ACTA ARITHMETICA
XXVI (1974)

The transcendence of linear forms in $\omega_1, \omega_2, \eta_1, \eta_2, 2\pi i, \log\gamma$

by

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I. Introduction. In a series of papers, Baker [1], [2] and Coates [3] have studied the transcendence of linear forms in the periods of elliptic functions. In this paper I will prove a theorem of the same type. Let $p(z)$ be a Weierstrass p-function with algebraic invariants g_2, g_3 and let $\omega_1, \omega_2, \eta_1, \eta_2$ be defined as usual. Let γ be a non-zero algebraic number.

THEOREM. *Assume $p(z)$ has complex multiplication. Then any non-vanishing linear form in $\omega_1, \omega_2, \eta_1, \eta_2, 2\pi i, \log\gamma$, with algebraic coefficients, is transcendental.*

The proof of this theorem is essentially the same as the proofs of the theorems in the papers referred to above. There are minor changes in the estimates, and the only serious difference is in the treatment of the determinant which appears at the end of the proof. For the present problem we employ a result of Tijdeman [4] on the number of zeros of exponential polynomials. Because of the similarity of this proof to the others, many of the following results are stated without proof. The above references contain proofs.

II. Lemmas on elliptic functions. Let K be the number field generated by g_2, g_3 over the rationals. Let n be an arbitrary integer > 1. Write $c, c_1, c_2, \ldots$ for positive constants which depend only on g_2, g_3, not on n.

LEMMA 1. *Assume $\frac{1}{4}g_2, \frac{1}{4}g_3$ are algebraic integers. Then*

$$p\left(\frac{\lambda_1\omega_1+\lambda_2\omega_2}{n}\right) \qquad (0 \leqslant \lambda_1, \lambda_2 < n;\ \lambda_1, \lambda_2 \text{ integers not both } 0)$$

is an algebraic number with the maximum of the absolute values of its conjugates at most $c_1 n^2$. Further, the leading coefficient of its minimal integral polynomial divides n^{c_2}.

Let K_n be the field obtained by adjoining to K all of the numbers $p(\omega)$, $p'(\omega)$, where ω denotes $(\lambda_1\omega_1+\lambda_2\omega_2)/n$ and λ_1, λ_2 range from 0 to $n-1$ excluding $\lambda_1 = \lambda_2 = 0$.

LEMMA 2. *The field K_n has degree $c_3 n^2$ over K and contains $e^{2\pi i/n}$.*

LEMMA 3. *Assume $\frac{1}{4}g_2$, $\frac{1}{4}g_3$ algebraic integers. Let ν_1, ν_2 be integers with $(\nu_1, \nu_2, 2n) = 1$. Then the number*

$$\xi(\nu_1, \nu_2, 2n) = \left(\frac{\nu_1\eta_1+\nu_2\eta_2}{2n}\right) - \left(\frac{\nu_1\omega_1+\nu_2\omega_2}{2n}\right)$$

belongs to the field K_{2n}, each of its conjugates has absolute value at most $c_3 n$, and the leading coefficient in its minimal integral polynomial divides $(2n)^{c_4}$.

LEMMA 4. *For any positive integer k, the j-th derivative of $p(z)^k$ can be expressed as*

$$\sum u p(z)^t p'(z)^{t'} p''(z)^{t''}$$

where the summation is over all non-negative integers t, t', t'' with $2t+3t'+4t'' = j+2k$, and $u = u(t, t', t'', j, k)$ denotes a rational integer with absolute value at most $j!\, c_5^{j+k}$.

LEMMA 5. *Let $f(z)$ be a function regular at a point z such that $f''(z) = -p'(z)$. Then for any positive integer k, the j-th derivative of $f(z)^k$ can be expressed as*

$$\sum u f(z)^\tau f'(z)^{\tau'} p(z)^t p''(z)^{t''}$$

where the summation is over all non-negative integers τ, τ', t, t', t'' with $\tau+2\tau'+2t+3t'+4t'' = j+k$, $\tau+\tau' \leqslant k$, and $u = u(\tau, \tau', t, t', t'', j, k)$ denotes a rational integer with absolute value at most $j!\, c_6^{j+k}$.

