Table 3

d	l	h	t	λ	d	l	h	t	λ
11	3	2	0	1	136	5	4	1	2
11	5	2	1	2	136	7	4	1	2
19	11	1	1	2	143	7	10	1	3
20	3	2	0	1	164	3	8	3	3
35	3	2	1	2	164	5	8	1	2
47	3	5	2	2	227	3	5	1	2
51	5	2	3	2	239	3	15	0	6
56	3	4	1	2	244	11	6	1	2
84	5	4	0	1	248	3	8	0	1
104	5	6	1	2	260	3	8	1	2

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Limit theorems for lacunary series and uniform distribution mod 1

by

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Dedicated to Professor Paul Erdös to his 60th birthday

1. Introduction. A sequence $\langle x_n\rangle$ of real numbers is called uniformly distributed mod 1 if its discrepancy

$$D_N = \sup_{0\leqslant\alpha<\beta\leqslant 1} |N^{-1}A(N,\alpha,\beta)-(\beta-\alpha)| \to 0. \tag{1.1}$$

Here $A(N,\alpha,\beta)$ is the number of indices $n\leqslant N$ with $\alpha\leqslant\{x_n\}<\beta$. (As usual, $\{\varepsilon\}$ denotes the fractional part of ε.) Let $\langle n_k, k\geqslant 1\rangle$ be a lacunary sequence of integers, i.e. a sequence of integers satisfying

$$n_{k+1}/n_k \geqslant q > 1 \qquad (k=1,2,\ldots). \tag{1.2}$$

It is well known (see [8]) that the sequence $\langle n_k x\rangle$ is uniformly distributed mod 1 for almost all x. A much sharper result is due to Erdös and Koksma [3]. They proved that for almost all x

$$ND_N(x) \ll \big(N\log^3 N\log\log N\omega(N)\big)^{1/2} \tag{1.3}$$

where $\omega(N)$ is any monotone sequence increasing to ∞. In 1954 Erdös and Gaal improved (1.3) to

$$ND_N(x) \ll N^{1/2}(\log\log N)^{5/2+\varepsilon} \quad \text{a.e.} \tag{1.4}$$

for any $\varepsilon>0$, but their result was never published. (See [1], p. 56.) As a matter of fact most workers in the field expected even a law of the iterated logarithm to hold which would replace the exponent $5/2+\varepsilon$ in (1.4) by $\frac{1}{2}$ which is best possible. The purpose of this paper is to prove this conjecture, often referred to as the Erdös–Gaal conjecture. More precisely, we shall prove the following theorem.

THEOREM 1. *For almost all* x

$$32^{-1/2} \leqslant \limsup_{N\to\infty} \frac{ND_N(x)}{\sqrt{N\log\log N}} \leqslant C \tag{1.5}$$

where

$$C \leqslant 166 + 664\,(q^{1/2}-1)^{-1}.$$

The lower bound in Theorem 1 (i.e. the left inequality in (1.5)) is well known since the publication of the paper by Erdös and Gaal [2]. For the reader's convenience I shall give a proof at the end of Section 3.

Concerning the constant C I have the following

CONJECTURE.

$$C^2 \leqslant 2 \sup_I \limsup_{N\to\infty} N^{-1} \int_0^1 \Big(\sum_{k\leqslant N} \big(1_I(n_k x) - |I|\big)\Big)^2 dx \tag{1.6}$$

where the supremum is extended over all intervals $I = [\alpha, \beta)$, $0 \leqslant \alpha < \beta \leqslant 1$, 1_I is the indicator of I, extended with period 1 and $|I|$ denotes the length of I.

The value of the constant is suggested by [10], corollary 4.2.2 where an even stronger result was proved for the special case $n_k = 2^k$. This result states that the lim sup in (1.5) actually equals a.e. the square root of the constant on the right-hand side of (1.6). Moreover, I gave trivial bounds on that constant.

For the case $n_k = 2^{l_k}$ ($l_1 < l_2 < \ldots$ integer) the upper bound in Theorem 1 is due to L. and S. Gaal [5]. They did not write down an explicit bound on C. In [10] I gave a different proof of this result and showed that $C \leqslant 2$.

