But the fractions \( p_i^a / \sigma(p_i^a) \) appear in reduced form, so \( p_i^a \neq p_i^d \) for \( i = 1, 2, \ldots \), a conclusion we have already seen is impossible. Hence Case 1 does not occur.

Assuming Case 2 holds, we note that
\[
a = \sigma(m_i^q p_i) - km_i p_i = (p_i + 1) \sigma(m_i) - km_i p_i = p_i \left[ \sigma(m_i) - km_i \right] + \sigma(m_i).
\]
Then if \( \sigma(m_i) = km_i \), we would have \( a = \sigma(m_i) \) and hence \( m_i p_i \notin S'(a) \), a contradiction. Hence we may assume \( \sigma(m_i) > km_i \). Then for \( i = 1, 2, \ldots \), we have
\[
a > p_i + \sigma(m_i) > p_i + m_i.
\]
But either \( \{p_i\} \) or \( \{m_i\} \) is unbounded, so we have a contradiction. This completes the proof of Theorem 5.

References

[9] A. Mąkowski, Remarques sur les fonctions \( \sigma(n) \), \( q(n) \) et \( \sigma(n) \), Mathesis 69 (1960), pp. 308-309.

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An "exact" formula for the 2n-th Bernoulli number

by

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Summary. In [1], Chowla and Hartung prove the following formula for the Bernoulli number \( B_{2n} \): The integer

\[
2 \left( \frac{2^{2n} - 1}{(2n)!} \right) (-1)^{n-1} B_{2n} = 1 + \sum_{k=1}^{2n} \left( \frac{2 \cdot (2^{2n} - 1)(2n)!}{2^{2n} - 1 - 2m} \right) \frac{\zeta(2m)}{(2n)!}.
\]

where \([x]\) as usual denotes the greatest integer \( \leq x \). The idea behind the above formula is to use the formula

\[
\zeta(2n) = \sum_{k=1}^{2n} \frac{k^{2n-2}}{2n} \left( \frac{2n}{2m} \right) \frac{\zeta(2m)}{(2n)!},
\]

and to sum the series for \( \zeta(2n) \) far enough to get the rational number \( B_{2n} \) out sufficiently accurate in order to have its precise value determined. According to heavy overestimation of the denominator of \( B_{2n} \), however, (1) sums the series in (2) unnecessarily far. The objective of the present paper is to show that a much smaller number of terms suffices in the series for \( \zeta(2n) \). It turns out as is natural to suspect, that the \( B_{2n} \)'s with large denominators will need more terms than the others in a formula of the Chowla–Hartung type; to make a comparison, our formula (13) needs only 4 terms for \( B_{2n} \), which has a large denominator \( 1920 \), where Chowla–Hartung's formula needs 54 terms. The number of terms needed to get \( B_{2n} \) at all precisely by the usual technique is in this case 3. We also deduce a corresponding formula with the denominators entirely removed by the use of the von Staudt–Clausen theorem. It needs still fewer terms from the series for \( \zeta(2n) \).

An upper bound for the denominator \( Q_{2n} \) of \( B_{2n} = P_{2n}/Q_{2n} \). As is well-known, the denominator of \( B_{2n} \) is

\[
Q_{2n} = \prod_{i=1}^{2n} P_i,
\]

where

\[
Q_{2n} = \prod_{i=1}^{2n} P_i.
\]
where the product is extended over all primes \( p \), for which \( p - 1 \) divides \( 2n \). The question is: How large might \( Q_{2n} \) get? First, all primes except 2 are odd, hence (apart from the trivial factor 1) only even divisors of \( 2n \) count. Now the even divisors of \( 2n \) are of the form \( 2 \times (n \text{ divisor of } n) \). Furthermore the number of divisors of \( n, d(n) \), is \( \leq 2 \lceil \sqrt{n} \rceil \), since the divisors occur in pairs, \( d \) and \( n/d \), and there are at most \( \lceil \sqrt{n} \rceil \) divisors \( \leq \sqrt{n} \). (If \( n = m^2 \) there is even one divisor less, since \( m = n/m \) in this case.) If all the even divisors \( 2d \) of \( 2n \) would lead to primes \( 2d + 1 \), we would have

\[
Q_{2n} = 2 \prod_{d \mid (2n) \text{ even}} (2d+1) \left( \frac{2n}{d} + 1 \right),
\]

and a fortiori

\[
Q_{2n} \leq 2 \prod_{d=1}^{\lceil \sqrt{2n} \rceil} (2d+1) \left( \frac{2n}{d} + 1 \right).
\]