Proof see Lemma 3 of [2].

III. Proof of the theorem. We suppose there exist algebraic integers $\alpha_0 \neq 0$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ such that

$$\alpha_1\omega_1+\alpha_2\omega_2+\beta_1\eta_1+\beta_2\eta_2+\alpha_3 2\pi i+\alpha_4\log\gamma = \alpha_0 \tag{2}$$

and will show that that assumption leads to a contradiction. We first note that there is no loss of generality in assuming that $\frac{1}{4}g_2$, $\frac{1}{4}g_3$ are algebraic integers. This is clear from the observation that, for every positive integer a, the invariants associated with the p-function with periods ω_1/a, ω_2/a are $a^4 g_2$, $a^6 g_3$ and the corresponding values of η_1, η_2 are $a\eta_1$, $a\eta_2$.

We shall use the following notation. We denote by $\mathscr{K}$ the field generated by

$$\begin{gathered}\alpha_i\ (0 \leqslant i \leqslant 4), \quad \beta_j\ (1 \leqslant j \leqslant 2), \quad g_2, g_3, \\ p_{ij} = p^{(i)}(\omega_j/2) \quad (0 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 2)\end{gathered} \tag{3}$$

over the rationals, and we write d for the degree of $\mathscr{K}$. We write $p(\lambda, \mu, z)$ for the μth derivative of the λth power of $p(z)$. By $c, c_1, c_2, \ldots$ we shall

signify positive numbers which depend only on ω_1, ω_2, and the numbers (3). Finally, for any function $F(z_1, z_2, z_3, z_4)$ of the complex variables z_1, z_2, z_3, z_4 and any non-negative integers m_1, m_2, m_3, m_4 we put

$$F_{m_1,m_2,m_3,m_4}(z_1, z_2, z_3, z_4) = \frac{\partial^{m_1+m_2+m_3+m_4}}{\partial z_1^{m_1}\partial z_2^{m_2}\partial z_3^{m_3}\partial z_4^{m_4}} F(z_1, z_2, z_3, z_4).$$

We denote by k a positive integer and we define

$$L = L_0 = L_1 = L_2 = L_3 = [k^{13/14}], \quad L_4 = [k^{13/28}], \quad h = [k^{1/28}]$$

where, as usual, $[x]$ denotes the integral part of x. We assume throughout that $k > c$, where c is chosen sufficiently large for the validity of the subsequent arguments. Further, we define

$$\begin{aligned} &f(z_1, z_2, z_3, z_4) \\ &\quad = \alpha_1\omega_1 z_1+\alpha_2\omega_2 z_2+\beta_1\zeta(\omega_1 z_1)+\beta_2\zeta(\omega_2 z_2)+\alpha_3 2\pi i z_3+\alpha_4\log\gamma z_4.\end{aligned}$$

The proof now proceeds by a series of lemmas.

LEMMA 6. *Let M, N be integers with $N > M > 0$, and let u_{ij} $(1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N)$ be integers with absolute values at most $U \geqslant 1$. Then there are integers $x_1, \ldots, x_N$, not all 0, with absolute values at most $(NU)^{M/(N-M)}$ such that*

$$\sum_{j=1}^{N} u_{ij}x_j = 0 \quad (1 \leqslant i \leqslant M).$$

LEMMA 7. *There exist integers $\varrho(\lambda_0, \ldots, \lambda_4)$, not all zero, with absolute values at most k^{12k} such that the function*