The proof of the upper bound in Theorem 1 is different from the ones given in [5] and [10]. It uses methods developed by Erdös and Gaal [2] and by Takahashi [12].

A well known theorem of Erdös and Turán [4] states that for any positive integer m

$$D_N \ll m^{-1} + \sum_{1\leqslant h\leqslant m} (hN)^{-1} \Big|\sum_{n\leqslant N} e^{2\pi i h x_n}\Big|. \tag{1.7}$$

Here D_N denotes the discrepancy of the sequence $\langle x_n \rangle$ as defined in (1.1). Moreover, by the main result in the paper of Erdös and Gaal [2]

$$\limsup_{N\to\infty} \frac{\Big|\sum_{k\leqslant N} e^{2\pi i n_k x}\Big|}{\sqrt{N \log\log N}} = 1 \quad \text{a.e.} \tag{1.8}$$

for any lacunary sequence $\langle n_k \rangle$ (not necessarily integer). (The result that the limes superior does not exceed 1 is due to Salem and Zygmund [11]). In many discussions on the subject it was suggested that a combination of (1.7) and (1.8) might yield a proof of the upper bound in Theorem 1. But since Erdös is a coauthor of both results it is safe to assume that this idea cannot be made to work.

As an immediate consequence of Theorem 1 and Koksma's inequality (3.10) below we obtain

THEOREM 2. *Let $f(x)$ be a function of bounded variation on $[0, 1]$ with*

$$f(x+1) = f(x) \quad \text{and} \quad \int_0^1 f(x)\,dx = 0. \tag{1.9}$$

Then for any lacunary sequence $\langle n_k \rangle$ of integers

$$\limsup_{N\to\infty} \frac{\Big|\sum_{k\leqslant N} f(n_k x)\Big|}{\sqrt{N \log\log N}} \ll 1 \quad \text{a.e.}$$

where the constant implied by $\ll$ depends on f and, perhaps, on q.

For functions $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ satisfying (1.9) Theorem 2 is due to Takahashi [12]. It is quite likely that a modified version of Theorem 2 holds for sums

$$\sum_{k\leqslant N} a_k f(n_k x) \tag{1.10}$$

(see problem 1 below). For a certain subclass of lacunary sequences theorems of this type have been announced by Gaposhkin ([6], [7]). It seems that the details have not yet been published.

As a matter of fact Theorem 1 and Koksma's inequality yield a result stronger than Theorem 2 and, except for the value of the constant, equivalent to Theorem 1.

THEOREM 3. *Let $\mathcal{F}$ be the class of functions of variation on $[0, 1]$ not exceeding V and satisfying (1.9). Then for any lacunary sequence $\langle n_k \rangle$ of integers*

$$V/8\sqrt{2} \leqslant \limsup_{N\to\infty} \frac{\sup_{f\in\mathcal{F}} \Big|\sum_{k\leqslant N} f(n_k x)\Big|}{\sqrt{N \log\log N}} \leqslant VC \quad \text{a.e.}$$

where a bound for C is given in Theorem 1.

Many interesting questions and problems remain. I shall state a few.

1) Give a detailed proof that

$$\limsup_{N\to\infty} \frac{\Big|\sum_{k\leqslant N} a_k f(n_k x)\Big|}{\sqrt{A_N^2 \log\log A_N}} \ll 1 \quad \text{a.e.}$$

where

$$A_N = \Big(\sum_{k\leqslant N} a_k^2\Big)^{1/2} \to \infty, \qquad a_N = o\,(A_N^2/\log\log A_N)^{1/2}.$$

2) Investigate to what extent Theorems 1 and 3 and problem 1 can be generalized to sequences of integers n_k with

$$n_{k+1}/n_k \geqslant 1+ck^{-a} \quad \text{for some } c>0 \text{ and } 0<a<\tfrac{1}{2}.$$

(The paper by Takahashi [13] might give some clues to this problem.)

3) Which of the theorems remain valid if the condition that n_k is an integer is dropped (see (1.8)).