But (5) is easy to overestimate accurately. First,

\[
\prod_{d=1}^{\lceil \sqrt{2n} \rceil} \frac{2d+1}{d} = \frac{(2s+1)!}{2^s \cdot s!} = \frac{(2s+1)2s+1 + \sqrt{2 \pi (2s+1)} \cdot e^{1/12s+1} \cdot e^{2s}}{\sqrt{2 \pi (2s+1)} \cdot 2^s \cdot e^{1/12s+1} \cdot e^{2s}}
\]

\[
\leq \frac{(2s+1)2s+1 + \sqrt{2 \pi (2s+1)} \cdot 2^s \cdot e^{1/12s+1} \cdot e^{2s}}{2 \sqrt{2 \pi (2s+1)} \cdot e^{1/12s+1} \cdot e^{2s}}< \frac{(2s+1)2s+1 + \sqrt{2 \pi (2s+1)} \cdot e^{1/12s+1} \cdot e^{2s}}{2 \sqrt{2 \pi (2s+1)} \cdot e^{1/12s+1} \cdot e^{2s}}.
\]

Here we have used Stirling's formula with remainder \( 0 < \theta_1 < 1 \), \( 0 < \theta_2 < 1 \), and the fact that \( (1+1/n)^n \) approaches its limit \( e \) from below, as \( n \to \infty \).

Next,

\[
\prod_{d=1}^{\lceil \sqrt{2n} \rceil} (2n+1) \left( 2n + \frac{\sqrt{1} + \frac{1}{2}}{2} \right) < \left( 2n + \frac{\sqrt{1} + \frac{1}{2}}{2} \right)^{\lceil \sqrt{2n} \rceil}
\]

according to the inequality between the geometric and arithmetic mean. Thus, finally

\[
Q_{2n} < 2 \left( 2n + \frac{\sqrt{1} + \frac{1}{2}}{2} \right)^{\lceil \sqrt{2n} \rceil} \left( 2s+1 \right)^{1/2} \cdot e^{1/12s+1} \cdot e^{2s} < 0.8 \left( 4n + s + 1 \right)^{1/2} \cdot e^{1/12s+1} \cdot e^{2s} \quad \text{with} \quad s = \lceil \sqrt{2n} \rceil.
\]

The numerator \( (-1)^{n+1}/P_{2n} \) of \( (-1)^{n+1} B_{2n} Q_{2n} \). Using (8), we now get the integer

\[
(-1)^{n+1} P_{2n} = (-1)^{n+1} B_{2n} Q_{2n} = \frac{(2n)!}{2^{2n-1} \pi^{2n} n} \cdot \left( 2n - \frac{\sqrt{1} + \frac{1}{2}}{2} \right)^{1/2} \cdot e^{1/12s+1} \cdot e^{2s}.
\]

Now the remainder

\[
\sum_{k=1}^{\infty} k^{-2n} < M^{-2n} + \int_{M^{-2n}}^{\infty} x^{-2n} dx = M^{-2n} + M^{-2n+2} \quad \text{if} \quad M \leq 2n - 1.
\]

In order to determine the integer \( (-1)^{n+1} P_{2n} \), precisely, by using only the \( M = 1 \) first terms of the series, it suffices to choose \( M \) large enough to make the remainder

\[
\frac{(2n)!}{2^{2n-1} \pi^{2n} n} \sum_{k=1}^{\infty} k^{-2n} < 1.
\]

