But the fractions $p_i^{x_i}/\sigma(p_i^{x_i})$ appear in reduced form, so $p_1^{x_1} = p_2^{x_2} = \ldots$, a conclusion we have already seen is impossible. Hence Case 1 does not occur.

Assuming Case 2 holds, we note that

$$a = \sigma(m_i p_i) - k m_i p_i = (p_i+1)\sigma(m_i) - k m_i p_i = p_i[\sigma(m_i) - k m_i] + \sigma(m_i).$$

Then if $\sigma(m_i) = k m_i$, we would have $a = \sigma(m_i)$ and hence $m_i p_i \notin S'(a)$, a contradiction. Hence we may assume $\sigma(m_i) > k m_i$. Then for $i = 1, 2, \ldots$, we have

$$a \geqslant p_i + \sigma(m_i) \geqslant p_i + m_i.$$

But either $\{p_i\}$ or $\{m_i\}$ is unbounded, so we have a contradiction. This completes the proof of Theorem 5.

References

[1] P. Cattaneo, *Sui numeri quasiperfetti*, Boll. Un. Mat. Ital. (3) 6 (1951), pp. 59–62.
[2] H. Davenport, *Über Numeri Abundantes*, Sitzungsberichte der Preussichen Akademie, Phys. Math. Klasse (1933), pp. 830–837.
[3] P. Erdös, *On primitive abundant numbers*, J. London Math. Soc. 10 (1935), pp. 49–58.
[4] — *On the density of some sequences of numbers: III*, J. London Math. Soc. 13 (1938), pp. 119–127.
[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fourth edition, Oxford 1960.
[6] B. Hornfeck und E. Wirsing, *Über die Häufigkeit vollkommener Zahlen*, Math. Ann. 133 (1957), pp. 431–438.
[7] H.-J. Kanold, *Über die Verteilung der vollkommenen Zahlen und allgemeinerer Zahlenmengen*, Math. Ann. 132 (1956), pp. 442–450.
[8] D. H. Lehmer, *On Euler's totient function*, Bull. Amer. Math. Soc. 38 (1932), pp. 745–757.
[9] A. Mąkowski, *Remarques sur les fonctions $\theta(n)$, $\varphi(n)$ et $\sigma(n)$*, Mathesis 69 (1960), pp. 302–303.
[10] I. J. Schoenberg, *Über die asymptotische Verteilung reeler Zahlen* mod 1, Math. Zeitschr. 28 (1928), pp. 171–200.
[11] W. Sierpiński, *Elementary Theory of Numbers*, Warszawa 1964.
[12] E. Wirsing, *Bemerkung zu der Arbeit über vollkommene Zahlen*, Math. Ann. 137 (1959), pp. 316–318.

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An "exact" formula for the $2n$-th Bernoulli number

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Summary. In [1], Chowla and Hartung prove the following formula for the Bernoulli number B_{2n}: The integer

$$(1)\qquad 2(2^{2n}-1)(-1)^{n-1}B_{2n} = 1 + \left[\frac{2(2^{2n}-1)(2n)!}{2^{2n-1}\pi^{2n}}\sum_{k=1}^{3n} k^{-2n}\right],$$

where $[x]$ as usual denotes the greatest integer $\leqslant x$. The idea behind the above formula is to use the formula

$$(2)\qquad \zeta(2n) = \sum_{k=1}^{\infty} k^{-2n} = \frac{2^{2n-1}\pi^{2n}(-1)^{n-1}B_{2n}}{(2n)!},$$

and to sum the series for $\zeta(2n)$ far enough to get the rational number B_{2n} out sufficiently accurate in order to have its precise value determined. According to heavy overestimation of the denominator of B_{2n}, however, (1) sums the series in (2) unnecessarily far. The objective of the present paper is to show that a much smaller number of terms suffices in the series for $\zeta(2n)$. It turns out as is natural to suspect, that the B_{2n}'s with large denominators will need more terms than the others in a formula of the Chowla–Hartung type; to make a comparison, our formula (13) needs only 4 terms for B_{36}, which has a large denominator 1919190, where Chowla–Hartung's formula needs 54 terms. The number of terms needed to get B_{36} at all precisely by the used technique is in this case 3. We also deduce a corresponding formula with the denominators entirely removed by the use of the von Staudt–Clausen theorem. It needs still fewer terms from the series for $\zeta(2n)$.