2. Inequalities for lacunary series. We shall consider only even functions f. Without loss of generality we assume that $\|f\|_\infty \leqslant 1$ and $0 \leqslant x \leqslant 1$. Throughout this chapter $\|f\|$ denotes $\|f\|_2 = (\int_0^1 f^2(x)\,dx)^{1/2}$ and λ denotes Lebesgue measure. The following inequality is fundamental for the proof of Theorem 1.

PROPOSITION. *Let $M \geqslant 0$, $N \geqslant 1$ be integers and let $R \geqslant 1$, $\delta > 0$. Suppose that $f(x)$ satisfies* (1.9) *and $\|f\| \geqslant N^{-1/4}$ and has variation* $\operatorname{Var} f \leqslant 2$. *Then as $N \to \infty$*

$$\lambda\Big\{x\colon \Big|\sum_{k=M+1}^{M+N} f(n_k x)\Big| \geqslant (1+2\delta)C_1 R\|f\|^{1/4}(N\log\log N)^{1/2}\Big\}$$
$$\leqslant \exp\big(-(1+\delta)\|f\|^{-1/2}R\log\log N\big)+R^{-2}N^{-3/4}$$

where

$$C_1 = \tfrac{1}{2}+2(q^{1/2}-1)^{-1}.$$

Here and throughout Chapters 2 and 3 the constant implied by $\ll$ only depends, perhaps, on q and δ. Hence it is enough to prove the proposition for $M = 0$ only since the sequence $\langle n_{M+k},\ k=1,2,\ldots\rangle$ is lacunary with the same factor q. Moreover, it suffices to prove the proposition without the absolute value bars on the left-hand side of the expression in the curly brackets since $-f$ also satisfies the hypotheses of the proposition.

We expand $f(x)$ in a Fourier series with partial sums

$$f_n(x) = \sum_{1\leqslant j\leqslant n} c_j \cos 2\pi jx.$$

Obviously,

$$c_0 = 0, \quad c_j \ll j^{-1} \quad (1\leqslant j\leqslant n). \tag{2.1}$$

Put

$$\varphi_n(x) = f(x)-f_n(x),$$
$$\psi(x) = \psi(x;m,n) = \sum_{m\leqslant j<n} c_j\cos 2\pi jx,$$
$$\Phi_N(x) = \Phi_N(x;m,n) = \sum_{k\leqslant N}\psi(n_k x).$$

LEMMA 1. $\|\Phi_N\|^2 \ll N\log(n/m)\cdot m^{-1}$.

Proof. By the definition of Φ_N

$$\begin{aligned}\|\Phi_N\|^2 &= \int_0^1\Big(\sum_{k\leqslant N}\sum_{m\leqslant j<n} c_j\cos 2\pi jn_k x\Big)^2 dx \\ &= \sum_{k,l\leqslant N}\sum_{m\leqslant i,j<n} c_i c_j\{in_k, jn_l\} \\ &\leqslant \sum_{k,l\leqslant N}\sum_{m\leqslant i,j<n}(c_i^2+c_j^2)\{in_k, jn_l\} \\ &\ll \sum_{k,l\leqslant N}\sum_{m\leqslant i,j<n} c_i^2\{in_k, jn_l\} \\ &\ll \sum_{k\leqslant N}\sum_{m\leqslant i<n} c_i^2 \sum_{l\leqslant N,\, m\leqslant j<n}\{in_k, jn_l\}.\end{aligned} \tag{2.2}$$

Here $\{u, v\}$ denotes the Kronecker symbol. But for fixed k and i with $m\leqslant i<n$ the number of solutions of the equation

$$in_k = jn_l, \quad l\leqslant N,\ m\leqslant j<n \tag{2.3}$$

does not exceed a constant times $\log(n/m)$. Indeed, let l_1 be the smallest index l so that n_l is a solution of (2.3) and let l_2 be the largest index with that property. Then

$$j_1 n_{l_1} = j_2 n_{l_2}, \quad m\leqslant j_2\leqslant j_1<n$$

and by (1.2)

$$q^{l_2-l_1}\leqslant n_{l_2}/n_{l_1} = j_1/j_2\leqslant n/m.$$

Hence

$$l_2-l_1 \ll \log(n/m).$$

Consequently, the inner sum in (2.2) is bounded by that quantity and thus by (2.1)