Using (8) and (10), we get the following condition for this:

\[
\frac{(2n)!}{2^{2n-1} \pi^{2n} n} \cdot \left( 2n - \frac{\sqrt{1} + \frac{1}{2}}{2} \right)^{1/2} \cdot e^{1/12s+1} \cdot e^{2s} < 0.8 \left( 4n + s + 1 \right)^{1/2} \cdot e^{1/12s+1} \cdot e^{2s} \quad \text{for} \quad 2n - 1 < 1,
\]

where \( s = \lceil \sqrt{2n} \rceil \). This gives

\[
M > \frac{\lceil (2n)! \rceil}{2^{2n-1} \pi^{2n} n} \cdot \frac{1}{2} \cdot \frac{3.2}{e^{1/12s+1} \cdot e^{2s}} \cdot \frac{1}{2n - \frac{\sqrt{1} + \frac{1}{2}}{2}}
\]

\[
\approx \frac{\lceil (2n)! \rceil}{2^{2n-1} \pi^{2n} n} \cdot \frac{3.2}{e^{1/12s+1} \cdot e^{2s}} \cdot \frac{1}{2n - \frac{\sqrt{1} + \frac{1}{2}}{2}}
\]

\[
\quad \text{if} \quad M \leq 2n - 1.
\]

Example. For \( 2n = 30 \), (13) gives with \( s = \lceil \sqrt{30} \rceil = 5 \)

\[
M > \frac{18}{\pi} \cdot (72)^{1/2} \cdot 0.8 \cdot 72 \cdot (3.2)^{1/2} \cdot 0.1171 = 4.01.
\]

In this case our deduction shows that 4 terms of the series would suffice to give the numerator of \( B_{30} \) with an error less than one unit. Knowing \( B_{30} = -26315971553053477373/1919190 \) we can check up on \( 1919190B_{30} \times \times k^{-30} \), and in this way we find that only the 3 first terms of the series actually are needed to determine \( B_{30} \) precisely. The asymptotic value of \( M \) in (13) is \( n/(6e) = 0.1171n \), compared to Chowla–Hartung's 3n.
Remark. The practical man’s approach to the problem would to discard the whole of the foregoing theoretical discussions, includ the complicated formula (13), and just compute the integer
\begin{equation}
P_{2n} = \frac{(2n)!}{2^{2n-1} \cdot 2^{2n}} (1 + 2^{-2n} + 3^{-2n} + \ldots)
\end{equation}
by taking just as many terms of the series as needed for this integer identify itself unambiguously.

A formula with still fewer terms. By the use of the von Staudt– Clausen theorem:
\begin{equation}
B_{2n} = - \sum_{(p-1)|2n} \frac{1}{p} \pmod{1},
\end{equation}
we know that
\begin{equation}
C_{2n} = B_{2n} + \sum_{(p-1)|2n} \frac{1}{p}
\end{equation}
is an integer. These integers have been computed by Knuth and Buckh [2]. In this way we get rid of the tedious deduction of an upper bound for \(Q_{2n}\), and we get
\begin{equation}
(-1)^{n-1} C_{2n} = (-1)^{n-1} B_{2n} + (-1)^{n-1} \sum_{p} \frac{1}{p}.
\end{equation}
\begin{align*}
&= (-1)^{n-1} \sum_{p} \frac{1}{p} + \frac{(2n)!}{2^{2n-1} \cdot 2^{2n}} \zeta(2n) \\
&= (-1)^{n-1} \sum_{p} \frac{1}{p} + \frac{(2n)!}{2^{2n-1} \cdot 2^{2n}} \left( \sum_{k=1}^{M-1} \frac{k}{k^{2n}} + \sum_{k=M}^{\infty} k^{-2n} \right)
\end{align*}
Using (10), we get the remainder
\begin{equation}
R \leq \frac{(2n)!}{2^{2n-1} \cdot 2^{2n}} 2M^{-2n}
\end{equation}
if \(M \leq 2n-1\). Now \(R < 1\) for all
\begin{equation}
M > \frac{((2n)!)^{1/(2n)}}{2\pi} \approx \frac{\sqrt{\pi}}{\sqrt{\pi} (64 \pi n)^{1/4n}}.
\end{equation}
With this formula the previous example \(3n = 36\) gives \(M > 2.36\), which shows that 3 terms suffice with this technique. As a matter of fact, \(-C_{36} \cdot 3^{-36} \approx 10^{-4}\) so that only 2 terms would suffice in this case. A asymptotic number of terms needed is the same as for the previous or \(n/(\epsilon\pi)\). The “practical man’s approach” also applies.