not have the proper limit distribution with* B_N *and* A_N.

Proof. Replacing ξ in (1) by $\xi\cdot B_N/\Delta_N$, we obtain

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{\Delta_N}}\frac{\log p}{p}\,\tau\left(\xi\,\frac{B_{N/p}}{\Delta_{N/p}}\,\frac{\Delta_{N/p}}{\Delta_N}\right)e^{i\xi\frac{A_{N/p}}{\Delta_{N/p}}\frac{\Delta_{N/p}}{\Delta_N}}$$

$$= \tau\left(\xi\,\frac{B_N}{\Delta_N}\right)e^{i\xi\frac{A_N}{\Delta_N}}+o(1).$$

Since B_N/Δ_N, A_N/Δ_N are bounded and $\Delta_{N_l^a}/\Delta_{N_l}\to 0$, then applying the same considerations as used in the proof of (2), we have

$$\frac{1}{\log N_l}\sum_{p\leqslant N_l} e^{i\xi\frac{g(p)}{\Delta_{N_l}}}\frac{\log p}{p} = \tau\left(\xi\,\frac{B_{N_l}}{\Delta_{N_l}}\right)e^{i\xi\frac{A_{N_l}}{\Delta_{N_l}}}+o(1).$$

In the analogous situation in the proof of Theorem 1

$$\frac{1}{\log N_l}\sum_{p\leqslant N_l} e^{i\xi\frac{g(p)}{\Delta_{N_l}}}\frac{\log p}{p} = 1+o(1)$$

was obtained, which contradicts the last formula if the limit distribution is proper.

COROLLARY. *If* $(g(n)-A_N)/B_N$ *has the proper limit distribution there exists a constant* m *such that*

$$\Delta_N = \max(|A_N|, |B_N|) \leqslant \log^m N.$$

Proof. Let

$$\frac{\log\Delta_N}{\log\log N} = \varrho(N),$$

and $\varrho(N)$ be unbounded. Now let us choose the sequence N_l so that $\varrho(N_l)\to\infty$, $\varrho(N)\leqslant\varrho(N_l)$ for $N\leqslant N_l$. Then for $u\leqslant 1$

$$\frac{\Delta_{N_l^u}}{\Delta_{N_l^u}} = \frac{(\log N_l^u)^{\varrho(N_l^u)}}{(\log N_l)^{\varrho(N_l)}}\leqslant u^{\varrho(N_l)}\to 0$$

for $N_l\to\infty$ uniformly in u in any closed subinterval of $(0, 1)$. Therefore, due to the above theorem, the limit distribution with such Δ_N does not exist for any additive function. Since $(g(n)-A_N)/B_N$ has a proper limit distribution, $\varrho(N)$ is bounded.

2. The necessary conditions. As seen from formula (3), the existence of a limit distribution of $g(n)/B_N$ in the case of B_{N^u}/B_N tending to 0, implies that the limit distribution exists for $g(p)/B_N$ with the weight

$\log p/p$. It becomes evident that in the more general case when $B_{N^u}/B_N \to \varphi(u)$, the existence of a limit distribution of $g(n)/B_N$ implies that the limit distribution exists for $g(p)/B_N$ with the same weight. However, in the case $\varphi(u) \not\equiv 0, 1$ more complicated considerations than in the last section are required.

THEOREM 2. *If there exists a limit distribution, proper or improper, for $g(n)/B_N$ and if $B_{N^a}/B_N \to \varphi(a)$ uniformly in a in any closed subinterval of $(0, +\infty)$, then there exist a number d and non-decreasing bounded functions $L_l(u)$, $l = 0, 1, 2, \ldots$, $L_l(\pm\infty) = \lim_{u\to\pm\infty} L_l(u)$ such that, in all points of continuity of $L_l(u)$,*

(5) $$\sum_{\substack{p\leqslant N\\ g(p)\leqslant uB_N}} \left\|\frac{g(p)}{B_N}\right\|^2 \frac{1}{p} \to L_0(u),$$

(6) $$\sum_{p\leqslant N} \left\|\frac{g(p)}{B_N}\right\| \frac{1}{p} \to d$$

and for $l = 1, 2, 3, \ldots$,

(7) $$\frac{l}{\log^l N} \sum_{\substack{p\leqslant N\\ g(p)\leqslant uB_N}} \frac{\log^l p}{p} \to L_l(u).$$

Proof. Suppose that

$$F(s) = \sum_{n=1}^{\infty} e^{i\xi\frac{g(n)}{B_N}} \frac{1}{n^s}$$

where

$$s = 1 + \frac{1}{\log N} + \frac{it}{\log N}, \quad |t| \leqslant K_1(N) = o(\log N).$$

By partial summation we obtain

$$F(s) = s\int_1^\infty \frac{1}{x}\sum_{n\leqslant x} e^{i\xi\frac{g(n)}{B_N}} \frac{dx}{x^s} = \log N \int_0^\infty \frac{1}{N^u} \sum_{n\leqslant N^u} e^{i\xi\frac{g(n)}{B_{N^u}}\cdot\frac{B_{N^u}}{B_N}} \frac{du}{e^{uz}} + O(K_1(N)),$$

where $z = 1 + it$. Since

$$\int_0^{1/K} \tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right)\frac{du}{e^{uz}} = O\left(\frac{1}{K}\right) \quad\text{and}\quad \int_K^\infty \tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right)\frac{du}{e^{uz}} = O\left(\frac{1}{K}\right)$$

we get

$$\frac{1}{\log N} F(s) = \int_{1/K}^{K} \tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right)\frac{du}{e^{uz}} + O\left(\frac{1}{K}\right) + o_K(1).$$

Taking into consideration $B_{N^u}/B_N \to \varphi(u)$ uniformly in u in the interval $[1/K, K]$, $\tau_N(\xi) \to \tau(\xi)$, and the boundedness of $\varphi(u)$ (see Introduction),

we find

$$\tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right) = \tau(\xi\varphi(u)) + o(1).$$

Consequently

$$\frac{F(s)}{\log N} = \int_{1/K}^{K} \tau(\xi\varphi(u))\frac{du}{e^{uz}} + O\left(\frac{1}{K}\right) + o_K(1),$$

from which, since K is arbitrary, we obtain

$$\frac{F(s)}{\log N} = \int_0^\infty \tau(\xi\varphi(u))\frac{du}{e^{uz}} + o(1).$$

Using $\zeta(s) \sim 1/(s-1)$ we have, for our s, for $|t| \leqslant K_1$ where $K_1 > 0$ is a constant,

$$\frac{F(s)}{\zeta(s)} = z\int_0^\infty \tau(\xi\varphi(u))\frac{du}{e^{uz}} + o(1)$$

uniformly in t. On the other hand, by the multiplicativity of $e^{i\xi\frac{g(n)}{B_N}}$, and by $B_N \to \infty$,

$$\frac{F(s)}{\zeta(s)} = (1+o(1))\exp\left(\sum_p \left(e^{i\xi\frac{g(p)}{B_N}} - 1\right) p^{-1-\frac{z}{\log N}}\right).$$

Thus

(8) $$\sum_p \left(e^{i\xi\frac{g(p)}{B_N}} - 1\right)p^{-1-\frac{z}{\log N}} = \log\left(z\int_0^\infty \tau(\xi\varphi(u))\frac{du}{e^{uz}} + o(1)\right).$$

Continuation of the proof consists of obtaining an estimate for the sum of coefficients of the Dirichlet series on the left-hand side of (8). For this, its behaviour as a function of z should be analysed first. The required information, in a small neighbourhood of $\xi = 0$, can be directly derived from (8). For $|z| \leqslant K_1$, $a = 1/K_1$ and $\sigma_N = 1 + 1/\log N$, we have

$$\sum_p \left(e^{i\xi\frac{g(p)}{B_N}} - 1\right)p^{-\sigma_N}\left(p^{-\frac{it}{\log N}} - 1\right)$$
$$= O\left(\sum_{p\leqslant N^a} \left|e^{i\xi\frac{g(p)}{B_N}} - 1\right| \frac{\log p}{p}\frac{|z|}{\log N}\right) + O\left(\sum_{N^a<p\leqslant N} \frac{1}{p}\right)$$
$$= O\left(\frac{|z|}{K_1}\right) + O(\log K_1) = O(\log K_1).$$

Therefore, for $|z| \leqslant K_1$,

(9) $$\sum_p \left(e^{i\xi\frac{g(p)}{B_N}} - 1\right)p^{-1-\frac{z}{\log N}} = \sum_p \left(e^{i\xi\frac{g(p)}{B_N}} - 1\right)p^{-\sigma_N} + O(\log K_1).$$

From (8), for $t = 0$ we have

$$\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-\sigma_N} = \log\Big(\int_0^\infty \tau\big(\xi\varphi(u)\big)\frac{du}{e^u}+o(1)\Big).$$

Since $\psi(\xi) \overset{\text{def}}{=} \int_0^\infty \tau\big(\xi\varphi(u)\big)\frac{du}{e^u}$ is the function continuous in ξ and $\psi(0) = 1$, there exists $\xi_0 > 0$ such that $|\psi(\xi)| > 1/2$ for $|\xi| \leqslant \xi_0$. Hence for $|\xi| \leqslant \xi_0$ uniformly in ξ,

(10) $$\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-\sigma_N} = O(1).$$

From (9) and (10) it follows the existence of $\xi_0 > 0$ independent of K_1, such that

(11) $$\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1-\frac{z}{\log N}} = O(\log K_1)$$

uniformly in ξ for $|\xi| \leqslant \xi_0$ and $|z| \leqslant K_1$. Therefore, for $|\xi| \leqslant \xi_0$, the first component in the argument of logarithm is $\neq 0$ for arbitrary z, hence

(12) $$\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1-\frac{z}{\log N}} = C(z, \xi)+o(1)$$

uniformly in z and ξ for $|z| \leqslant K_1$ and $|\xi| \leqslant \xi_0$, where

$$C(z, \xi) = \log\Big(z\int_0^\infty \tau\big(\xi\varphi(u)\big)\frac{du}{e^{uz}}\Big)$$

is the function continuous in ξ and z.