$$\|\Phi_N\|^2 \ll N\log(n/m)\sum_{i\geqslant m} c_i^2 \ll N\log(n/m)m^{-1}.$$

LEMMA 2. *For any positive integer T*

$$\int_0^1\Big(\sum_{k\leqslant N}\varphi_T(n_k x)\Big)^2 dx \ll NT^{-1}.$$

Proof. Let h_0 be the smallest integer with $T\leqslant 2^{h_0}$. We apply Lemma 1 with $m = T$, $n = 2^{h_0}$ and $m = 2^h$, $n = 2^{h+1}$ $(h = h_0, h_0+1, \ldots)$ and obtain

$$\Big\|\sum_{k\leqslant N}\varphi_T(n_k x)\Big\| \leqslant \Big\|\sum_{k\leqslant N}\psi(n_k x; T, 2^{h_0})\Big\| + \sum_{h\geqslant h_0}\Big\|\sum_{k\leqslant N}\psi(n_k x; 2^h, 2^{h+1})\Big\|$$
$$\ll N^{1/2}\Big(T^{-1/2}+\sum_{h\geqslant h_0}2^{-\frac{1}{4}h}\Big) \ll N^{1/2}T^{-1/2}. \ \blacksquare$$

Now let H be any integer with

$$q^H > 3H^6. \tag{2.4}$$

Following Takahashi [12] we put

$$g(x) = \sum_{j=1}^{H^6} c_j \cos 2\pi jx \quad \text{and} \quad U_m(x) = \sum_{k=Hm+1}^{H(m+1)} g(n_k x). \tag{2.5}$$

We observe that by (2.1)

$$\sum_{j\geq n} c_j^2 \leq n^{-1}$$

and

$$\|g\|_\infty \leq \operatorname{Var} f + \|f\|_\infty \leq 3. \tag{2.6}$$

LEMMA 3. *Let $\varkappa > 0$ be with $\varkappa H^{3/2} < 1$. Then for any positive integer P*

$$\int_0^1 \exp\Big\{\varkappa \sum_{m=0}^{P-1} U_{2m}(x)\Big\}\,dx \leq \exp\big(\tfrac{1}{2}(1+\delta)C_1\varkappa^2\|f\|HP\big)$$

and

$$\int_0^1 \exp\Big\{\varkappa \sum_{m=1}^{P} U_{2m-1}(x)\Big\}\,dx \leq \exp\big(\tfrac{1}{2}(1+\delta)C_1\varkappa^2\|f\|HP\big)$$

where C_1 is defined in the proposition.

This is a modification of [12], Lemma 1. Its proof can be easily modeled after [12]. The only difference of any significance is that we employ the inequality

$$e^z \leq 1 + z + \tfrac{1}{2}(1+\delta)z^2 \quad \text{for} \quad |z| \leq z_0$$

instead of the one used in [12]. We then obtain

$$\int_0^1 \exp\Big\{\varkappa \sum_{m=0}^{P-1} U_{2m}(x)\Big\}dx \leq \int_0^1 \prod_{m=0}^{P-1} \big(1 + \varkappa U_{2m}(x) + \tfrac{1}{2}\varkappa^2(1+\delta)\,U_{2m}^2(x)\big)\,dx.$$

Except for the factor $1+\delta$ the integral on the right-hand side is estimated in [12], pp. 102–103. In [12] relation (2.8) a factor 2 must be added. We also observe that the constant B_2 in Takahashi's paper [12] does not exceed $C_1\|f\|$. Taking the factor $1+\delta$ into account we obtain the first inequality in Lemma 3. The second one can be proved in the same way. ∎

