Addendum to the paper “On the product of the conjugates outside the unit circle of an algebraic number”


by

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The aim of this Addendum is to formulate two theorems which go further than Theorems 2 and 3 of [1] (*) and have been practically proved in that paper, but the fact has been overlooked by the writer. The notation of [1] is retained. In particular for a given polynomial \( F \) we denote by \( \deg F \) its degree, by \( C(F) \) its content and by \( \|F\| \) the sum of squares of the absolute values of the coefficients.

**Theorem 2'.** Let \( K \) be a totally real algebraic number field or a totally complex quadratic extension of such a field and \( F \in K[z] \) a polynomial with the leading coefficients \( p_0 \) such that \( z^{\deg F} F(z^{-1}) \neq \text{const} \cdot P(z), \; P(0) \neq 0 \).

Let \( |K| \) be the degree of \( K \); \( P(i) \) (\( i = 1, \ldots, |K| \)) the polynomials conjugate to \( F(z) \) and \( \alpha_0 \) the zeros of \( P(0)(z) \) then

\[
\prod_{i=1}^{|K|} \prod_{|a_i| \geq 1} |a_i| \geq \begin{cases} 
\left( \frac{1 + \sqrt{5}}{2} \right)^{|K|/2} \left( \frac{N_{K \to P(C)^{1/2}}}{p_0} \right) \cdot \left( \frac{N_{K \to C(F)^{1/2}}}{p_0} \right)^{|\alpha_0|/2} & \text{if } |P(0)| \neq |p_0|, \\
\left( \frac{1 + \sqrt{17}}{4} \right)^{|K|} \left( \frac{N_{K \to (P(0)C(F), p_0C(F))^{1/2}}}{(p_0 \alpha_0)^{1/2}} \right) & \text{if } |P(0)| = |p_0|.
\end{cases}
\]

**Corollary 1'.** If \( z^{\deg F} F(z^{-1}) \neq \text{const} \cdot P(z), \; P(0) \neq 0 \) then

\[
\prod_{i=1}^{|K|} \prod_{|a_i| \geq 1} |a_i| \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{|K|/2} \frac{C(F)}{p_0}.
\]

(*) Misprints of that paper are listed at the end of the Addendum.
THEOREM 3'. Let $K$ satisfy the assumptions of Theorem 2', $L$ be a subfield of $K$, $f(z) \in L[z]$. The number $n$ of irreducible factors $P$ of $f$ such that $z^{[L]} P(z^{-1}) \not\equiv \text{const} P(z)$, $P(0) \neq 0$ counted with their multiplicities satisfies the inequality

$$(*) \quad (1 + \frac{1+V^2}{2})^{n[\mathbf{L}]} + (1 + \frac{1+V^2}{2})^{-n[\mathbf{L}]} \leq N_{K/L} \|f\| N_{K/L}^{-2} C(f)$$

with the equality attained only if either $L = Q$, $f(z) = c(z^{[L]} - 1)$ or $K \subset Q(V, \zeta_m)$, $L = Q$.

(4') $z^{[L]} f(z) f(\frac{1}{z}) = c \left( z^{[L]} - \left[ \left| \frac{1+V^2}{2} \right| z^{[L]} + \left| \frac{1-V^2}{2} \right| z^{[L]} \right] - z^{[L]} \right)$

for $l, m$ integers, $m$ odd.

COROLLARY 2'. The number $n$ occurring in Theorem 3' satisfies the inequality

$$n \leq \frac{\log (N_{K/L} \|f\| N_{K/L}^{-2} C(f))}{[L] \log \frac{1+V^3}{2}}$$

where the constant $\frac{1+V^3}{2}$ is best possible.

To see Corollary 1' it is enough to note that by (28) on p. 394 of [1] $P(0) \| f \| P(0)$ is an integer divisible by $C(f)$.

(In particular if $P(0) C(f) = \langle P_0 \rangle C(P)$ then $P_0 \| f \| P(0)$ is divisible by $C(P)$. 

Hence:

$$|N_{K/L} a_0| \geq N_{K/L} \frac{C(P)}{\langle P_0 \rangle P(0)}$$

and the assertion of Theorem 2' in the case $|P(0)| = |P_0|$ follows from the formula

$$\prod_{j=1}^{[L]} \prod_{a_j=1}^{[L]} a_j = \prod_{j=1}^{[L]} a_j^{-1} \geq \left( \frac{1+\sqrt{17}}{4} \right)^{[K]} N_{K/L} a_0^{-1} 1/4$$

(see [1], p. 394, line 10 from below). The case $|P(0)| \neq |P_0|$ has been settled in [1].

To see Corollary 1' it is enough to note that

$$\left( \frac{1+\sqrt{17}}{4} \right)^{[L]} N_{K/L} \frac{C(P)}{\langle P_0 \rangle P(0)} \geq \left( \frac{1+\sqrt{5}}{2} \right)^{[L]} N_{K/L} \frac{C(P)}{\langle P_0 \rangle P(0)}$$

Theorem 3' follows from Corollary 1' in the same way as Theorem 3 from Theorem 2 in [1] under the assumption about prime ideal factors of $\langle f_0, f(0) \rangle C(f)^{-1}$, where $f_0$ is the leading coefficient of $f$.

Corollary 2' follows directly from (4') and the existence of polynomials satisfying (4'), e.g.

$$f(z) = z^{[L]} - \left[ \left| \frac{1+V^2}{2} \right| z^{[L]} + \left| \frac{1-V^2}{2} \right| z^{[L]} \right] - 1.$$