Thus we have proved that there exists a neighbourhood of the point $\xi = 0$ in which, for the series (8), the equation (12) is valid. For this goal it appeared to be sufficient to get the estimate (10). In the following, (10) and therefore (12) will be found, from other considerations, uniformly in ξ in the domain $|\xi| \leqslant C$, C arbitrary.

Let the estimate (12) hold uniformly in ξ for $|\xi| \leqslant \alpha$. Multiplying (12) by $\frac{1}{2\pi i}\frac{e^z}{z}$ and integrating in t from $-K$ to K, we obtain

(13) $$I \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}e^{-\frac{z\log p}{\log N}}dz = \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}C(z, \xi)\,dz+o_K(1).$$

On the other hand

$$I = \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1-\frac{z}{\log N}}dz + \sum\nolimits_2 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}\frac{1}{2\pi i}\int_{1-iK}^{1+iK}e^{z\frac{\log(N/p)}{\log N}}\frac{dz}{z}+O\Big(\sum\nolimits_3 \frac{1}{p}\int_{1-iK}^{1+iK}\frac{|dz|}{|z|}\Big),$$

where summations $\sum_1, \sum_2, \sum_3$ are taken respectively for $p \leqslant N^{1/K^2}$, $N^{1/K^2} \leqslant p \leqslant N^{1-1/\sqrt{K}}$ and $p \geqslant N^{1+1/\sqrt{K}}$, and for $N^{1-1/\sqrt{K}} < p < N^{1+1/\sqrt{K}}$. Since

$$\frac{1}{2\pi i}\int_{1-iK}^{1+iK}e^{z\frac{\log(N/p)}{\log N}}\frac{dz}{z} = O\Big(\frac{\log N}{Kp^{\sigma_N-1}|\log(N/p)|}\Big)+\begin{cases}1 & \text{if} \quad p < N,\\ 0 & \text{if} \quad p > N,\end{cases}$$

then

$$I = \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1-\frac{z}{\log N}}dz + \sum_{N^{1/K^2}\leqslant p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}+O\Big(\frac{\log K}{\sqrt{K}}\Big)$$

uniformly in ξ in the interval $|\xi| \leqslant \alpha$. The first integral on the right-hand side is equal to

$$\frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}dz\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1} + O\Big(\int_{1-iK}^{1+iK}\Big|\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)(p^{-\frac{z}{\log N}}-1)p^{-1}\Big|\frac{|dz|}{|z|}\Big)$$
$$= \sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}+O\Big(\frac{1}{K}\Big|\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}\Big|\Big)+O\Big(\frac{1}{K}\Big),$$

and since

$$\Big|\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}\Big| = \Big|\sum_p (e^{i\xi\frac{g(p)}{B_N}}-1)p^{-\sigma_N}-\sum_{p>N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-\sigma_N}- \sum_{p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}(p^{1-\sigma_N}-1)- \sum_{N^{1/K^2}\leqslant p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}\Big| = O(\log K)$$

(in the proof, (12) with $t = 0$ was used), it is also equal to

$$\sum\nolimits_1 (e^{i\xi\frac{g(p)}{B_N}} - 1)p^{-1} + O\left(\frac{\log K}{K}\right).$$

Hence

$$I = \sum_{p\leqslant N} (e^{i\xi\frac{g(p)}{B_N}} - 1)p^{-1} + O\left(\frac{\log K}{\sqrt{K}}\right)$$

uniformly in ξ for $|\xi| \leqslant a$. By this and by (13),

$$\psi_N(\xi) \stackrel{\text{def}}{=} \sum_{p\leqslant N} (e^{i\xi\frac{g(p)}{B_N}} - 1)p^{-1}$$
$$= \frac{1}{2\pi i}\int_{1-iK}^{1+iK} \frac{e^z}{z} C(z, \xi)\,dz + O\left(\frac{\log K}{\sqrt{K}}\right) + o_K(1).$$

Applying Cauchy's criterion we find that the sequence of continuous functions $\psi_N(\xi)$ converges uniformly in ξ for $|\xi| \leqslant a$. So that for $|\xi| \leqslant a$,

$$\psi_N(\xi) = \sum_{p\leqslant N} (e^{i\xi\frac{g(p)}{B_N}} - 1)p^{-1} = \psi(\xi) + o(1). \tag{14}$$

So far the last equality has been proved for $|\xi| \leqslant \xi_0$. On the other hand

$$\psi_N(\xi) = \int_{-\infty}^{+\infty} \frac{e^{i\xi u} - 1 - i\xi\|u\|}{\|u\|^2}\, d\sum_{\substack{p\leqslant N\\ g(p)\leqslant uB_N}} \left\|\frac{g(p)}{B_N}\right\|^2 \frac{1}{p} + \tag{15}$$
$$+ i\xi \sum_{p\leqslant N} \left\|\frac{g(p)}{B_N}\right\| \frac{1}{p},$$

from which, in the usual way (e.g. [3], XVII, § 1), we can prove that

$$\sum_{p\leqslant N} \left\|\frac{g(p)}{B_N}\right\|^2 \frac{1}{p} \quad \text{and} \quad \sum_{p\leqslant N} \left\|\frac{g(p)}{B_N}\right\| \frac{1}{p}$$

are bounded, and therefore (10), (12) and (14) hold uniformly in ξ in any interval $|\xi| \leqslant C$. From (14) and (15) follow (e.g. [3], XVII, § 1) (5) and (6). Condition (7) can be derived from the known relationship (14) between characteristic functions and distribution functions by partial summation.

3. The sufficient conditions. It turns out that the assumptions of Theorem 2 are also sufficient for the existence of a limit distribution of $g(n)/B_N$. The proof of this fact (Lemma 3) is largely based upon a minor generalization of one of the results of [5], in which the Halász method

[4] is used essentially. If in advance we assume that $B_{N^u}/B_N \to \varphi(u)$, then the sufficient conditions can be easily obtained from Lemma 3, and they appear to be less complicated (due to the lack of the condition (7)).

In order not to place limitations upon the generality of the choice of B_N, we will prove the following:

LEMMA 2. *Let $\tau(\xi) \not\equiv 1$ be a characteristic function and let D_N be a sequence such that D_{N^u}/D_N is uniformly bounded in u for $0 \leqslant u \leqslant 1$ ($D_N > 0$ if $\tau(\xi) = \tau(-\xi)$),*

$$\tau\left(\xi\frac{D_{N^u}}{D_N}\right) = \psi(\xi, u) + o(1) \tag{16}$$

uniformly in ξ and u for $|\xi| \leqslant C$, $0 < a \leqslant u \leqslant 1$, where $\psi(\xi, u)$ is continuous in u, then there exists the function $\varphi(u)$ for which

$$\frac{D_{N^u}}{D_N} \to \varphi(u)$$

uniformly in u in any closed subinterval of (0, 1]. *Note that either $\varphi(u) \equiv 0$ or $\varphi(u) \neq 0$ for $0 < u < 1$.*

Proof. Suppose that there exists u such that D_{N^u}/D_N does not have a limit, then

$$\varliminf_{N\to\infty} \frac{D_{N^u}}{D_N} = \varphi_1(u) < \varphi_2(u) = \varlimsup_{N\to\infty} \frac{D_{N^u}}{D_N}.$$

From (16) $\tau(\xi\varphi_1(u)) = \tau(\xi\varphi_2(u))$. Since the equality holds for all ξ,

$$\tau(\xi) = \tau\left(\xi\left(\frac{\varphi_1(u)}{\varphi_2(u)}\right)^q\right) \quad \text{and} \quad \tau(\xi) = \tau\left(\xi\left(\frac{\varphi_2(u)}{\varphi_1(u)}\right)^q\right)$$

for any integer $q \geqslant 0$. This does not hold for $\varphi_1(u)/\varphi_2(u) \neq -1$ since $\tau(\xi) \not\equiv 1$. But if $\varphi_1(u)/\varphi_2(u) = -1$, then $\tau(\xi) \equiv \tau(-\xi)$, in which case $D_N > 0$, and hence $0 < \varphi_1(u) < \varphi_2(u)$, which contradicts the assumption that $\varphi_1(u)/\varphi_2(u) = -1$. Therefore $D_{N^u}/D_N \to \varphi(u)$ for all $0 \leqslant u \leqslant 1$.

We will prove that $\varphi(u) \equiv 0$ or that is not equal to 0 for $0 < u \leqslant 1$. Assume that there exist points at which $\varphi(u) = 0$. Let $u_0 = \sup\limits_{\substack{0<u\leqslant 1\\ \varphi(u)=0}} u$, then for $u_0 < u \leqslant 1$, $\varphi(u) \neq 0$, and for $u < u_0$, $\varphi(u) = \varphi\left(\frac{u}{u_1}\cdot u_1\right)$ $= \varphi\left(\frac{u}{u_1}\right)\cdot\varphi(u_1) = 0$, if $u < u_1 < u_0$ and $\varphi(u_1) = 0$ (the multiplicative feature of $\varphi(u)$ for $0 \leqslant u \leqslant 1$ is a result of its definition). Thus

$$\varphi(u)\begin{cases} \equiv 0 & \text{if} \quad u < u_0, \\ \neq 0 & \text{if} \quad u_0 < u \leqslant 1. \end{cases}$$

On the other hand $D_N/D_{N^u} = 1/\varphi(u) + o(1)$ if $\varphi(u) \neq 0$.