We now can prove the proposition. Again we use ideas of Takahashi [12]. Let $N \geq N_0$ be given. Put $H = [N^{1/6}]$. If N_0 is sufficiently large then (2.4) is satisfied. With this choice of H we define the functions $g(x)$ and $U_m(x)$ by (2.5). For any $Q > 0$ we have

$$\lambda\Big\{x:\ \sum_{k\leq N} f(n_k x) \geq (1+2\delta)Q\Big\} \leq \lambda(A_1) + \lambda(A_2) \tag{2.7}$$

where

$$A_1 = \Big\{x:\ \sum_{k\leq N} g(n_k x) \geq (1+\delta)Q\Big\}, \qquad A_2 = \Big\{x:\ \sum_{k\leq N} \varphi_T(n_k x) \geq \delta Q\Big\}$$

and

$$T = [N^{1/6}]^6. \tag{2.8}$$

We put

$$Q = 2C_1\|f\|^{1/4}R(N\log\log N)^{1/2}$$

and apply the inequality

$$\lambda\{x:\ h(x) \geq (1+\delta)Q\} \leq \exp\big(-(1+\delta)Q\varkappa\big)\cdot\int_0^1 \exp\big(\varkappa h(x)\big)\,dx \tag{2.9}$$

with

$$h(x) = \sum_{k\leq N} g(n_k x) \quad \text{and} \quad \varkappa = (\|f\|^{-3/2}N^{-1}\log\log N)^{1/2}.$$

To estimate the integral in (2.9) we define the integer P by

$$H(2P+1) \leq N < H(2P+3). \tag{2.10}$$

Then by (2.6), (2.10) and the hypothesis on $\|f\|$

$$\varkappa\Big|\sum_{k\leq N} g(n_k x) - \sum_{m=0}^{2P} U_m(x)\Big| \leq \varkappa \sum_{k=H(2P+1)}^{N} |g(n_k x)| \leq 6H\varkappa = o(1).$$

Hence

$$\int_0^1 \exp\{\varkappa h(x)\}\,dx \ll \int_0^1 \exp\Big\{\varkappa \sum_{m=0}^{2P} U_m(x)\Big\}dx \ll \exp\big((1+\delta)C_1\varkappa^2\|f\|N\big)$$

by Cauchy's inequality, Lemma 3 and (2.10) since $\varkappa H^{3/2} = o(1)$. Thus by (2.9)

$$\lambda(A_1) \ll \exp\big(-(1+\delta)C_1R\|f\|^{-1/2}\log\log N\big). \tag{2.11}$$

Next, by Lemma 2 and (2.8)

$$\lambda(A_2) \ll Q^{-2} \ll R^{-2}N^{-3/4}. \tag{2.12}$$

The proposition follows now from (2.7), (2.11) and (2.12).

3. Proof of Theorem 1. It is possible to prove a general theorem in probability theory where the hypothesis is just about the conclusion of the proposition of Section 2 and then to derive Theorem 1 from it. This procedure is then similar to the one followed in [9] (see in particular Satz 1) and [10] (in particular Theorem 1.3.1 and Chapter 4) and has a wide range of applications. But for the sake of simplicity I shall give a more direct proof of Theorem 1 using a method of Erdös and Gaal [2].

Let $\eta,\delta>0$ and let $N\geqslant N_0(\eta,\delta)$ be given. Put

$$H=[\log N/\log 4]+1. \tag{3.1}$$

Any $0\leqslant a\leqslant 1$ can be written in dyadic expansion

$$a=\sum_{j=1}^{\infty}2^{-j}\varepsilon_j,\quad \varepsilon_j=0,1.$$

Obviously,

$$\sum_{j=1}^{H}2^{-j}\varepsilon_j\leqslant a\leqslant\sum_{j=1}^{H}2^{-j}\varepsilon_j+2^{-H}.$$

Put

$$\varrho_h(x)=\varrho_h(x,a)=1\Big\{\sum_{j=1}^{h}2^{-j}\varepsilon_j\leqslant x<\sum_{j=1}^{h+1}2^{-j}\varepsilon_j\Big\},\quad 1\leqslant h<H,$$

$$\sigma_H(x)=\sigma_H(x,a)=1\Big\{\sum_{j=1}^{H}2^{-j}\varepsilon_j\leqslant x<\sum_{j=1}^{H}2^{-j}\varepsilon_j+2^{-H}\Big\}.$$