$$\begin{aligned}\Phi(z_1, z_2, z_3, z_4) = \sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\sum_{\lambda_3=0}^{L_3}\sum_{\lambda_4=0}^{L_4} \varrho(\lambda_0, \ldots, \lambda_4) f(z_1, z_2, z_3, z_4)^{\lambda_0} \times \\ \times p(\omega_1 z_1)^{\lambda_1} p(\omega_2 z_2)^{\lambda_2} e^{2\pi i\lambda_3 z_3}\gamma^{\lambda_4 z_4}\end{aligned}$$

satisfies

$$\Phi_{m_1,m_2,m_3,m_4}(s+\tfrac{1}{2}, \ldots, s+\tfrac{1}{2}) = 0 \tag{4}$$

for all integers s with $1 \leqslant s \leqslant h$, and all non-negative integers m_1, m_2, m_3, m_4 with $m_1+m_2+m_3+m_4 \leqslant k$.

Proof. This follows by a standard argument using Siegel's lemma, and by our choice of L, h.

LEMMA 8. *Suppose $Z \geqslant 6$, and let*

$$\varphi(z_1, z_2, z_3, z_4) = \Phi(z_1, z_2, z_3, z_4)\prod_{j=1}^{2}\prod_{\Omega_j}(\omega_j z_j - \Omega_j)^{3L}, \tag{5}$$

where Ω_j runs over all periods of $p(z)$ with

$$|\Omega_j| \leqslant |\omega_j| Z. \tag{6}$$

Then $\varphi(z_1, \ldots, z_4)$ *is regular in the* disc $|z_j| \leqslant Z$ $(j = 1, 2, 3, 4)$. *For any* z *with* $|z| \leqslant \frac{1}{2}Z$, *and for any non-negative integers* $m_1, \ldots, m_4$ *with* $m_1 + \ldots + m_4 \leqslant k$ *we have*

$$(7) \qquad |\varphi_{m_1,\ldots,m_4}(z, z, z, z)| \leqslant k^{14k} Z^{c_8 LZ^2}.$$

Proof. This is the same as Lemma 8 of [3] with k^{10k} replaced by k^{12k}. It is necessary to assume, as one can without loss of generality, that $|\gamma| \leqslant 1$.

LEMMA 9. *Let* Q, S, Z *be numbers such that* $1 < Q < S < Z-1$, *and let* $m_1, \ldots, m_4$ *be integers such that* $m_1 + \ldots + m_4 \leqslant k$. *Suppose* q, r, s *are integers, with* q *even,* $(r, q) = 1$, *and*

$$(8) \qquad 1 \leqslant q \leqslant Q, \quad 1 \leqslant s \leqslant S, \quad 1 \leqslant r < q$$

such that

$$(9) \qquad \varphi_{\mu_1,\mu_2,\mu_3,\mu_4}\left(s + \frac{r}{q}, \ldots, s + \frac{r}{q}\right) = 0$$

for all non-negative integers $\mu_1, \ldots, \mu_4$ *with* $\sum_{j=1}^{4} \mu_j < \sum_{j=1}^{4} m_j$. *Then either* (9) *holds when* $\mu_j = m_j$ $(j = 1, 2, 3, 4)$ *or we have*

$$(10) \qquad \left|\varphi_{m_1,m_2,m_3,m_4}\left(s + \frac{r}{q}, \ldots, s + \frac{r}{q}\right)\right| > (kS)^{-c_9(k+L_4S)Q^3}.$$

Proof. The hypotheses imply that when $\mu_j = m_j$ $(j = 1, \ldots, 4)$ the number on the left of (10) is given by $W_1 W_2 \Phi$, where

$$W_j = \prod_{\Omega_j} \left\{\omega_j\left(s + \frac{r}{q}\right) - \Omega_j\right\}^{3L} \quad (j = 1, 2),$$

$$\Phi = \Phi_{m_1,m_2,m_3,m_4}\left(s + \frac{r}{q}, \ldots, s + \frac{r}{q}\right).$$