Next, taking such a v that $u_0 < u_1 \cdot v < 1$, $1 < v < 1/u_0$, $u_1 < u_0$, we obtain

$$\varphi(u_1 \cdot v) + o(1) = \frac{D_{N^{u_1 \cdot v}}}{D_N} = \frac{D_{N^{u_1 \cdot v}}}{D_{N^v}} \cdot \frac{D_{N^v}}{D_{(N^v)^{1/v}}}$$
$$= \varphi(u_1) \cdot \frac{1}{\varphi(1/v)} + o(1) = o(1),$$

i.e. $\varphi(u_1 \cdot v) = 0$, which contradicts the choice of u_0. Therefore either $\varphi(u) \equiv 0$ for $u < 1$ or $\varphi(u) \neq 0$ for $0 < u \leqslant 1$.

We will show that $D_{N^u}/D_N \to \varphi(u)$ uniformly in u for $0 < a \leqslant u \leqslant 1$. Assume that D_{N^u}/D_N does not tend uniformly to $\varphi(u)$ in this interval; then such ε_0 and sequences N_l and u_l, where $0 < a \leqslant u_l \leqslant 1$, can be found that

$$\left| \frac{D_{N_l^{u_l}}}{D_{N_l}} - \varphi(u_l) \right| \geqslant \varepsilon_0 \tag{17}$$

and $u_l \to u_0$, $D_{N_l^{u_l}}/D_{N_l} \to b$ (because of the boundedness of D_{N^u}/D_N). By this and by (16),

$$\tau(\xi \cdot b) = \tau\big(\xi \cdot \varphi(u_0)\big),$$

but due to (17), $b \neq \varphi(u_0)$.

Repeating the considerations used above, we obtain a contradiction with $\tau(\xi) \not\equiv 1$.

LEMMA 3. *Let $f(n;\ N,\ \xi)$ be a function multiplicative in n satisfying the conditions*

$$|f(n;\ N,\ \xi)| \leqslant 1$$

and

$$\sum_{p \leqslant N} \frac{1 - \operatorname{Re} f(p;\ N,\ \xi)}{p} = O(1); \tag{18}$$

then

$$\frac{1}{N} \sum_{n \leqslant N} f(n;\ N,\ \xi) \tag{19}$$
$$= \frac{1}{2\pi i} \int_{1-iK}^{1+iK} \frac{e^z}{z} \prod_{p \leqslant N^{1/u}} \left(1 - \frac{1}{p e^{z \frac{\log p}{\log N}}}\right) \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r;\ N,\ \xi)}{p^r e^{z \frac{\log p^r}{\log N}}}\right) dz +$$
$$+ O\left(\frac{1}{K^{1/48}}\right) + o_K(1)$$

uniformly in u for $0 < \alpha \leqslant u \leqslant 1$ for all $a > 0$ and $K > 1$.

Proof. We define the multiplicative function $f_u(n;\ N,\ \xi)$ by

$$f_u(p^r;\ N,\ \xi) = \begin{cases} f(p^r;\ N,\ \xi) & \text{if} \quad p \leqslant N^{1/u}, \\ 1 & \text{if} \quad p > N^{1/u}. \end{cases}$$

The generating Dirichlet series for $f_u(n;\ N,\ \xi)$ will be denoted by $F(s)$ with $s = \sigma_N + it$. $f_u(n;\ N,\ \xi)$ being multiplicative, $F(s)$ can be written in the form

$$F(s) = \prod_{p \leqslant N^{1/u}} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r;\ N,\ \xi)}{p^{rs}}\right) \cdot \prod_{p > N^{1/u}} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{20}$$
$$= \left(1 + \sum_{r=1}^{\infty} \frac{f(2^r;\ N,\ \xi)}{2^{rs}}\right) \times$$
$$\times \exp\left[\sum_{3 \leqslant p \leqslant N^{1/u}} \frac{f(p;\ N,\ \xi)}{p^s} + \sum_{p > N^{1/u}} \frac{1}{p^s} + H_1(s)\right],$$

where $H_1(s)$ is uniformly bounded.

We will now analyse the behaviour of $F(s)$ in the domain $K/\log N \leqslant |t| \leqslant K$. We have

$$\frac{F(s)}{\zeta(\sigma_N)} \ll \exp\left(- \sum_{p \leqslant N^{1/u}} \frac{1 - \operatorname{Re} f(p;\ N;\ \xi) p^{-it}}{p^{\sigma_N}}\right).$$

By (18) and by inequality

$$1 - \operatorname{Re} f(p;\ N,\ \xi) p^{-it} = 1 - \operatorname{Re} p^{-it} + \operatorname{Re}\big(p^{-it} - f(p;\ N,\ \xi) p^{-it}\big)$$
$$\geqslant 1 - \operatorname{Re} p^{-it} - \big(1 - \operatorname{Re} f(p;\ N,\ \xi)\big) -$$
$$- 2\sqrt{1 - \operatorname{Re} f(p;\ N,\ \xi)} \cdot \sqrt{1 - \operatorname{Re} p^{-it}},$$

we find

$$\frac{|F(s)|}{\zeta(\sigma_N)} \ll \exp\Bigg[- \sum_{p \leqslant N^{1/u}} \frac{1 - \operatorname{Re} p^{-it}}{p^{\sigma_N}} + \sum_{p \leqslant N} \frac{1 - \operatorname{Re} f(p;\ N,\ \xi)}{p^{\sigma_N}} + \tag{21}$$
$$+ 2 \sqrt{\sum_{p \leqslant N} \frac{1 - \operatorname{Re} f(p;\ N,\ \xi)}{p^{\sigma_N}}} \cdot \sqrt{\sum_{p \leqslant N^{1/u}} \frac{1 - \operatorname{Re} p^{-it}}{p^{\sigma_N}}}\Bigg]$$
$$\ll \exp\left[-\frac{1}{2} \sum_p \left(\frac{1}{p^{\sigma_N}} - \operatorname{Re} \frac{1}{p^{\sigma_N + it}}\right)\right] \ll \left|\frac{\zeta(\sigma_N + it)}{\zeta(\sigma_N)}\right|^{1/2} \ll \frac{1}{\sqrt{K}}$$

uniformly in t and u in $K/\log N \leqslant |t| \leqslant K$, $0 < a \leqslant u \leqslant 1$. By applying Perron's formula and (20), we obtain for $a \leqslant u \leqslant 1$,

$$(22)\quad \frac{1}{N}\sum_{n\leqslant N} f(n;\,N,\,\xi) = \frac{1}{N}\sum_{n\leqslant N} f_u(n;\,N,\,\xi)$$

$$= \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\prod_{p\leqslant N^{1/u}}\Bigg(1-\frac{1}{pe^{z\frac{\log p}{\log N}}}\Bigg)\Bigg(1+\sum_{r=1}^{\infty}\frac{f(p^r;\,N,\,\xi)}{p^r e^{z\frac{\log p^r}{\log N}}}\Bigg)dz +$$

$$+O\left(\frac{K}{\log N}\right)+O\left(\left|\int_J \frac{N^{s-1}}{s}F(s)\,ds\right|\right),$$

where

$$J = J_1\cup J_2,$$

$$J_1 = \left\{t:\ \log^2 N \geqslant t \geqslant \frac{K}{\log N}\right\} \quad\text{and}\quad J_2 = \left\{t:\ -\log^2 N \leqslant t \leqslant \frac{-K}{\log N}\right\}.$$

The remaining part of the proof of the lemma will be devoted to deriving an estimate of the integral on J. It will be proved that this integral is $O(1/K^{1/48})$. To estimate the integral on J it is sufficient to estimate it on J_1, since the estimation on J_2 can be obtained by repeating the same considerations.

