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Addendum to the paper "On the product of the conjugates outside the unit circle of an algebraic number"

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by

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The aim of this Addendum is to formulate two theorems which go further than Theorems 2 and 3 of [1] ([1]) and have been practically proved in that paper, but the fact has been overlooked by the writer. The notation of [1] is retained. In particular for a given polynomial F we denote by $|F|$ its degree, by $C(F)$ its content and by $\|F\|$ the sum of squares of the absolute values of the coefficients.

THEOREM 2′. *Let K be a totally real algebraic number field or a totally complex quadratic extension of such a field and $P \in K[z]$ a polynomial with the leading coefficient p_0 such that $z^{|P|}\bar{P}(z^{-1}) \neq \text{const}\, P(z)$, $P(0) \neq 0$.*

Let $|K|$ be the degree of K, $P^{(i)}$ $(i = 1, \ldots, |K|)$ the polynomials conjugate to $P(z)$ and a_{ij} the zeros of $P^{(i)}(z)$. Then

$$\prod_{i=1}^{|K|} \prod_{|a_{ij}|>1} |a_{ij}| \geqslant \begin{cases} \left(\dfrac{1+\sqrt{5}}{2}\right)^{|K|/2} \left(N_{K/Q} \dfrac{C(P)}{(p_0)}\right)^{1/2+1/2\sqrt{5}} \cdot \left(N_{K/Q} \dfrac{P(0)}{C(P)}\right)^{1/2-1/2\sqrt{5}} & \text{if } |P(0)| \neq |p_0|, \\ \left(\dfrac{1+\sqrt{17}}{4}\right)^{|K|} \left(N_{K/Q} \dfrac{(P(0)C(P),\, p_0 C(\bar{P}))}{(p_0 \bar{p}_0)}\right)^{1/\sqrt{17}} & \text{if } |P(0)| = |p_0|. \end{cases}$$

COROLLARY 1′. *If $z^{|P|}\bar{P}(z^{-1}) \neq \text{const}\, P(z)$, $P(0) \neq 0$ then*

$$\prod_{i=1}^{|K|} \prod_{|a_{ij}|>1} |a_{ij}| \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{|K|/2} N_{K/Q} \frac{C(P)}{(p_0)}.$$

([1]) Misprints of that paper are listed at the end of the Addendum.

THEOREM 3′. *Let K satisfy the assumptions of Theorem 2′, L be a subfield of K, $f(z)\in L[z]$. The number n of irreducible factors P of f such that $z^{|P|}\overline{P}(z^{-1})\neq \mathrm{const}\,\overline{P}(z)$, $P(0)\neq 0$ counted with their multiplicities satisfies the inequality*

$$(*)\qquad \left(\frac{1+\sqrt{5}}{2}\right)^{n|L|}+\left(\frac{1+\sqrt{5}}{2}\right)^{-n|L|}\leqslant N_{L/Q}\|f\|N_{L/Q}^{-2}C(f)$$

with the equality attained only if either $L=Q$, $f(z)=c(z^{|f|}\pm 1)$ or $K\supset Q(\sqrt{5},\zeta_m)$, $L=Q$.

$$(\overset{*}{*})\qquad z^{|f|}f(z)f\left(\frac{1}{z}\right)=c\left(z^{4lm}-\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2m}+\left(\frac{1-\sqrt{5}}{2}\right)^{2m}\right]z^{2lm}+1\right),$$

l, m integers, m odd.

COROLLARY 2′. *The number n occurring in Theorem 3′ satisfies the inequality*

$$n<\frac{\log\left(N_{L/Q}\|f\|N_{L/Q}^{-2}C(f)\right)}{|L|\log\dfrac{1+\sqrt{5}}{2}}$$

where the constant $\log\dfrac{1+\sqrt{5}}{2}$ is best possible.

To see Theorem 1 it is enough to note that by (28) on p. 394 of [1] $p_0^{(i)}\overline{p}_0^{(i)}a_k^{(i)}$ is an integer divisible by

$$\left(\overline{P^{(i)}(0)}\,C(P^{(i)}),\,p_0^{(i)}C(\overline{P}^{(i)})\right).$$

(In particular if $\overline{P}(0)C(P)=(p_0)C(\overline{P})$ then $\overline{p}_0^{(i)}a_k^{(i)}$ is divisible by $C(\overline{P}^{(i)})$.) Hence:

$$|N_{K/Q}a_k^{(1)}|\geqslant N_{K/Q}\frac{\left(\overline{P(0)}\,C(P),\,p_0C(\overline{P})\right)}{(p_0\overline{p}_0)}$$

and the assertion of Theorem 2′ in the case $|P(0)|=|p_0|$ follows from the formula

$$\prod_{i=1}^{|K|}\prod_{|a_{ij}|>1}|a_{ij}|=\prod_{i=1}^{|K|}|c_{i0}|^{-1}\geqslant\left(\frac{1+\sqrt{17}}{4}\right)^{|K|}|N_{K/Q}a_k^{(1)}|^{1/\sqrt{17}}$$

(see [1], p. 394, line 10 from below). The case $|P(0)|\neq|p_0|$ has been settled in [1].

To see Corollary 1′ it is enough to note that

$$\left(\frac{1+\sqrt{17}}{4}\right)^{|K|}N_{K/Q}\left(\frac{\left(\overline{P(0)}\,C(P),\,p_0C(\overline{P})\right)}{(p_0\overline{p}_0)}\right)^{1/\sqrt{17}}>\left(\frac{1+\sqrt{5}}{2}\right)^{|K|/2}\left(N_{K/Q}\frac{C(P)}{(p_0)}\right)^{2/\sqrt{17}}.$$

Theorem 3′ follows from Corollary 1′ in the same way as Theorem 3 from Theorem 2 in [1] under the assumption about prime ideal factors of $\left(f_0,f(0)\right)C(f)^{-1}$, where f_0 is the leading coefficient of f.

Corollary 2′ follows directly from $(*)$ and the existence of polynomials satisfying $(\overset{*}{*})$, e.g.

$$f(z)=z^{2lm}\pm\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m}+\left(\frac{1-\sqrt{5}}{2}\right)^{m}\right]z^{lm}-1.$$

Note that the bound given in Corollary 2′ is independent of K.

Reference

[1] A. Schinzel, *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arith. 24 (1973), pp. 385–399.

Corrigenda to [1]

page 384
Formula (5) of Lemma 1 is due to F. Wiener, see H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. (2) 13 (1914), pp. 1–5. (I owe this reference to Prof. E. Bombieri.)

page 386, line 2 and page 393, line 7

for	$1/2+1/\sqrt{5}$	read	$1/2+1/2\sqrt{5}$
for	$1/2-1/\sqrt{5}$	read	$1/2-1/2\sqrt{5}$

page 388, line 6 from below

for	$\pm\prod_{\lvert a_j\rvert=1}(-a_j)$	read	$\pm\dfrac{p_0}{P(0)}$
for	$\pm P(0)$	read	$\pm$

page 389, line 7 from below

for	(5) and (6)	read	(7) and (8)

page 393, line 2

for	$-1)=$	read	$-1)^{\lvert K\rvert}=$

page 394, line 9 from below

for	$p_0^{(i)}$	read	$\overline{p}_0^{(i)}$

page 394, line 8 from below

for	$P^{(i)}$	read	$\overline{P}^{(i)}$