Here $1\{\alpha\leqslant x<\beta\}$ denotes the indicator of the interval $[\alpha,\beta)$ extended with period 1. Then

$$\sum_{h=1}^{H-1}\varrho_h(x)\leqslant 1\{0\leqslant x<a\}\leqslant\sum_{h=1}^{H-1}\varrho_h(x)+\sigma_H(x). \tag{3.2}$$

For fixed h there are only 2^h different functions ϱ_h and there are only 2^H different functions σ_H as a varies between 0 and 1. We denote these functions by $\varrho_h^{(j)}$ $(1\leqslant j\leqslant 2^h)$ and $\sigma_H^{(j)}$ $(1\leqslant j\leqslant 2^H)$. Moreover, these functions have the same structure. Consequently, it makes sense to define

$$\varphi_h^{(j)}=\begin{cases}\varrho_h^{(j)}, & 1\leqslant j\leqslant 2^h\quad\text{if}\quad 1\leqslant h<H,\\ \sigma_H^{(j)}, & 1\leqslant j\leqslant 2^H\quad\text{if}\quad h=H.\end{cases} \tag{3.3}$$

For integer $1\leqslant j\leqslant 2^h$, $1\leqslant h\leqslant H$, $N\geqslant 1$, $M\geqslant 0$ we write

$$F(M,N,j,h)=\Big|\sum_{k=M+1}^{M+N}\Big(\varphi_h^{(j)}(n_kx)-\int_0^1\varphi_h^{(j)}(x)\,dx\Big)\Big|. \tag{3.4}$$

LEMMA 4. *Define n by $2^n\leqslant N<2^{n+1}$. Then there are integers m_l with $0\leqslant m_l<2^{n-l}$ $(1\leqslant l\leqslant n)$ such that*

$$F(0,N,j,h)\leqslant F(0,2^n,j,h)+\sum_{\frac{1}{2}n\leqslant l\leqslant n}F(2^n+m_l2^l,2^{l-1},j,h)+N^{1/3}.$$

This is a slight modification of [5], Lemma 3.10. We put

$$\phi(N)=2(1+2\delta)C_1(N\log\log N)^{1/2} \tag{3.5}$$

and define the sets

$$G(n,j,h)=\{x\colon F(0,2^n,j,h)\geqslant 2^{-h/8}\phi(2^n)\},$$

$$H(n,j,h,l,m)=\{x\colon F(2^n+m2^l,2^{l-1},j,h)\geqslant 2^{-h/8}2^{(l-n-3)/6}\phi(2^n)\},$$

$$G_n=\bigcup_{h\leqslant H}\bigcup_{j\leqslant 2^h}G(n,j,h),\qquad H_n=\bigcup_{h\leqslant H}\bigcup_{j\leqslant 2^h}\bigcup_{\frac{1}{2}n\leqslant l\leqslant n}\bigcup_{m\leqslant 2^{n-l}}H(n,j,h,l,m).$$

LEMMA 5. *There is an $n_0=n_0(\eta,\delta)$ such that*

$$\lambda\Big(\bigcup_{n\geqslant n_0}(G_n\cup H_n)\Big)<\eta.$$

Proof. We observe that by (3.1)

$$N^{-1/2}\leqslant 2^{-h-1}\leqslant\Big\|\varphi_h^{(j)}-\int_0^1\varphi_h^{(j)}\Big\|^2\leqslant 2^{-h}\qquad(1\leqslant j\leqslant 2^h,\ 1\leqslant h\leqslant H). \tag{3.6}$$

We apply the proposition with $M=0$, $N=2^n$ and $R=1$. Of course, the functions $\varphi_h^{(j)}$ are not even. But any function $\alpha(x)$ with period 1, $\operatorname{Var}\alpha\leqslant 2$ and $\int_0^1\alpha(x)\,dx=0$ can be written in the form $\alpha(x)=\alpha_1(x)+\alpha_2(x)$, where α_i $(i=1,2)$ have the same properties as α, α_1 is even and α_2 is odd. (Simply put $\alpha_1(x)=\frac{1}{2}(\alpha(x)+\alpha(-x))$.) Since the proposition also holds for odd functions satisfying the remaining hypotheses we conclude that the application of the proposition to $\varphi_h^{(j)}-\int\varphi_h^{(j)}$ is legitimate if we replace C_1 by $2C_1$. The factor 2 has been taken care of in (3.5). Hence by (3.6)