We will rewrite $F(s)$ in the form

$$F(s) = \left(1+\sum_{r=1}^{\infty}\frac{f(2^r;\,N,\,\xi)}{2^{rs}}\right)\sum_{\substack{n=1\\(n,2)=1}}^{\infty}\frac{f_u(n;\,N,\,\xi)}{n^s} = g(s)F_2(s)$$

and we will integrate the estimated integral by parts

$$\frac{1}{2\pi i}\int_{J_1}\frac{N^{s-1}}{s}F(s)\,ds = -\frac{e}{2\pi}\int_{J_1}F_2(\sigma_N+it)\,d\int_t^{\log^2 N}\frac{N^{iu}}{\sigma_N+iu}g(\sigma_N+iu)\,du$$

$$= \frac{e}{2\pi}F\left(\sigma_N+i\frac{K}{\log N}\right)\int_{J_1}\frac{N^{it}}{\sigma_N+it}g(s)\,dt +$$

$$+\frac{e}{2\pi}\int_{J_1}F_2'(s)\int_t^{\log^2 N}\frac{N^{iu}}{\sigma_N+iu}g(\sigma_N+iu)\,du\,dt.$$

$g(s)$ and its differentials being uniformly bounded in t, integrating several times by parts we obtain

$$\int_t^{\log^2 N}\frac{N^{iu}}{\sigma_N+iu}g(\sigma_N+iu)\,du = O\left(\frac{1}{|s|\log N}\right).$$

From the last two equations, applying (21), we find

$$(23)\quad \frac{1}{2\pi i}\int_{J_1}\frac{N^{s-1}}{s}F(s)\,ds = O\left(\frac{1}{\log N}\int_{J_1}|F_2'(s)|\frac{|ds|}{|s|}+\frac{1}{\sqrt{K}}\right).$$

By application of Hölder's inequality we get

$$(24)\quad \int_{J_1}\left|\frac{F_2'(s)}{s}\right||ds| \leqslant \left(\int_{|t|\leqslant\log^2 N}\left|\frac{F_2'}{F_2}(s)\right|^{12}|dt|\right)^{1/12}\left(\int_{J_1}\left|\frac{F_2(s)}{s}\right|^{12/11}|dt|\right)^{11/12}.$$

To begin with we will estimate the second integral of (24). Due to (21) we have

$$(25)\quad \int_{J_1}\left|\frac{F_2(s)}{s}\right|^{12/11}dt$$

$$\leqslant \max_{K/\log N\leqslant t\leqslant K}|F_2(s)|^{1/22}\int_{-\infty}^{+\infty}\frac{|F_2(s)|^{23/22}}{|s|^{12/11}}\,dt + \frac{2}{K^{1/22}}\int_{-\infty}^{+\infty}\frac{|F_2(s)|^{12/11}}{|s|^{23/22}}\,dt$$

$$\ll \left(\frac{\log N}{\sqrt{K}}\right)^{1/22}\int_{-\infty}^{+\infty}\left|\frac{F_2(s)}{s}\right|^{23/22}dt \ll \left(\frac{\log N}{\sqrt{K}}\right)^{1/22}\sum_{-\infty}^{+\infty}\int_{|t-m|\leqslant 1/2}\left|\frac{F_2(s)}{s}\right|^{23/22}|dt|$$

$$\ll \left(\frac{\log N}{\sqrt{K}}\right)^{1/22}\max_{-\infty<m<+\infty}\left(\int_{|t-m|\leqslant 1/2}|F_2(s)|^{23/22}\,dt\sum_{-\infty}^{+\infty}\frac{1}{\left||n|-\frac{1}{2}\right|^{23/22}}\right)$$

$$\ll \left(\frac{\log N}{\sqrt{K}}\right)^{1/22}\max_{-\infty<m<+\infty}\left(\int_{-\infty}^{+\infty}|F_2(s+im)|^{23/22}\frac{dt}{|s|^2}\right).$$

$f(n;\,N,\,\xi)$ being multiplicative, $F_2(s)$ can be represented in the form

$$(26)\quad F_2(s) = \exp\left[\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)}{p^s}+H_2(s)\right],$$

where $H_2(s)$ is bounded uniformly in t, ξ and N. On account of (26) we have

$$(F_2(s+im))^{23/44} = \exp\left[\frac{23}{44}\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)}{p^{s+im}}+\frac{23}{44}H_2(s+im)\right].$$

The function

$$\exp\left[\frac{23}{44}\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)p^{-im}}{p^s}\right]$$

is a Dirichlet series with coefficients

$$\lambda(n) = \lambda_1(n)\prod_{p^a\| n}f_u^a(p;\,N,\,\xi)p^{-ima},$$

where $\lambda_1(n)$ are coefficients of the Dirichlet series

$$\exp\left[\frac{23}{44}\sum_{p\geqslant 3}\frac{1}{p^s}\right],$$

$p^a\|n$ meaning $p^a|n$, and $p^{a+1}\nmid n$. On the other hand, applying (26) to $\zeta_2(s)$, we obtain

$$\left(\zeta_2(s)\right)^{23/44} \stackrel{\text{def}}{=} \Bigg(\sum_{\substack{n=1\\(n,2)=1}}^{\infty}\frac{1}{n^s}\Bigg)^{23/44} = \exp\left[\frac{23}{44}\sum_{p\geqslant 3}\frac{1}{p^s}+\frac{23}{44}H_3(s)\right],$$

where $H_3(s)$ is bounded uniformly in t and N. Furthermore, since

$$\frac{1}{s}\sum_{n}\frac{\lambda(n)}{n^s} = -\int_0^{\infty}\Big(\sum_{n<e^u}\lambda(n)\Big)\cdot e^{-iu(\sigma_N+it)}du,$$

by Parseval's equality, we find

$$\int_{-\infty}^{+\infty}|F_2(s+im)|^{23/22}\frac{dt}{|s|^2}$$

$$= \int_{-\infty}^{+\infty}\left|\exp\left[\frac{23}{44}\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)p^{-im}}{p^s}\right]\right|^2\cdot\left|\exp\left[\frac{23}{22}H_2(s+im)\right]\right|\frac{dt}{|s|^2}$$

$$\ll \int_{-\infty}^{+\infty}\left|\frac{1}{s}\sum_{n}\frac{\lambda(n)}{n^s}\right|^2dt \ll \int_0^{\infty}\Big|\sum_{n<e^u}\lambda(n)\Big|^2e^{-2\sigma_N u}\,du$$

$$\ll \int_0^{+\infty}\Big(\sum_{n<e^u}\lambda_1(n)\Big)^2e^{-2\sigma_N u}\,du \ll \int_{-\infty}^{+\infty}|\zeta_2(s)|^{23/22}\frac{dt}{|s|^2}.$$

Hence, with the help of known estimates for $\zeta(s)$, we obtain

$$\int_{-\infty}^{+\infty}|F_2(s+im)|^{23/22}\frac{dt}{|s|^2} \ll (\log N)^{1/22}.$$

Applying (24) and (25), we get

$$\int_{J_1}|F_2'(s)|\frac{|ds|}{|s|} \ll \Bigg(\int_{|t|\leqslant\log^2 N}\left|\frac{F_2'}{F_2}(s)\right|^{12}|dt|\Bigg)^{1/12}\left(\frac{\log N}{\sqrt[4]{K}}\right)^{1/12}.$$

This relation and equality (23) show that to complete this proof it is sufficient to obtain the estimate

$$\int_{|t|\leqslant\log^2 N}\left|\frac{F_2'}{F_2}(s)\right|^{12}|dt| = O(\log^{11}N). \tag{27}$$

The proof of this last assertion is, in fact, contained in [4] (see Lemma 1). The application of this lemma to our purposes, will be preceded by a few remarks. From equality (26) it follows that

$$\frac{F_2'}{F_2}(s) = -\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)}{p^s}\log p + H_2'(s), \tag{28}$$

where $H_2'(s)$ is bounded uniformly in N, ξ and t.

We will estimate the number M of points t_k for which

$$\left|\frac{F_2'}{F_2}(\sigma_N+it_k)\right| > \frac{w}{\sigma_N-1},$$

where $w \geqslant A(\sigma_N-1)^{1/4}$, $|t_k-t_l|\geqslant\eta(\sigma_N-1)$, $|t_k|\leqslant\log^2 N$,

$$\eta = \frac{2B}{w^2}\leqslant\frac{1}{\sqrt{\sigma_N-1}}.$$

Let

$$\varrho_k = \exp\left[-\arg\frac{F_2'}{F_2}(\sigma_N+it_k)\right],$$

then on the basis of (28) and the analogous equality for $\frac{\zeta'}{\zeta}(s)$, we have

$$\frac{wM}{\sigma_N-1}\leqslant\sum_{k=1}^{M}\varrho_k\frac{F_2'}{F_2}(\sigma_N+it_k) = \sum_{k=1}^{M}\varrho_k\Big[-\sum_{p\geqslant 3}\frac{f_u(p;\,N,\,\xi)}{p^s}\log p+H_2'(s)\Big]$$

$$\leqslant\sum_{p}\frac{\log p}{p^{\sigma_N}}\left|\sum_{k=1}^{M}\frac{\varrho_k}{p^{it_k}}\right|+a_1M$$

$$\leqslant a_1M+\sqrt{\sum_{p}\frac{\log p}{p^{\sigma_N}}\cdot\sum_{k=1}^{M}\sum_{l=1}^{M}\varrho_k\bar{\varrho}_l\sum_{p}\frac{\log p}{p^{\sigma_N+i(t_k-t_l)}}}$$

$$\leqslant a_1M+\sqrt{\frac{2}{\sigma_N-1}\left[M\frac{\zeta'}{\zeta}(\sigma_N)+M^2\max_{\eta(\sigma_N-1)\leqslant|t|\leqslant\log^2N}\left|\frac{\zeta'}{\zeta}(\sigma_N+it)\right|\right]}$$

$$\leqslant a_1M+\sqrt{\frac{4M}{(\sigma_N-1)^2}+\frac{2M^2}{\eta(\sigma_N-1)^2}}.$$

Since $w\geqslant A(\sigma_N-1)^{1/4}$,

$$\frac{1}{2}\frac{wM}{\sigma_N-1}\leqslant\sqrt{\frac{4M}{(\sigma_N-1)^2}+\frac{2M^2}{\eta(\sigma_N-1)^2}}.$$

From the last inequality, we obtain

$$w^2M^2\leqslant 16M+8M^2/\eta = 16M+\tfrac{1}{2}M^2w^2,$$

if we choose $B = 8$. Therefore $M \leqslant 32/w^2$. Hence, in the same way as in [4], it follows that the measure of the set of those t, for which

$$\left|\frac{F_2'}{F_2}(\sigma_N+it)\right| > \frac{w}{\sigma_N-1} \qquad (|t| \leqslant \log^2 N)$$

does not exceed $\frac{a_2}{w^4}(\sigma_N-1)$. Now, by exact repetition of the simple considerations of [4] (see pp. 392–393) with $a = 12$, we obtain (27).