Now $\Omega_1 = n_1\omega_1 + n_2\omega_2$ for some integers n_1, n_2, and, as $I(\omega_2/\omega_1) \neq 0$, it follows that

$$\left|\omega_1\left(s + \frac{r}{q}\right) - \Omega_1\right| = |\omega_1| \left|\left(s - \eta_1 + \frac{r}{q}\right) - n_2(\omega_2/\omega_1)\right|$$

exceeds c_{10} unless $n_1 = s$ or $s+1$ and $n_2 = 0$; in the latter cases we have the lower bound c_{11}/Q. Similar estimates hold for the factors of W_2. Hence $|\omega_1\omega_2| > (c_{12}/Q)^{6L}$, and it therefore suffices to prove either $\Phi = 0$ or

$$|\Phi| > (kS)^{-c_{13}(k+L_4S)Q^3}.$$

We now define $p_{ij} = p^{(i)}\left(\frac{r\omega_j}{q}\right)$ $(0 \leqslant i \leqslant 2, 1 \leqslant j \leqslant 2)$. It is plain from (2) that

$$(11) \qquad f\left(s + \frac{r}{q}, \ldots, s + \frac{r}{q}\right) = \alpha_0\left(s + \frac{r}{q}\right) - \beta_1\xi_1 - \beta_2\xi_2$$

where, in the notation of Lemma 3,

$$\xi_1 = \xi(sq+r, 0, \tfrac{1}{2}q), \qquad \xi_2 = \xi(0, sq+r, \tfrac{1}{2}q).$$

We conclude from Lemma 3 that

$$\Psi = \omega_1^{-m_1}\omega_2^{-m_2}(2\pi i)^{-m_3}(\log\gamma)^{-m_4}\Phi$$

is contained in the field generated by

$$g_2, g_3, \quad \alpha_i \ (0 \leqslant i \leqslant 4), \quad \beta_j \ (1 \leqslant j \leqslant 2), \quad p^{(i)}(\omega_j/q) \ (0 \leqslant i \leqslant 2, 1 \leqslant j \leqslant 2),$$
$$e^{2\pi i/q}, \quad \gamma^{1/q}$$

over the rationals. By Lemma 2, this field has degree at most $c_{14}q^3$, and so, in particular, Ψ has degree at most $c_{14}q^3$. Further, it is clear from Lemmas 1 and 3, and the equations $p'(z)^2 = p(z)^3 - g_2 p(z) - g_3$ and $p''(z) = 6p'(z)^2 - \frac{1}{2}g_2$, that we can find an integer c_{15} such that $q^{c_{15}}$ times p_{ij} and (11) are both algebraic integers with the maximum of their absolute values at most $q^{c_{16}}$. Denoting by g the leading coefficient of the minimal integral polynomial of γ, it is clear from the explicit expression for Ψ that

$$(12) \qquad g^{2L_4S} q^{c_{17}(m_1+m_2+m_3+m_4+L)} \Psi$$

is an algebraic integer with the maximum of the absolute values of its conjugates at most

$$(L+1)^4(L_4+1)2^k k^{12k}(m_1!\, m_2!\, m_3!\, m_4!)^4 (c_{18}S)^{c_{19}(k+L)} c_{20}^{L_4S} < (kS)^{c_{21}(k+L_4S)}.$$

Since Ψ has degree at most $c_{14}Q^3$, the assertion of the lemma follows on noting that either $\Psi = 0$ or the norm of (12) is at least 1.

LEMMA 10. *Let* J *be an integer satisfying* $0 \leqslant J \leqslant 110$. *Then*

$$\Phi_{m_1,m_2,m_3,m_4}\left(s + \frac{r}{q}, \ldots, s + \frac{r}{q}\right) = 0$$

for all integers q, r, s *with* q *even,* $(r, q) = 1$,

$$1 \leqslant q \leqslant 2h^{J/8}, \quad 1 \leqslant s \leqslant h^{(J/4)+1}, \quad 1 \leqslant r < q$$

and all non-negative integers m_1, m_2, m_3, m_4 *with* $m_1 + m_2 + m_3 + m_4 \leqslant k/2^J$.