An upper bound for the denominator Q_{2n} of $B_{2n} = P_{2n}/Q_{2n}$. As is well-known, the denominator of B_{2n} is

$$(3)\qquad Q_{2n} = \prod_{(p-1)|2n} p,$$

where the product is extended over all primes p, for which $p-1$ divides $2n$. The question is: How large might Q_{2n} get? First, all primes except 2 are odd, hence (apart from the trivial factor 1) only even divisors of $2n$ count. Now the even divisors of $2n$ all are of the form $2\times$(a divisor of n). Furthermore the number of divisors of n, $d(n)$, is $\leqslant 2[\sqrt{n}]$, since the divisors occur in pairs, d and n/d, and there are at most $[\sqrt{n}]$ divisors $\leqslant [\sqrt{n}]$. (If $n = m^2$ there is even one divisor less, since $m = n/m$ in this case.) If *all* the even divisors $2d$ of $2n$ would lead to primes $2d+1$, we would have

$$Q_{2n} = 2\prod_{\substack{d\mid n\\ d\leqslant[\sqrt{n}]}}(2d+1)\left(\frac{2n}{d}+1\right), \tag{4}$$

and *a fortiori*

$$Q_{2n} \leqslant 2\prod_{d=1}^{[\sqrt{n}]}(2d+1)\left(\frac{2n}{d}+1\right). \tag{5}$$

But (5) is easy to overestimate accurately. First,

$$\begin{aligned}\prod_{d=1}^{s}\frac{2d+1}{d} &= \frac{(2s+1)!}{2^s\cdot(s!)^2} = \frac{(2s+1)^{2s+1}\sqrt{2\pi(2s+1)}\cdot e^{\theta_1/12(2s+1)}\cdot e^{2s}}{e^{2s+1}\cdot 2^s\cdot s^{2s}\cdot 2\pi s\cdot e^{\theta_2/6s}}\\ &\leqslant \frac{(2s+1)^{2s}(\sqrt{2s+1})^3}{(2s)^{2s}\cdot s\cdot e\sqrt{2\pi}}\cdot 2^s\cdot e^{1/12(2s+1)}\\ &= \frac{1}{e}\left(1+\frac{1}{2s}\right)^{2s}\cdot\frac{(2s+1)^{3/2}\cdot 2^s}{s\sqrt{2\pi}}\cdot e^{1/12(2s+1)} < \frac{(2s+1)^{3/2}\cdot 2^s}{s\sqrt{2\pi}}\cdot e^{1/24s}.\end{aligned} \tag{6}$$

Here we have used Stirling's formula with remainder ($0<\theta_1<1$, $0<\theta_2<1$), and the fact that $(1+1/n)^n$ approaches its limit e from below, as $n\to\infty$. Next,

$$\prod_{d=1}^{s}(2n+d) < \left(2n+\frac{s+1}{2}\right)^s, \tag{7}$$

according to the inequality between the geometric and arithmetic mean. Thus, finally

$$\begin{aligned}Q_{2n} &< 2\left(2n+\frac{s+1}{2}\right)^s 2^s(2s+1)^{3/2}\frac{e^{1/24s}}{s\sqrt{2\pi}}\\ &< 0.8(4n+s+1)^s(2s+1)^{3/2}\cdot e^{1/24s}/s, \quad \text{with} \quad s=[\sqrt{n}].\end{aligned} \tag{8}$$

The numerator $(-1)^{n-1}P_{2n}$ of $(-1)^{n-1}B_{2n}$. Using (8), we now get the integer

$$\begin{aligned}(-1)^{n-1}P_{2n} &= (-1)^{n-1}B_{2n}Q_{2n} = \frac{(2n)!\,Q_{2n}}{2^{2n-1}\pi^{2n}}\zeta(2n)\\ &= \frac{(2n)!\,Q_{2n}}{2^{2n-1}\pi^{2n}}\left(\sum_{k=1}^{M-1}k^{-2n}+\sum_{k=M}^{\infty}k^{-2n}\right).\end{aligned} \tag{9}$$

Now the remainder

$$\begin{aligned}\sum_{M}^{\infty}k^{-2n} &< M^{-2n}+\int_M^{\infty}x^{-2n}dx = M^{-2n}+M^{-(2n-1)}/(2n-1)\\ &= \left(1+\frac{M}{2n-1}\right)M^{-2n}\leqslant 2M^{-2n}, \quad \text{if} \quad M\leqslant 2n-1.\end{aligned} \tag{10}$$