THEOREM 3 ([1]). *If there exist the non-decreasing bounded functions $L_l(u)$, $l = 0, 1, 2, \ldots$, and a constant d, satisfying the conditions of Theorem 2 in all points of continuity of $L_l(u)$, with the possible exception of $u = 0$, then there exists the limit distribution of $g(n)/B_N$.*

Besides $L_0(+\infty) \neq 0$, $\tau(\xi) \not\equiv 1$ and

$$D_N = \begin{cases} |B_N| & \text{if} \quad \tau(\xi) \equiv \tau(-\xi), \\ B_N & \text{in the opposite case,} \end{cases}$$

then there exists a function $\varphi(u)$ bounded in $0 \leqslant u \leqslant 1$ such that $D_{N^u}/D_N \to \varphi(u)$ uniformly in u in any closed subinterval of $(0, +\infty)$.

Proof. From (5) it follows that

$$\sum_{p\leqslant N} (1-\operatorname{Re} e^{i\xi\frac{g(p)}{B_N}})\, p^{-1} = O(1),$$

since $1-\cos\xi\alpha \leqslant (\xi^2+1)\cdot\|\alpha\|^2$. Applying Lemma 3 to the function $f(n; N, \xi) = e^{i\xi\frac{g(p)}{B_N}}$, we obtain

$$\tau_N(\xi) \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n\leqslant N} e^{i\xi\frac{g(n)}{B_N}}$$

$$= \frac{1}{2\pi i}\int_{1-iK}^{1+iK} \frac{e^z}{z} \prod_{p\leqslant N^{1/u}} \left(1-\frac{1}{pe^{z\frac{\log p}{\log N}}}\right)\left(1+\sum_{r=1}^{\infty}\frac{e^{i\xi\frac{g(p^r)}{B_N}}}{p^r e^{z\frac{\log p^r}{\log N}}}\right)dz + O\left(\frac{1}{K^{1/48}}\right) + o_K(1).$$

As $B_N \to \infty$ we have

$$\prod_{p\leqslant N^{1/u}}\left(1-\frac{1}{p^s}\right)\left(1+\sum_{r=1}^{\infty}\frac{e^{i\xi\frac{g(p\ )}{B_N}}}{p^{rs}}\right) = (1+o(1))\exp\Big[\sum_{p\leqslant N^{1/u}}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-s}\Big],$$

([1]) Theorem will be true, if the conditions at $L_l(u)$ will be fulfilled only for $l = 0,1$.

due to which

$$(29) \qquad \tau_N(\xi) = \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\exp\Big[\sum_{p\leqslant N^{1/u}}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}e^{-z\frac{\log p}{\log N}}\Big]dz + O\left(\frac{1}{K^{1/48}}\right) + o_K(1).$$

We will now prove that

$$(30) \qquad f_N(z,\xi) \stackrel{\text{def}}{=} \sum_{p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1}e^{-z\frac{\log p}{\log N}} = f(z,\xi)+o(1).$$

Due to (7)

$$\tau_{l,N}(\xi) \stackrel{\text{def}}{=} \frac{l}{\log^l N}\sum_{p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)\frac{\log^l p}{p} = \tau_l(\xi)+o(1),$$

from which

$$f_N(z,\xi) = \sum_{p\leqslant N}(e^{i\xi\frac{g(p)}{B_N}}-1)p^{-1} + \sum_{l=1}^{\infty}\frac{(-z)^l}{l!\,l}\tau_{l,N}(\xi)$$

$$= \int_{-\infty}^{+\infty}\frac{e^{i\xi u}-1-i\xi\|u\|}{\|u\|^2}\, d\sum_{\substack{p\leqslant N\\ g(p)\leqslant uB_N}}\left\|\frac{g(p)}{B_N}\right\|^2\frac{1}{p} + i\xi\sum_{p\leqslant N}\left\|\frac{g(p)}{B_N}\right\|\frac{1}{p} + \sum_{l=1}^{\infty}\frac{(-z)^l}{l!\,l}\tau_l(\xi) + \sum_{l=1}^{\infty}\frac{(-z)^l}{l!\,l}\big(\tau_{l,N}(\xi)-\tau_l(\xi)\big)$$

$$= \int_{-\infty}^{+\infty}\frac{e^{i\xi u}-1-i\xi\|u\|}{\|u\|^2}\, dL_0(u) + i\xi d + \sum_{l=1}^{\infty}\frac{(-z)^l}{l!\,l}\tau_l(\xi) + o_K(1)$$

uniformly in z for $|z| \leqslant K$. Above we used (6), (7) and the following considerations.

Let $M \geqslant 2K \geqslant 2|z|$, then

$$\left|\sum_{l=1}^{\infty}\frac{(-z)^l}{l!\,l}\big(\tau_{l,N}(\xi)-\tau_l(\xi)\big)\right| \leqslant \sum_{l=1}^{M}\frac{K^l}{l!\,l}|\tau_{l,N}(\xi)-\tau_l(\xi)| + \sum_{l=M+1}^{\infty}\frac{K^l}{l!}$$

$$\leqslant \sup_{\substack{1\leqslant l\leqslant M\\ |\xi|\leqslant C}}|\tau_{l,M}(\xi)-\tau_l(\xi)|\, l^K + (\tfrac{1}{2})^M.$$

The last expression can be made $<\varepsilon$ ($\varepsilon > 0$ arbitrary). To do so it is necessary that M is sufficiently large and that N tends to infinity. From (29) and

(30), for $u = 1$ we have

$$\tau_N(\xi) = \frac{1}{2\pi i}\int_{1-iK}^{1+iK} \frac{e^z}{z}\exp\big(f(z,\xi)\big)\,dz + O\left(\frac{1}{K^{1/48}}\right) + o_K(1). \tag{31}$$

Hence, K being arbitrary, applying Cauchy's criterion we find

$$\tau_N(\xi) = \tau(\xi) + o(1),$$

i.e. $g(n)/B_N$ has the limit distribution.

We will first show that if the conditions of the theorem are satisfied, then D_{N^u}/D_N is bounded uniformly in u for $0 \leqslant u \leqslant 1$. Note that

$$D_N = \begin{cases} |B_N| & \text{if} \quad \tau(-\xi) = \tau(\xi), \\ B_N & \text{in the opposite case.} \end{cases}$$

We will assume the opposite. Then the sequences N_k and $u_k \leqslant 1$ can be found such that $N_k^{u_k} \to \infty$ and $D_{N_k^{u_k}}/D_{N_k} \to \infty$ for $K \to \infty$. Since $L_0(\pm\infty) = \lim_{u\to\pm\infty} L_0(u)$, for any $\varepsilon(N) \to \infty$ we have

$$\sum_{\substack{p\leqslant N \\ |g(p)|\geqslant \varepsilon(N) B_N}} \left\|\frac{g(p)}{B_N}\right\|^2 \frac{1}{p} \to 0 \quad \text{for} \quad N \to \infty,$$

from which

$$\sum_{p\leqslant N_k} \left\|\frac{g(p)}{D_{N_k^{u_k}}}\right\|^2 \frac{1}{p}$$

$$\leqslant \sum_{\substack{p\leqslant N_k \\ |g(p)| < \sqrt{|D_{N_k^{u_k}} \cdot D_{N_k}|}}} \left(\frac{g(p)}{D_{N_k^{u_k}}}\right)^2 \frac{1}{p} + \sum_{\substack{p\leqslant N_k \\ |g(p)| \geqslant \sqrt{|D_{N_k^{u_k}} \cdot D_{N_k}|}}} \left\|\frac{g(p)}{D_{N_k^{u_k}}}\right\|^2 \frac{1}{p}$$

$$\leqslant \left|\frac{D_{N_k}}{D_{N_k^{u_k}}}\right| \Bigg(\sum_{\substack{p\leqslant N_k \\ |g(p)| < |D_{N_k}|}} \frac{g^2(p)}{D^2_{N_k}} \frac{1}{p} + \sum_{\substack{p\leqslant N_k \\ |g(p)| > |D_{N_k}|}} \frac{1}{p} \Bigg) + o(1)$$

$$= \left|\frac{D_{N_k}}{D_{N_k^{u_k}}}\right| \big(L_0(+\infty) + o(1)\big) + o(1) = o(1).$$

On the other hand, from (5) and by $L_0(+\infty) \neq 0$,

$$\sum_{p\leqslant N_k^{u_k}} \left\|\frac{g(p)}{D_{N_k^{u_k}}}\right\|^2 \frac{1}{p} = L_0(+\infty) + o(1) \geqslant c > 0.$$

Since $u_k \leqslant 1$, the last two inequalities are contradictory. Let $0 < a \leqslant u \leqslant 1$. Using formula (29), after having carried out in it the following replacements, $N \to N^u$, $\xi \to \xi \dfrac{B_{N^u}}{B_N}$ (this last is permitted since B_{N^u}/B_N is bounded), and $K \to Ku$,

$$\tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right) = \frac{1}{2\pi i}\int_{1-iKu}^{1+iKu} \frac{e^z}{z}\exp\Big[\sum_{p\leqslant N} \big(e^{i\xi\frac{g(p)}{B_N}} - 1\big) p^{-1} e^{-z\frac{\log p}{u\log N}}\Big] dz + $$
$$+ O\left(\frac{1}{K^{1/48}}\right) + o_K(1)$$
$$= \frac{1}{2\pi i}\int_{1/u-iK}^{1/u+iK} \frac{e^{uz}}{z}\exp\Big[\sum_{p\leqslant N} \big(e^{i\xi\frac{g(p)}{B_N}} - 1\big) p^{-1} e^{-z\frac{\log p}{\log N}}\Big] dz + $$
$$+ O\left(\frac{1}{K^{1/48}}\right) + o_K(1)$$
$$= \frac{1}{2\pi i}\int_{1/u-iK}^{1/u+iK} \frac{e^{uz}}{z}\exp\big(f(z,\xi)\big)\,dz + O\left(\frac{1}{K^{1/48}}\right) + o_K(1).$$

