The transcendence of linear forms in \( \omega_1, \omega_2, \eta_1, \eta_2, 2\pi i, \log \gamma \)

by

Richard Franklin (Providence, R. I.)

I. Introduction. In a series of papers, Baker [1], [2] and Coates [3] have studied the transcendence of linear forms in the periods of elliptic functions. In this paper I will prove a theorem of the same type. Let \( p(z) \) be a Weierstrass \( p \)-function with algebraic invariants \( g_2, g_3 \) and let \( \omega_1, \omega_2, \eta_1, \eta_2 \) be defined as usual. Let \( \gamma \) be a non-zero algebraic number.

Theorem. Assume \( p(z) \) has complex multiplication. Then any non-vanishing linear form in \( \omega_1, \omega_2, \eta_1, \eta_2, 2\pi i, \log \gamma \), with algebraic coefficients, is transcendental.

The proof of this theorem is essentially the same as the proofs of the theorems in the papers referred to above. There are minor changes in the estimates, and the only serious difference is in the treatment of the determinant which appears at the end of the proof. For the present problem we employ a result of Tijdeman [4] on the number of zeros of exponential polynomials. Because of the similarity of this proof to the others, many of the following results are stated without proof. The above references contain proofs.

II. Lemmas on elliptic functions. Let \( K \) be the number field generated by \( g_2, g_3 \) over the rationals. Let \( n \) be an arbitrary integer \( > 1 \). Write \( c, c_1, c_2, \ldots \) for positive constants which depend only on \( g_2, g_3 \), not on \( n \).

Lemma 1. Assume \( \frac{1}{2} g_2, \frac{1}{2} g_3 \) are algebraic integers. Then

\[
p \left( \frac{\lambda_1 \omega_2 + \lambda_2 \omega_1}{n} \right) \quad (0 \leq \lambda_2; \lambda_2 < \omega; \lambda_1, \lambda_2 \text{ integers not both } 0)
\]

is an algebraic number with the maximum of the absolute values of its conjugates at most \( c_1 n^2 \). Further, the leading coefficient of its minimal integral polynomial divides \( n^6 \).

Let \( K_n \) be the field obtained by adjoining to \( K \) all of the numbers \( p(\omega), p'(\omega) \), where \( \omega \) denotes \( (\lambda_1 \omega_1 + \lambda_2 \omega_2)/n \) and \( \lambda_1, \lambda_2 \) range from 0 to \( n-1 \) excluding \( \lambda_1 = \lambda_2 = 0 \).

---

* — Acta Arithmetica XXVI. 2
Lemma 2. The field $K_n$ has degree $a_n n^2$ over $K$ and contains $e^{2\pi i/3}$.

Lemma 3. Assume $\frac{1}{3} \gamma_1, \frac{1}{3} \gamma_2$ algebraic integers. Let $\gamma_1, \gamma_2$ be integers with $(\gamma_1, \gamma_2, 2n) = 1$. Then the number

$$\xi(\gamma_1, \gamma_2, 2n) = \frac{\gamma_1 \omega_1 + \gamma_2 \omega_2}{2n}$$

belongs to the field $K_{n^2}$, each of its conjugates has absolute value at most $\alpha_n$, and the leading coefficient in its minimal integral polynomial divides $(2n)^{n^2}$.

Lemma 4. For any positive integer $k$, the $j$-th derivative of $p(x)^k$ can be expressed as

$$\sum_{0 \leq i \leq k} \binom{k}{i} p(x)^{(i)} p''(x)(x)^{(j-i)}$$

where the summation is over all non-negative integers $i, j, k$ with $i + j \leq k$, and $u = u(i, j, k)$. Then for any positive integer $k$, the $j$-th derivative of $f(x)^k$ can be expressed as

$$\sum_{0 \leq i \leq k} \binom{k}{i} p(x)^{(i)} f(x)(x)^{(j-i)}$$

where the summation is over all non-negative integers $i, j, k$ with $i + j \leq k$, and $u = u(i, j, k)$. Then $u$ denotes a rational integer with absolute value at most $c_k$.

Proof. See Lemma 3 of [2].

III. Proof of the theorem. We suppose there exist algebraic integers $a_2 \neq 0, a_0, a_2, a_3, a_4, a_7, a_7$ such that

$$a_2 a_0 a_1 + a_2 a_3 a_5 + a_2 a_4 a_5 + a_2 a_6 a_7 + a_2 a_8 a_7 = a_0$$

and will show that this assumption leads to a contradiction. We first note that there is no loss of generality in assuming that $\frac{1}{3} \gamma_1, \frac{1}{3} \gamma_2$ are algebraic integers. This is clear from the observation that, for every positive integer $a$, the invariants associated with the $p$-function with periods $\omega_1 \omega_2, \omega_1 \omega_3$ are rational integers $a_2 a_3, a_2 a_4$ and the corresponding values of $\eta_1, \eta_2$ are $a_2 a_3, a_2 a_4$.