$$\lambda\big(G(n,j,h)\big)\ll\exp\big(-(1+\delta)2^{h/4}\log n\big)+N^{-3/4}$$

and so by (3.1)

$$\lambda(G_n)\ll n^{-1-\frac{1}{2}\delta} \tag{3.7}$$

if N is sufficiently large. A similar application of the proposition with $M=2^n+m2^l$, $N=2^{l-1}$ and $R=R_l=2^{(n-l)/3}$ yields

$$\lambda\big(H(n,j,h,l,m)\big)\ll\exp\big(-(1+\delta)2^{h/4}R\log n\big)+2^{2(l-n)/3}N^{-3/4}.$$

Hence

$$\lambda(H_n)\ll n^{-1-\frac{1}{2}\delta}. \tag{3.8}$$

The lemma follows now from (3.7) and (3.8). ∎

We now can finish the proof of Theorem 1. Let $0 \leqslant a \leqslant 1$ be arbitrary. By (3.1)–(3.3), Lemmas 4 and 5 for $N \geqslant N_0$ (recall that the indicator $1\{0 \leqslant x < a\}$ was extended with period 1)

$$\Big|\sum_{k\leqslant N} 1\{0 \leqslant n_k x < a\} - Na\Big| \leqslant \sum_{h=1}^{H} \Big|\sum_{k\leqslant N} \varphi_h^{(j)}(n_k x) - N\int_0^1 \varphi_h^{(j)}\Big| + 2^{-H}N$$

$$\leqslant \sum_{h\leqslant H}\Big\{F(0, 2^n, j, h) + \sum_{\frac{1}{2}n\leqslant l\leqslant n} F(2^n + m_l 2^{l-1}, j, h)\Big\} + 2N^{1/2}$$

$$\leqslant \phi(2^n)\sum_{h\leqslant H} 2^{-h/8}\Big\{1 + \sum_{\frac{1}{2}n\leqslant l\leqslant n} 2^{(l-n-3)/6}\Big\} + 2N^{1/2}$$

$$\leqslant \phi(N)(2^{1/8}-1)^{-1}\big(1 + 2^{-1/2}(1-2^{-1/6})^{-1}\big) + 2N^{1/2}$$

$$\leqslant (1+4\delta)\big(83 + 332(q^{1/2}-1)^{-1}\big)(N\log\log N)^{1/2}$$

for all x except, perhaps, a set of measure less than η, no matter how a was chosen. Hence for those x

$$\Big|\sum_{k\leqslant N} 1\{a \leqslant n_k x < \beta\} - N(\beta - a)\Big|$$

$$\leqslant (1+4\delta)\big(166 + 664(q^{1/2}-1)^{-1}\big)(N\log\log N)^{1/2}.$$

We divide by $(N\log\log N)^{1/2}$, take the supremum over all $0 \leqslant a < \beta \leqslant 1$, take the limes superior, put $\delta = 0$ and then $\eta = 0$, all in that order. This proves the upper bound.

The lower bound is an immediate consequence of (1.8) and the following inequality

$$\Big|\sum_{k\leqslant N} e^{2\pi i x_k}\Big| \leqslant \sqrt{32}\, N D_N \qquad (3.9)$$

valid for any sequence $\langle x_k\rangle$ with discrepancy D_N. This follows easily from Koksma's inequality (see [8])

$$\Big|\sum_{k\leqslant N} f(x_k) - N\int_0^1 f(x)\,dx\Big| \leqslant V(f)\, N D_N \qquad (3.10)$$

which holds for any sequence $\langle x_n\rangle$ with discrepancy D_N and for any function f with period 1 and variation $V(f)$. We apply (3.10) with $f(x) = \sin x$ and $f(x) = \cos x$ and obtain (3.9).

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Note added later. Recently Niederreiter proved that the constant $\sqrt{32}$ in (3.9) can be improved to 4. Consequently the left side in (1.5) can be improved to 1/4 and the left side in Theorem 3 to $V/8$.

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