On the basis of Cauchy's criterion, there exists the function $\psi(\xi, u)$ continuous in u such that

$$\tau_{N^u}\left(\xi\frac{B_{N^u}}{B_N}\right) = \psi(\xi, u) + o(1) \tag{31'}$$

uniformly in u for $0 < a \leqslant u \leqslant 1$. From the above-proved formula (31′) and the boundedness of B_{N^u}/B_N, we obtain

$$\tau\left(\xi\frac{B_{N^u}}{B_N}\right) = \psi(\xi, u) + o(1).$$

Recalling the definition of D_N, we have

$$\tau\left(\xi\frac{D_{N^u}}{D_N}\right) = \psi(\xi, u) + o(1).$$

Applying Lemma 2 we find that there exists a function $\varphi(u)$ such that $D_{N^u}/D_N \to \varphi(u)$ uniformly in u for $0 < a \leqslant u \leqslant 1$. Since by the above the existence of the limit distribution has been proved, and $\tau(\xi) \not\equiv 1$ due to the assumption of the theorem, on basis of Theorem 1 $\varphi(u) \not\equiv 0$ for $0 < u < 1$. Therefore on basis of Lemma 2 $\varphi(u) \neq 0$ for $0 < u < 1$. Hence for $u \geqslant 1$,

$$\frac{D_{N^u}}{D_N} = \frac{1}{\varphi(1/u)} + o(1)$$

uniformly in u for $0 < a \leqslant 1/u \leqslant 1$, from which the second part of the theorem follows.

4. The necessary and sufficient conditions. In this section $B_N \to \infty$ is not assumed. The following is a straightforward generalization of the Erdös–Wintner theorem.

THEOREM 4. *Let $B_{Nu}/B_N \to \varphi(u)$ uniformly in any closed subinterval of $(0, +\infty)$. Then the following two assumptions are equivalent:*

(i) *$g(n)/B_N$ has a limit distribution.*

(ii) *There exist $L_0(u)$ and $d = \text{const}$ such that*

$$\sum_{\substack{p \leqslant N \\ g(p) \leqslant uB_N}} \left\| \frac{g(p)}{B_N} \right\|^2 \frac{1}{p} \to L_0(u) \tag{5}$$

and

$$\sum_{p \leqslant N} \left\| \frac{g(p)}{B_N} \right\| \frac{1}{p} \to d \tag{6}$$

in all points of continuity of $L_0(u)$, with the possible exception of $u = 0$, and B_N has either finite or infinite limit.

Proof. We will show that (i) implies (ii). Indeed, if $B_N \to \infty$ then, on basis of Theorem 2, conditions (5) and (6) are satisfied. From [5], Theorem 4, it follows that B_N has either finite or infinite limit. If the limit is finite then, due to the theorem on the convergence of types [6], it can be assumed that $B_N \equiv 1$ and that (ii) follows from the Erdös–Wintner theorem. We will now assume that condition (ii) is satisfied. From (5) and (6) we find

$$\sum_{p \leqslant N} (e^{i\xi \frac{g(p)}{B_N}} - 1) \frac{1}{p} = \int_{-\infty}^{+\infty} \frac{e^{i\xi u} - 1 - i\xi \|u\|}{\|u\|^2} \, dL_0(u) + i\xi d + o(1).$$

Since $B_{Nu}/B_N \to \varphi(u)$, by partial summation we establish the analogue of (7) for corresponding characteristic functions. And finally, using the relationship between characteristic and distribution functions, we establish (7). If $B_N \to \infty$, then to finish the proof we use the first part of Theorem 3. If however, B_N tends to a finite limit then we apply the Erdös–Wintner theorem.

The proof of Theorem 5 is a little more complicated as for the proof of the sufficiency nothing is presumed about the behaviour of B_{Nu}/B_N. Furthermore, in Theorem 5 the necessary and sufficient conditions for the existence of non-trivial limit distribution are given. Theorem 5 can also be considered as a generalization of the Erdös–Wintner theorem, even though, due to its generality, its conditions appear more complicated.

THEOREM 5. *For the existence of a proper limit distribution of $g(n)/B_N$ for $B_{Nu}/B_N \to \varphi(u)$ (uniformly in u in any closed subinterval of $(0, +\infty)$) it is necessary and sufficient that B_N has either finite or infinite limit, and*

that there exist the bounded non-decreasing functions $L_l(u)$, $l = 0, 1, 2, \ldots$,(2) *and a constant d, satisfying the assumptions of Theorem 2, moreover $0 = L_0(-\infty) < L_0(+\infty)$, and if $L_1(u) = E(u)$, then*

$$\sum_{p \leqslant N} \left\| \frac{g(p)}{B_N} + a\frac{B_{N/p}}{B_N} - a \right\|^2 \frac{\log p}{p} = \Omega(\log N) \tag{32}$$

for any a.

Proof of necessity.

Remark. It is sufficient to consider the case $B_N \to \infty$ for $N \to \infty$ as, if the limit distribution exists, B_N has either finite or infinite limit (see [5], Theorem 4) and the case of finite limit resolves itself into the Erdös–Wintner theorem. The necessity of conditions (5)–(7) follows from Theorem 2. The necessity of $L_0(+\infty) > 0$ can be proved in the following way.

If $L_0(+\infty) = 0$, then taking into consideration the inequality $1 - \cos \xi\alpha \leqslant (\xi^2+1)\|\alpha\|^2$ we obtain

$$\begin{aligned} \left| \frac{l}{\log^l N} \sum_{p \leqslant N} (e^{i\xi \frac{g(p)}{B_N}} - 1) \frac{\log^l p}{p} \right| &\leqslant \frac{l}{\log^l N} \sum_{p \leqslant N} \sqrt{2\left(1 - \cos \xi \frac{g(p)}{B_N}\right)} \frac{\log^l p}{p} \\ &\leqslant \frac{l}{\log^l N} \sqrt{2 \sum_{p \leqslant N} \frac{1 - \cos \xi \frac{g(p)}{B_N}}{p}} \cdot \sqrt{\sum_{p \leqslant N} \frac{\log^{2l} p}{p}} \\ &\leqslant \sqrt{l} \sqrt{(\xi^2+1) \sum_{p \leqslant N} \left\| \frac{g(p)}{B_N} \right\|^2 \frac{1}{p}} \to 0. \end{aligned} \tag{33}$$

From which,

$$L_l(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ 1 & \text{for} \quad u > 0, \end{cases}$$

and $\tau_l(\xi) \equiv 0$ for $l = 1, 2, 3, \ldots$ Therefore, from (30) and (31)

$$\tau_N(\xi) = \frac{1}{2\pi i} \int_{1-iK}^{1+iK} \frac{e^z}{z} e^{i\xi d} dz + O\left(\frac{1}{K^{1/48}}\right) + o_K(1) = e^{i\xi d} + O\left(\frac{1}{K^{1/48}}\right) + o_K(1).$$

As K is arbitrary, $\tau(\xi) = e^{i\xi d}$, which contradicts the existence of proper limit distribution.

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(2) See footnote on page 352.

Let now $L_1(u) \not\equiv E(u)$. We will show that then $\varphi(u) \not\equiv 1$. In fact let $\varphi(u) \equiv 1$. Then by Lemma 1

$$(34)\qquad \frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p} = 1+o(1).$$

Transition from (1) to (34) is realized by the scheme used for the proof of (2). On the other hand, since $L_1(u) \not\equiv E(u)$,

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\frac{g(p)}{B_N}}\frac{\log p}{p} = \lambda(\xi)+o(1),$$

where $\lambda(\xi) \not\equiv 1$. Therefore $\varphi(u) \not\equiv 1$. From the boundedness (for $u \leqslant 1$) and the multiplicativity of $\varphi(u)$, we have $\varphi(u) \leqslant 1$. Hence there exists at least one point $\delta < 1$ at which $\varphi(\delta) = \gamma < 1$. Then due to the multiplicativity, $\varphi(u) \leqslant \gamma^m$ for $\delta^{m+1} \leqslant u \leqslant \delta^m$. Assuming that (32) is not satisfied we obtain

$$\frac{1}{\log N}\sum_{p\leqslant N}\left|e^{i\xi\left(\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right)}-1\right|\frac{\log p}{p} = o(1).$$

On the other hand, by Lemma 1,

$$\frac{1}{\log N}\sum_{p\leqslant N}\left(e^{i\xi\left(\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right)}-1\right)e^{i\xi\left(-a\frac{B_{N/p}}{B_N}+a\right)}\tau\left(\xi\frac{B_{N/p}}{B_N}\right)\frac{\log p}{p}+$$
$$+\frac{1}{\log N}\sum_{p\leqslant N}e^{i\xi\left(-a\frac{B_{N/p}}{B_N}+a\right)}\tau\left(\xi\frac{B_{N/p}}{B_N}\right)\frac{\log p}{p} = \tau(\xi)+o(1).$$