Proof. The lemma is valid for $J = 0$ by Lemma 7. We suppose that I is an integer with $0 \leqslant I < 110$, and we assume that the lemma holds for $J = 0, \ldots, I$. We proceed to deduce its validity for $J = I+1$. We define

$$Q_J = 2h^{J/8}, \quad S_J = h^{(J/4)+1}, \quad T_J = [k/2^J] \quad (J = 0, \ldots, I)$$

and we assume that there are integers q', r', s', with q' even, $(r', q') = 1$,

$$1 \leqslant q' \leqslant Q_{I+1}, \quad 1 \leqslant s' \leqslant S_{I+1}, \quad 1 \leqslant r' < q',$$

and non-negative integers m_1', m_2', m_3', m_4' with $m_1'+m_2'+m_3'+m_4' \leqslant T_{I+1}$ satisfying

$$\Phi_{m_1', m_2', m_3', m_4'}\left(s'+\frac{r'}{q'}, \ldots, s'+\frac{r'}{q'}\right) \neq 0$$

and we shall derive a contradiction. Further, we assume that m_1', m_2', m_3', m_4' are chosen minimally so that

$$\Phi_{\mu_1, \mu_2, \mu_3, \mu_4}\left(s'+\frac{r'}{q'}, \ldots, s'+\frac{r'}{q'}\right) = 0$$

for all non-negative integers $\mu_1, \ldots, \mu_4$ with $\sum_{j=1}^{4} \mu_j < \sum_{j=1}^{4} m_j'$.

Let $Z = 10S_{J+1}$, and let $\varphi(z_1, z_2, z_3, z_4)$ be the function defined in Lemma 8 for this choice of Z. Let $\psi(z) = \varphi_{m_1', m_2', m_3', m_4'}(z, z, z, z)$. Then, by our inductive hypothesis, we see that, for all integers q, r, s, with q even, $(r, q) = 1$,

$$1 \leqslant q \leqslant Q_I, \quad 1 \leqslant s \leqslant S_I, \quad 1 \leqslant r < q,$$

and each integer m satisfying $0 \leqslant m \leqslant T_{I+1}$, we have

$$\psi_m\left(s+\frac{r}{q}\right) = 0, \tag{13}$$

for $\psi_m\left(s+\frac{r}{q}\right)$ is given by

$$\sum_{\substack{j_1=0 \\ \Sigma j = m}}^{m_1} \cdots \sum_{j_4=0}^{m_4} \frac{m!}{j_1!\,j_2!\,j_3!\,j_4!}\, \varphi_{m_1'+j_1, \ldots, m_4'+j_4}\left(s+\frac{r}{q}, \ldots, s+\frac{r}{q}\right)$$

and the partial derivatives vanish here because

$$m_1'+j_1+\ldots+m_4'+j_4 \leqslant T_I.$$

Now write

$$F(z) = \prod_{\substack{q=1 \\ q\,\text{even}}}^{Q_I} \prod_{s=1}^{S_I} \prod_{\substack{r=1 \\ (r,q)=1}}^{q} \left(z-s-\frac{r}{q}\right)^{T_{I+1}}.$$

Then by (13), $\psi(z)F(z)$ is regular in the disc $|z| \leqslant 5S_{I+1}$. Hence, denoting by θ and Θ the upper bound of $|\psi(z)|$ and the lower bound of $|F(z)|$, respectively, on the circle $|z| = 5S_{I+1}$, we conclude from the maximum modulus principle that

$$\left|\psi\left(s'+\frac{r'}{q'}\right)\right| \leqslant \left|F\left(s'+\frac{r'}{q'}\right)\right| \frac{\theta}{\Theta}. \tag{14}$$