In order to determine the integer $(-1)^{n-1}P_{2n}$ precisely, by using only the $M-1$ first terms of the series, it suffices to choose M large enough to make the remainder

$$\frac{(2n)!\,Q_{2n}}{2^{2n-1}\pi^{2n}}\sum_{M}^{\infty}k^{-2n} < 1. \tag{11}$$

Using (8) and (10), we get the following condition for this:

$$\frac{(2n)!\,0.8(4n+s+1)^s(2s+1)^{3/2}e^{1/24s}}{2^{2n-1}\pi^{2n}s}\,2M^{-2n} < 1, \tag{12}$$

where $s=[\sqrt{n}]$. This gives

$$\begin{aligned}M &> \frac{\{(2n)!\}^{1/2n}\cdot 3.2^{1/2n}(4n+s+1)^{s/2n}}{2\pi\cdot s^{1/2n}}(2s+1)^{3/4n}e^{1/48ns}\\ &\approx \frac{n(4\pi n)^{1/4n}}{e\pi}\left(\frac{3.2}{s}\right)^{1/2n}(4n+s+1)^{s/2n}(2s+1)^{3/4n}e^{1/48ns},\end{aligned} \tag{13}$$

if $M\leqslant 2n-1$.

EXAMPLE. For $2n=36$, (13) gives with $s=[\sqrt{18}]=4$

$$M > \frac{18}{e\pi}(72\pi)^{1/72}0.8^{1/36}77^{1/9}9^{3/72}e^{1/3456} = 4.01. \tag{14}$$

In this case our deduction shows that 4 terms of the series would suffice to give the numerator of B_{36} with an error less than one unit. Knowing $B_{36} = -26315\,27155\,30534\,77373/1919190$ we can check upon $-1919190B_{36}\times\times k^{-36}$, and in this way we find that only the 3 first terms of the series actually are needed to determine P_{36} precisely. The asymptotic value of M in (13) is $n/(e\pi) = 0.1171n$, compared to Chowla–Hartung's $3n$.

Remark. The practical man's approach to the problem would to discard the whole of the foregoing theoretical discussions, includ the complicated formula (13), and just compute the integer

$$P_{2n} = \frac{(2n)!\,Q_{2n}}{2^{2n-1}\pi^{2n}}(1+2^{-2n}+3^{-2n}+\ldots) \tag{15}$$

by taking just as many terms of the series as needed for this integer identify itself unambiguously.

A formula with still fewer terms. By the use of the von Stau Clausen theorem:

$$B_{2n} \equiv -\sum_{(p-1)|2n} 1/p \pmod 1, \tag{16}$$

we know that

$$C_{2n} = B_{2n} + \sum_{(p-1)|2n} 1/p \tag{17}$$

is an integer. These integers have been computed by Knuth and Buckh [2]. In this way we get rid of the tedious deduction of an upper bou for Q_{2n}, and we get

$$\begin{aligned} (-1)^{n-1}C_{2n} &= (-1)^{n-1}B_{2n} + (-1)^{n-1}\sum \frac{1}{p} \\ &= (-1)^{n-1}\sum \frac{1}{p} + \frac{(2n)!}{2^{2n-1}\pi^{2n}}\zeta(2n) \\ &= (-1)^{n-1}\sum \frac{1}{p} + \frac{(2n)!}{2^{2n-1}\pi^{2n}}\left(\sum_{k=1}^{M-1} k^{-2n} + \sum_{k=M}^{\infty} k^{-2n}\right) \end{aligned} \tag{18}$$

Using (10), we get the remainder

$$R \leqslant \frac{(2n)!}{2^{2n-1}\pi^{2n}}\,2M^{-2n} \tag{19}$$

if $M \leqslant 2n-1$. Now $R < 1$ for all

$$M > \frac{\{(2n)!\}^{1/2n}4^{1/2n}}{2\pi} \approx \frac{n}{e\pi}(64\pi n)^{1/4n}. \tag{20}$$

With this formula the previous example $2n = 36$ gives $M > 2.36$, wh shows that 3 terms suffice with this technique. As a matter of f $-C_{36}\cdot 3^{-36} \approx 10^{-4}$ so that only 2 terms would suffice in this case. T asymptotic number of terms needed is the same as for the previous ca $n/(e\pi)$. The "practical man's approach" also applies.

Formulas with still fewer terms. One might use other relations between B_{2n} and $\zeta(2n)$ to get similar results. Using e.g.

$$\sum_{k=1}^{\infty}(2k-1)^{-2n} = (1-2^{-2n})\zeta(2n) = \frac{(-1)^{n-1}(2^{2n}-1)\pi^{2n}B_{2n}}{2(2n)!} \tag{21}$$

would give formulas for B_{2n} that need only approximately half as many terms as the ones exposed above.

References

[1] S. Chowla and P. Hartung, *An "exact" formula for the m-th Bernoulli number*, Acta Arith. 22 (1972), pp. 113–115.

[2] D. E. Knuth and T. J. Buckholz, *Computation of Tangent, Euler and Bernoulli numbers*, Math. Comp. 21 (1967), pp. 663–688.

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