From the last two equalities, we get

$$(35)\qquad \frac{1}{\log N}\sum_{p\leqslant N} f\left(\xi\frac{B_{N/p}}{B_N}\right)\frac{\log p}{p} = f(\xi)+o(1),$$

where $f(\xi) = \operatorname{Re}\tau(\xi)e^{-i\xi a}$. Let ξ_1 be a point at which $f(\xi)$ has the smallest value in the interval $|\xi| \leqslant 1$. Thus such $\xi_2 > 0$ can be found that for $|\xi| \leqslant \xi_2$, $f(\xi)-f(\xi_1) \geqslant \varepsilon > 0$. If p lies inside the interval $(N^{\sigma(m+1)}, N^{\sigma(m)})$, $\sigma(m) = 1/(1-\delta^m)$, then for $|\xi_1|\gamma^m \leqslant \xi_2$ and for sufficiently large N,

$$f\left(\xi_1\frac{B_{N/p}}{B_N}\right)-f(\xi_1) \geqslant \varepsilon > 0.$$

If, however, p lies outside this interval, then according to choice of ξ_1

$$f\left(\xi_1\frac{B_{N/p}}{B_N}\right)-f(\xi_1) \geqslant 0.$$

Hence

$$o(1) = \frac{1}{\log N}\sum_{p\leqslant N}\left(f\left(\xi_1\frac{B_{N/p}}{B_N}\right)-f(\xi_1)\right)\frac{\log p}{p}$$
$$\geqslant \frac{\varepsilon}{\log N}\sum_{N^{\sigma(m+1)}\leqslant p\leqslant N^{\sigma(m)}}\frac{\log p}{p} \geqslant \frac{\varepsilon}{2}\left(\frac{1}{1-\delta^m}-\frac{1}{1-\delta^{m+1}}\right) > 0.$$

Since contradiction has been obtained, the necessity of condition (32) is proved.

Proof of sufficiency. If B_N tends to a finite limit, the sufficiency of the conditions follows from the Erdös–Wintner theorem. If, however, $B_N \to \infty$, the existence of a limit distribution follows from Theorem 3. We will show that its non-triviality is a result of the conditions of Theorem 4.

We will carry out the proof by contradiction. Let $\tau(\xi) = e^{i\xi a}$. We will prove that then, either

$$L_0(+\infty) = 0 \quad \text{for} \quad L_1(u) \equiv E(u) \overset{\text{def}}{=} \begin{cases} 1 & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \leqslant 0, \end{cases}$$

or

$$(36)\qquad \frac{1}{\log N}\sum_{p\leqslant N}\left\|\frac{g(p)}{B_N}a+\frac{B_{N/p}}{B_N}-a\right\|^2\frac{\log p}{p} \to 0$$

for $L_1(u) \not\equiv E(u)$. Let, in fact, $L_1(u) \equiv E(u)$. Passing on to characteristic functions for real parts, we obtain

$$\frac{1}{\log N}\sum_{p\leqslant N}\left(1-\cos\xi\frac{g(p)}{B_N}\right)\frac{\log p}{p} = o(1).$$

Therefore, for $l \geqslant 1$,

$$|\tau_{l,N}(\xi)| = \left|\sum_{p\leqslant N}\left(e^{i\xi\frac{g(p)}{B_N}}-1\right)\frac{\log^l p}{p}\cdot\frac{l}{\log^l N}\right|$$
$$\leqslant \frac{2l}{\log N}\sqrt{\sum_{p\leqslant N}\left(1-\cos\xi\frac{g(p)}{B_N}\right)\frac{\log p}{p}}\sqrt{\sum_{p\leqslant N}\frac{\log p}{p}} \to 0.$$

Hence, by (30) and (31), for $\tau_l(\xi) \equiv 0$, $l = 1, 2, 3, \ldots$, we have

$$e^{i\xi a} = \frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\exp\left(\int_{-\infty}^{+\infty}\frac{e^{i\xi u}-1-i\xi\|u\|}{\|u\|^2}\,dL_0(u)+i\xi d\right)dz+$$
$$+O\left(\frac{1}{K^{1/48}}\right)+o_K(1).$$

Since K is arbitrary,

$$e^{i\xi a} = \exp\left(\int_{-\infty}^{+\infty} \frac{e^{i\xi u}-1-i\xi\|u\|}{\|u\|^2}\, dL_0(u)+i\xi d\right).$$

From the well-known theorem ([6], p. 314) $L_0(+\infty)=0$, $d=a$. Let now $L_1(u)\not\equiv E(u)$, then $L_0(+\infty)\neq 0$ (see (33)). If $L_0(+\infty)\neq 0$, then B_{N^u}/B_N is uniformly bounded for $0\leqslant u\leqslant 1$, as shown in the proof of Theorem 3, and by Lemma 1,

$$\frac{1}{\log N}\sum_{p\leqslant N} e^{i\xi\left(\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right)}\frac{\log p}{p} = 1+o(1)$$

or

$$\frac{1}{\log N}\sum_{p\leqslant N}\left(e^{i\xi\left(\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right)}-1\right)\frac{\log p}{p}$$

$$= \int_{-\infty}^{+\infty}\frac{e^{i\xi u}-1-i\xi\|u\|}{\|u\|^2}\, d\,\frac{1}{\log N}\sum_{\substack{p\leqslant N\\ \frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\leqslant u}}\left\|\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right\|^2\frac{\log p}{p}+$$

$$+\frac{i\xi}{\log N}\sum_{p\leqslant N}\left\|\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}-a\right\|\frac{\log p}{p}\to 0.$$

Applying the theorem from [3] (§ 1, XVII), we obtain (36). To finish the proof of Theorem 5 it only remains to show that $B_{N^u}/B_N\to\varphi(u)$ uniformly in u in any closed subinterval of $(0,+\infty)$. If $\tau(\xi)\not\equiv\tau(-\xi)$, then it follows from Lemma 2. If, however, $\tau(\xi)\equiv\tau(-\xi)$, there also exists the limit distribution for $g(n)/|B_N|$. Hence all the conditions of Theorem 5 are satisfied for $|B_N|$, and so it is natural in the case $\tau(\xi)\equiv\tau(-\xi)$ to assume, from the outset, that B_N is positive, in which case $B_{N^u}/B_N\to\varphi(u)$, as seen from Lemma 2.

Minor alteration to all the foregone leads to the following theorem, the proof of which will be omitted.

THEOREM 6 (3). *For the existence of proper limit distribution of* $\dfrac{g(n)-A_N}{B_N}$ *with* $A_N=b\log N+A_N^*$, *where* $(A_{N^u}^*-A_N^*)/B_N\to\psi(u)$ *and* $B_{N^u}/B_N\to\varphi(u)$ *uniformly in* u *in any closed subinterval of* $(0,+\infty)$, *it is necessary and sufficient that* B_N *has either a finite or an infinite limit, that*

$$A_N = b\log N+B_N\sum_{p\leqslant N}\left\|\frac{g(p)-b\log p}{B_N}\right\|\frac{1}{p}+CB_N+o(B_N),$$

(3) See footnote on page 352.

and that there exist the bounded non-decreasing functions $L_l(u)$, $l=0,1,2,\ldots$, *satisfying the conditions*

$$L_l(\pm\infty)=\lim_{u\to\pm\infty}L_l(u),$$

$$\sum_{\substack{p\leqslant N\\ g(p)-b\log p\leqslant uB_N}}\left\|\frac{g(p)-b\log p}{B_N}\right\|^2\frac{1}{p}\to L_0(u),$$

$$\frac{l}{\log^l N}\sum_{\substack{p\leqslant N\\ g(p)-b\log p\leqslant uB_N}}\frac{\log^l p}{p}\to L_l(u),\qquad l=1,2,\ldots,$$

$0=L_0(-\infty)<L_0(+\infty)$ *and if* $L_1(u)\not\equiv E(u)$, *then for any* a,

$$\sum_{p\leqslant N}\left\|\frac{g(p)}{B_N}+a\frac{B_{N/p}}{B_N}+\frac{A_{N/p}-A_N}{B_N}-a\right\|^2\frac{\log p}{p}=\Omega(\log N).$$

5. Examples. To illustrate the results obtained in this work we will give a few examples.

I. The first group of examples shows that for the normalizations studied in the present work, limit distributions can exist for some functions. In all hitherto known examples, normalizations $B_N=L(\log N)$, where $L(u)$ is slowly varying in the Karamata sense. They have here the form $B_N^{(\varrho)}=\log^\varrho N\,L(\log N)$, where $\varrho>0$ and $L(u)$ is slowly varying in the Karamata sense.