Now, by Lemma 8, we have

$$\theta \leqslant k^{14k}(10S_{I+1})^{-3L(10S_{I+1})^2}. \tag{15}$$

Further, since for any z with $|z| = 5S_{I+1}$,

$$\left|z-s-\frac{r}{q}\right| \geqslant 2\left|\left(s'+\frac{r'}{q'}\right)-\left(s+\frac{r}{q}\right)\right|,$$

and since also the number of sets q, r, s which occur in the definition of $F(z)$ is at least

$$\tfrac{1}{2}S_I(\varphi(2)+\varphi(4)+\ldots+\varphi(Q_I')) > c_{22}Q_I^2 S_I,$$

where Q_I' denotes the largest even integer not exceeding Q_I and φ denotes Euler's function (cf. [1], p. 155), we clearly have

$$\Theta > 2^{c_{21}Q_I^2 S_I T_{I+1}}\left|F\left(s'+\frac{r'}{q'}\right)\right|. \tag{16}$$

As

$$Q_I^2 = 4S_{I+1}h^{-5/4}, \quad S_I = S_{I+1}h^{-1/4}, \quad T_{I+1} > c_{23}k, \quad L \leqslant kh^{-2},$$

it follows readily from (14), (15), (16) that

$$\left|\psi\left(s'+\frac{r'}{q'}\right)\right| < 2^{-c_{24}Q_I^2 S_I T_{I+1}}. \tag{17}$$

On the other hand, the hypotheses of Lemma 9 are satisfied with

$$Q = Q_{I+1}, \quad S = S_{I+1}, \quad q = q', \quad r = r', \quad s = s', \quad m_j = m_j' \ (1 \leqslant j \leqslant 4),$$

and, by virtue of our initial assumption, we have $\psi(s'+r'/q') \neq 0$. We conclude from Lemma 9 that

$$\left|\psi\left(s'+\frac{r'}{q'}\right)\right| > (kS_{I+1})^{-c_9(k+L_4 S_{I+1})Q_{I+1}^8}. \tag{18}$$

But, as $L \leqslant kh^{-2}$, $L_4 \leqslant kh^{-15}$, $I < 110$, we see that the estimates (17) and (18) are contradictory for k sufficiently large. This completes the proof of the lemma.

IV. Completion of the proof. Let $P_k(z)$, $1 \leqslant k \leqslant n$ be polynomials of degree $\varrho_k - 1$ respectively. Put $\sigma = \sum_{k=1}^{n} \varrho_k$. Let $\omega_1, \ldots, \omega_k$ be arbitrary complex numbers, and put $\Delta = \max_k |\omega_k|$.

LEMMA 11. *The number of zeroes of the function*

$$F(z) = \sum_{k=1}^{n} P_k(z) e^{\omega_k z} \tag{19}$$

in an arbitrary circular disk of radius R in the complex plane is at most

$$3(\sigma - 1) + 4R\Delta. \tag{20}$$

Proof see [4], p. 58.

Now Lemma 10 implies

$$\Phi_{m_1, m_2, m_3, m_4}(s + \tfrac{1}{4}, \ldots, s + \tfrac{1}{4}) = 0 \tag{21}$$

for all integers s with $1 \leqslant s \leqslant L+1$, and all integers $m_1, \ldots, m_4$ with $0 \leqslant m_1, \ldots, m_4 \leqslant L$. Putting

$$G(z) = \Phi_{m_1, \ldots, m_4}(z, \ldots, z)$$

and noting that $L + k^{1/2} < 2^{1/110} k$, it is clear from Lemma 10 that

$$G_m(s + \tfrac{1}{4}) = 0 \quad (1 \leqslant s \leqslant L+1,\ 0 \leqslant m \leqslant k^{1/2}). \tag{22}$$

The left side of (21) can be written in the form

$$\sum_{\lambda_0=0}^{L_0} \sum_{\lambda_4=0}^{L_4} \sum_{\mu_1=0}^{m_1} \cdots \sum_{\mu_4=0}^{m_4} \binom{m_1}{\mu_1} \cdots \binom{m_4}{\mu_4} r(\lambda_0, \lambda_4, \mu, s) q(\lambda_0, \lambda_4, m_1 - \mu_1, \ldots, m_3 - \mu_3)$$

where $r(\lambda_0, \lambda_4, \mu, s)$ denotes the function

$$\left\{ \frac{\partial^{\mu_1+\mu_2+\mu_3+\mu_4}}{\partial z_1^{\mu_1} \ldots \partial z_4^{\mu_4}} f(z_1, \ldots, z_4)^{\lambda_0} \right\} \left\{ \frac{\partial^{m_4-\mu_4}}{\partial z_4^{m_4-\mu_4}} \gamma^{\lambda_4 z_4} \right\}$$

evaluated at the point $z_1 = \ldots = z_4 = s + \frac{1}{4}$, and

$$q(\lambda_0, \lambda_4, \nu_1, \nu_2, \nu_3) = \sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_3=0}^{L_3} \varrho(\lambda_0, \ldots, \lambda_4) p\left(\lambda_1, \nu_1, \frac{\omega_1}{4}\right) p\left(\lambda_2, \nu_2, \frac{\omega_2}{4}\right) (2\pi i \lambda_3)^{\nu_3} (i)^{\lambda_3}.$$

In particular, for $m_1 = \ldots = m_4 = 0$ we see that

$$\sum_{\lambda_0=0}^{L_0} \sum_{\lambda_4=0}^{L_4} f(s + \tfrac{1}{4}, \ldots, s + \tfrac{1}{4})^{\lambda_0} \gamma^{\lambda_4(s+\frac{1}{4})} q(\lambda_0, \lambda_4, 0, 0, 0) = 0 \tag{23}$$

for $1 \leqslant s \leqslant L+1$. Furthermore, (22) implies that each of these zeroes has multiplicity at least $k^{1/2}$. On the other hand, (23) can be put in the form of (19) with

$$P_{\lambda_4}(z) = \sum_{\lambda_0=0}^{L_0} f(s + \tfrac{1}{4}, \ldots, s + \tfrac{1}{4})^{\lambda_0} q(\lambda_0, \lambda_4, 0, 0, 0) \gamma^{1/4},$$

and $\omega_{\lambda_4} = \lambda_4 \log \gamma$. By Lemma 11, (23) should have no more than

$$3(L_4 + 1)(L + 1) + 4(L + \tfrac{5}{4}) L_4 \log \gamma \tag{24}$$

zeroes in the circle $|z| \leqslant L + \frac{5}{4}$. But for sufficiently large k, the number (24) is clearly less than $Lk^{1/2}$, so (23) is identically zero, and we conclude that

$$q(\lambda_0, \lambda_4, c, c, c) = 0 \quad (0 \leqslant \lambda_0 \leqslant L,\ 0 \leqslant \lambda_4 \leqslant L_4).$$

Now let ν_1, ν_2, ν_3 be any three integers with $0 \leqslant \nu_1, \nu_2, \nu_3 \leqslant L$, and suppose that $q(\lambda_0', \lambda_4', \nu_1', \nu_2', \nu_3') = 0$ for all integers $\lambda_0', \lambda_4', \nu_1', \nu_2', \nu_3'$ with

$$0 \leqslant \lambda_0', \lambda_4' \leqslant L, \quad 0 \leqslant \nu_1', \nu_2', \nu_3' \leqslant L, \quad \sum_{j=1}^{3} \nu_j' < \sum_{j=1}^{3} \nu_j.$$

Then (21) with $m_1 = \nu_1$, $m_2 = \nu_2$, $m_3 = \nu_3$, $m_4 = 0$ gives

$$\sum_{\lambda_0=0}^{L_0} \sum_{\lambda_4=0}^{L_4} r(\lambda_0, \lambda_4, 0, s) q(\lambda_0, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (1 \leqslant s \leqslant L+1),$$

and, as above, we conclude $q(\lambda_0, \lambda_4, \nu_1, \nu_2, \nu_3) = 0$ for $0 \leqslant \lambda_0, \lambda_4 \leqslant L$. It follows by induction that

$$q(\lambda_0, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (0 \leqslant \lambda_0, \lambda_4 \leqslant L,\ 0 \leqslant \nu_1, \nu_2, \nu_3 \leqslant L).$$