LEMMA 4. *Let* $g_\varrho(n)$ *be an additive function such that* $g_\varrho(p)=\log^\varrho p\times\times L(\log p)$, *where* $\varrho>0$, $\varrho\neq 1$ *and* $L(u)$ *is a monotonously increasing function, slowly varying in the Karamata sense. Then* $g_\varrho(n)/\log^\varrho N\,L(\log N)$ *has a proper limit distribution.*

Proof. Since $L(u)$ slowly varies in the Karamata sense, there exists $\varepsilon(N)\to\infty$ such that

$$\frac{L(\log p)}{L(\log N)}=1+o(1),\qquad\text{for}\qquad N\to\infty,$$

uniformly in p for $N^{\varepsilon(N)}\leqslant p\leqslant N$. From which, taking into account $L(\log p)/L(\log N)\leqslant 1$, we obtain

$$\sum_{\substack{p\leqslant N\\ \left(\frac{\log p}{\log N}\right)^\varrho\frac{L(\log p)}{L(\log N)}\leqslant u}}\left\|\left(\frac{\log p}{\log N}\right)^\varrho\frac{L(\log p)}{L(\log N)}\right\|^2\frac{1}{p}$$

$$=O\left(\sum_{p\leqslant N^{\varepsilon(N)}}\left(\frac{\log p}{\log N}\right)^{2\varrho}\frac{1}{p}\right)+\sum_{\substack{p\leqslant N\\ \log p\leqslant(u(1+o(1)))^{1/\varrho}\log N}}\left(\frac{\log p}{\log N}\right)^{2\varrho}\frac{1}{p}\to$$

$$\to L_0(u) = \begin{cases} 0 & \text{if} \quad u \leqslant 0, \\ \dfrac{1}{2\varrho} u^2 & \text{if} \quad 0 \leqslant u \leqslant 1, \\ \dfrac{1}{2\varrho} & \text{if} \quad u > 1. \end{cases}$$

In exactly the same way we find that for $N \to \infty$,

$$\sum_{p \leqslant N} \left\| \frac{g_\varrho(p)}{B_N^{(\varrho)}} \right\| \frac{1}{p} = \frac{1}{\varrho} + o(1)$$

and

$$\frac{l}{\log^l N} \sum_{\substack{p \leqslant N \\ g_\varrho(p) \leqslant u B_N^{(\varrho)}}} \frac{\log^l p}{p} \to \begin{cases} 0 & \text{if} \quad u \leqslant 0, \\ u^{l/\varrho} & \text{if} \quad 0 < u \leqslant 1, \\ 1 & \text{if} \quad u > 1. \end{cases}$$

Therefore conditions (5)–(7) are satisfied, hence $g_\varrho(n)/B_N^{(\varrho)}$ has a limit distribution. It only remains to prove that it is proper. Since $L_1(u) \not\equiv E(u)$, it must be shown that for any a,

$$C_N = \frac{1}{\log N} \sum_{p \leqslant N} \left\| \frac{g_\varrho(p)}{B_N^{(\varrho)}} + a \frac{B_{N/p}^{(\varrho)}}{B_N} - a \right\|^2 \frac{\log p}{p} = \Omega(1).$$

Bearing in mind the definitions of $g_\varrho(p)$ and $B_N^{(\varrho)}$, we have

$$C_N = \frac{1}{\log N} \sum_{p \leqslant N} \left\| \left(\frac{\log p}{\log N} \right)^\varrho + a \left(1 - \frac{\log p}{\log N} \right)^\varrho - a \right\|^2 \frac{\log p}{p} + o(1).$$

For any a, the function $y(u) = u^\varrho + a(1-u)^\varrho - a$ is continuous and $y(1) = 1 - a$. Hence, if $a \neq 1$ there exists $\varepsilon(a) < 1$ such that $|y(u)| > (1-a)/2$ for $\varepsilon(a) \leqslant u \leqslant 1$. Therefore

$$C_N \geqslant \frac{(1-a)^2}{10 \log N} \sum_{N^{\varepsilon(a)} \leqslant p \leqslant N} \frac{\log p}{p} \geqslant \frac{(1-a)^2}{20 \log N} \cdot \log N \cdot \big(1 - \varepsilon(a)\big) > 0.$$

For $a = 1$, $y(1/2) = 2(1/2)^\varrho - 1$, and using the multiplicity of the point $u = 1/2$, we obtain an analogous estimate from which, and from Theorem 5, the lemma follows. In particular, if $L(u) \equiv 1$ we obtain a solution of the Erdös problem mentioned in Introduction.

II. It can also be exactly proved that with normalization $\log^\varrho N$, $\varrho > 0$, there exists a proper distribution for any additive function, satisfying the following conditions.

(i) $g(p) = \log^\varrho p$, $\varrho > 0$ if $p \in U_1$, where U_1 is a set of prime numbers such that

$$\sum_{\substack{p \leqslant x \\ p \in U_1}} \frac{\log p}{p} \sim \delta \log x, \qquad 0 < \delta < 1.$$

(ii) $$\sum_{\substack{p \leqslant x \\ p \notin U_1}} \left\| \frac{g_\varrho(p)}{\log^\varrho x} \right\|^2 \frac{1}{p} \to 0, \quad \text{for} \quad x \to \infty.$$

We will consider in detail only one particular case. Take the additive function

$$g(p) = \begin{cases} \log p & \text{if} \quad p \equiv 1 \pmod 4, \\ 0 & \text{if} \quad p \equiv 3 \pmod 4). \end{cases}$$

Then $g(n)/\log N$ has a proper limit distribution. In fact,

$$\sum_{\substack{p \leqslant N \\ g(p) \leqslant u \log N}} \left\| \frac{g(p)}{\log N} \right\|^2 \frac{1}{p} = \sum_{p \leqslant N^u, p \equiv 1 \pmod 4} \left(\frac{\log p}{\log N} \right)^2 \frac{1}{p}$$

$$\to L_0(u) = \begin{cases} 0 & \text{if} \quad u \leqslant 0, \\ u/2 & \text{if} \quad 0 < u \leqslant 1, \\ 1/2 & \text{if} \quad u > 1. \end{cases}$$

In the same way we can prove (6) and (7). Hence from the theorem we find that $g(n)/\log N$ has the proper limit distribution, since for arbitrary a

$$\frac{1}{\log N} \sum_{p \leqslant N} \left\| \frac{g(p)}{\log N} + a \left(1 - \frac{\log p}{\log N} \right) - a \right\|^2 \frac{\log p}{p}$$

$$= \frac{1}{\log N} \sum_{\substack{p \leqslant N \\ p \equiv 1 \pmod 4}} \left\| (1-a) \frac{\log p}{\log N} \right\|^2 \frac{\log p}{p} + \frac{1}{\log N} \sum_{\substack{p \leqslant N \\ p \equiv 3 \pmod 4}} \left\| a \frac{\log p}{\log N} \right\|^2 \frac{\log p}{p}$$

$$= \Omega(1).$$

III. Corollary of Theorem 1′ shows that A_N and B_N with which the limit distribution can exist, cannot increase quicker than any power of $\log N$. It is on this, that the third group of examples of additive functions for which do not exist proper limit distributions for arbitrarily chosen B_N and A_N, is based. In particular, functions equal for prime numbers to $\exp(\log^\alpha \log p)$, $\alpha > 1$, belong to this group.

Lemma 5. *Let $g(n)$ be an additive function such that*

$$\sum_{p \leqslant N} \left\| \frac{g(p)}{\log^m N} \right\|^2 \frac{1}{p} = \Omega(1)$$

for any positive m. Then $(g(n)-A_N)/B_N$ *does not have the proper limit distribution.*

Proof. We will assume that there exist A_N and B_N such that $(g(n)-A_N)/B_N$ has a proper limit distribution. Then as a result of Theorem 1′ we have

$$\Delta_N = \max(|A_N|, |B_N|) \leqslant \log^{m_0} N.$$

Therefore $g(n)/\log^{m_1} N$, where $m_1 > m_0$ has the limit distribution. Applying Theorem 2, we find that there exists a function $L_0(u)$, $L_0(\pm\infty) = \lim_{u\to\pm\infty} L_0(u)$ such that

$$\sum_{\substack{p\leqslant N\\ g(p)\leqslant u\log^{m_1}N}} \left\| \frac{g(p)}{\log^{m_1} N}\right\|^2 \frac{1}{p} \to L_0(u)$$

in all points of continuity of $L_0(u)$. From this, for $m_2 > m_1$ we have

$$\sum_{p\leqslant N} \left\| \frac{g(p)}{\log^{m_2} N}\right\|^2 \frac{1}{p} \to 0,$$

which contradicts the condition of the lemma.

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The cyclotomic numbers of order eleven*

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1. Introduction. Let e be an integer greater than 1 and let p be a prime $\equiv 1 \pmod e$, say, $p = ef+1$. Let g be a primitive root (mod p). The number of solutions (s, t) with $0 \leqslant s, t \leqslant f-1$ of the congruence

$$g^{es+h}+1 \equiv g^{et+k} \pmod p, \tag{1.1}$$

where h, k are integers usually taken such that $0 \leqslant h, k \leqslant e-1$, is denoted by $(h, k)_e$. The numbers $(h, k)_e$ are called cyclotomic numbers of order e and in addition to h, k and e depend upon p and g. A central problem in the theory of cyclotomy is to evaluate the cyclotomic numbers in terms of the solutions of certain diophantine systems involving quadratic forms. The cases $e = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 24$ and 30 have been treated by several authors (see for example Dickson ([2], [3] and [4]), Lehmer ([6], $e = 8$), Whiteman ([14], [15] and [16], $e = 10, 12, 16$), Muskat ([8] and [9], $e = 14, 24, 30$), Baumert and Fredricksen ([1], $e = 9, 18$), Muskat and Whiteman ([10], $e = 20$), and Leonard and Williams ([7], $e = 7$).

In this paper we give the first complete treatment of the case $e = 11$, and we begin by stating, for $e = 11$, some results from the theory of cyclotomy. All results are stated when $e = 11$ as this is the only case we consider. For more general results and proofs the reader is referred to Dickson [2], [3] and Storer [11].

Let p be a prime of the form $p = 11f+1$, so that f is even. The cyclotomic numbers $(h, k) = (h, k)_{11}$ are periodic in both h and k (mod 11). They also have the following two well known properties:

$$(h, k) = (11-h, k-h) \tag{1.2}$$

and

$$(h, k) = (k, h). \tag{1.3}$$

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