Now choose λ_0, λ_4 so that $\varrho(\lambda_0, \ldots, \lambda_4) \neq 0$ for some $\lambda_1, \lambda_2, \lambda_3$. We conclude that the determinant Δ of coefficients of the $(L+1)^3$ equations

$$q(\lambda_0, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (0 \leqslant \nu_1, \nu_2, \nu_3 \leqslant L)$$

must vanish. But it is well known (¹) that

$$\Delta = (\Delta_1 \Delta_2 \Delta_3)^{L+1},$$

where

$$\Delta_j = 2! \ldots L! \, p'(\tfrac{1}{4}\omega_j)^{\frac{1}{2}L(L+1)} \quad (j = 1, 2),$$

$$\Delta_3 = 2! \ldots L! (-2\pi)^{\frac{1}{2}L(L+1)}.$$

Since $p'(w_j) \neq 0$ $(j = 1, 2)$, it follows that $\Delta \neq 0$, which is a contradiction. Thus (2) cannot be valid, and the proof of the theorem is complete.

(¹) Cf. Lemmas 6 and 7 of [1].

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Received on 10. 10. 1973 (471)

ACTA ARITHMETICA
XXVI (1974)

Scharfe untere Abschätzung für die Anzahlfunktion der *B*-Zwillinge

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1. Einleitung. Es sei $\mathscr{B}$ die Menge aller natürlichen Zahlen, die sich als Summe zweier Quadrate von ganzen Zahlen darstellen lassen. Die Elemente von $\mathscr{B}$ heißen B-Zahlen. Das Paar $(n, n+1)$ nennen wir B-Zwilling, wenn sowohl $n \in \mathscr{B}$ als auch $n+1 \in \mathscr{B}$ ist. Nach G. Rieger [6] gilt für die Anzahl (¹) $B_2(x) = \#\{n \leqslant x \colon n \in \mathscr{B}, n+1 \in \mathscr{B}\}$ der B-Zwillinge unterhalb x die obere Abschätzung

$$B_2(x) \ll x(\log x)^{-1}. \tag{1.1}$$

Bezüglich der Abschätzung von $B_2(x)$ nach unten ist bisher nur (vgl. [2], [7])

$$B_2(x) \geqslant c(\varepsilon)x(\log x)^{-(2\log 2+\varepsilon)} \qquad (\varepsilon > 0) \tag{1.2}$$

bekannt, wobei die Konstante $c(\varepsilon)$ nur von ε abhängt. In dieser Note soll mit Hilfe des Selberg'schen Siebes eine Abschätzung von $B_2(x)$ nach unten gegeben werden, welche (1.2) verbessert und die richtige Größenordnung von $B_2(x)$ angibt. Wir zeigen

SATZ. *Für die Anzahl der B-Zwillinge unterhalb x gilt*

$$B_2(x) \gg x(\log x)^{-1}. \tag{1.3}$$

Der Beweis beruht auf zwei Lemmata, welche Abschätzungen von Spezialfällen des linearen Siebes nach unten und des großen Siebes nach oben enthalten.

$c_1, c_2, \ldots$ bezeichnen positive Konstanten.

2. Hilfssätze. Sei $b(\cdot)$ die charakteristische Funktion von $\mathscr{B}$, $\mathscr{D}_1 := \{m \in N\colon p \mid m,\ p \text{ prim} \Rightarrow p \equiv 1\ (4)\}$ und $\mathscr{D}_3 := \{m \in N\colon p \mid m,\ p \text{ prim} \Rightarrow p \equiv 3\ (4)\}$.

(¹) $\#\{n\colon \ldots\}$ bezeichnet die Anzahl der n mit den Eigenschaften...