We shall use the following notation. We denote by $K$ the field generated by

$$a_1 (0 \leq i \leq 4), \quad a_j (1 \leq j \leq 2), \quad a_i, a_j$$

over the rationals, and we write $\delta$ for the degree of $K$. We write $p(\lambda, \mu, z)$ for the $\mu$-th derivative of the $\lambda$th power of $p(z)$. By $a_2 a_3, a_2 a_4, \ldots$ we shall signify positive numbers which depend only on $a_1, a_2, a_3, a_4$, and the numbers $(3)$. Finally, for any function $F(x_1, x_2, x_3, x_4)$ of the complex variables $x_1, x_2, x_3, x_4$ and any non-negative integers $m_1, m_2, m_3, m_4$ we put

$$F_{m_1, m_2, m_3, m_4}(x_1, x_2, x_3, x_4) = \frac{\partial^m_1 \partial^m_2 \partial^m_3 \partial^m_4}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_4^{m_4}} F(x_1, x_2, x_3, x_4).$$

We denote by $k$ a positive integer and we define

$$L = L_a = L_1 = L_2 = L_3 = [k^{1/2}], \quad L_4 = [k^{1/4}], \quad k = [k^{1/4}]$$

where, as usual, $[x]$ denotes the integral part of $x$. We assume throughout that $k > a$, where $c$ is chosen sufficiently large for the validity of the subsequent arguments. Further, we define

$$f(x_1, x_2, x_3, x_4) = a_2 a_1 x_1 + a_2 a_3 x_2 + a_1 x_1 x_2 + a_2 x_3 x_4 + a_2 a_4 x_5 + a_2 a_5 x_6.$$

The proof now proceeds by a series of lemmas.

Lemma 6. Let $M, N$ be integers with $N > M > 0$, and let $u_0 (1 \leq i \leq M, 1 \leq j \leq N)$ be integers with absolute values at most $U \geq 1$. Then there are integers $u_0, \ldots, u_M$, not all zero, with absolute values at most $(N/M)^{M(N-M)}$ such that

$$\sum_{i=0}^M u_i u_j = 0 \quad (1 \leq i \leq M).$$

Lemma 7. There exist integers $g(\lambda_0, \ldots, \lambda_4)$ not all zero, with absolute values at most $k^{N}$ such that the function

$$\Phi(x_1, x_2, x_3, x_4) = \sum_{\lambda_0=0}^N \sum_{\lambda_1=0}^N \sum_{\lambda_2=0}^N \sum_{\lambda_3=0}^N \sum_{\lambda_4=0}^N \Phi_{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4}(x_1, x_2, x_3, x_4)$$

satisfies

$$\Phi_{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4}(s + \frac{1}{2}, \ldots, s + \frac{1}{2}) = 0$$

for all integers $s$ with $1 \leq s \leq h$, and all non-negative integers $m_1, m_2, m_3, m_4$ with $m_1 + m_2 + m_3 + m_4 \leq k$. Proof. This follows by a standard argument using Siegel's lemma, and by our choice of $L, h$.

Lemma 8. Suppose $Z \geq 6$, and let

$$\psi(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_3, x_4) \prod_{j=1}^Z (\alpha_j x_j - \Omega_j)^{Z_2},$$

where $\Omega_j$ runs over all periods of $p(s)$ with

$$|\Omega_j| \leq |\alpha_j| Z.$$
Then $\varphi(x_1, \ldots, x_4)$ is regular in the disc $|x_i| < Z$ ($j = 1, 2, 3, 4$). For any $x$ with $|x| < \frac{1}{2}Z$, and for any non-negative integers $m_1, \ldots, m_4$ with $m_1 + \ldots + m_4 \leq k$, we have

$$|\varphi_{m_1, \ldots, m_4}(x_1, x_2, x, s)| \leq k^{12k} Z^{q_{22} Q_{22}}.$$

**Proof.** This is the same as Lemma 8 of [3] with $k^{12k}$ replaced by $k^{22k}$. It is necessary to assume, as one can without loss of generality, that $|\gamma| \leq 1$.

**Lemma 9.** Let $Q$, $S$, $Z$ be numbers such that $1 < Q < S < Z - 1$, and let $m_1, \ldots, m_k$ be integers such that $m_1 + \ldots + m_k \leq k$. Suppose $g, r, s$ are integers, with $q | g$, even, $(r, q) = 1$, and

$$1 \leq q < Q, \quad 1 \leq s < S, \quad 1 \leq r < q$$

such that

$$\varphi_{m_1, m_2, m_3, m_4}(s + \frac{r}{q}, \ldots, s + \frac{r}{q}) = 0$$

for all non-negative integers $m_1, \ldots, m_4$ with $\sum_{j=1}^{4} m_j < \sum_{j=1}^{4} m_j$. Then either (9) holds when $m_j = m_j$ ($j = 1, 2, 3, 4$) or we have

$$|\varphi_{m_1, m_2, m_3, m_4}(s + \frac{r}{q}, \ldots, s + \frac{r}{q})| > (kS)^{-\gamma^{2k} Z^{q_{22} Q_{22}}}.\tag{10}$$

**Proof.** The hypotheses imply that when $m_j = m_j$ ($j = 1, \ldots, 4$) the number on the left of (10) is given by $W_1W_2\Omega_2$, where

$$W_j = \prod_{2} \left\{ a_j \left( s + \frac{r}{q} \right) - \Omega_j \right\}^{\frac{1}{4}} \quad (j = 1, 2),$$

$$\Omega_1 = \omega_1 Q_1, \quad \Omega_2 = \omega_2 Q_1,$$

Now $\Omega_1 = \omega_1 Q_1 + n_2 \omega_2$ for some integers $\omega_1, \omega_2$. Then $I(\omega_2/\omega_1) = 0$, it follows that

$$\left| a_j \left( s + \frac{r}{q} \right) - \Omega_j \right| = \left| \omega_j \right| \left| s + \frac{r}{q} \right| - n_2 (\omega_2/\omega_1)$$

exceeds $c_{10}$ unless $n_2 = s$ or $s + 1$ and $n_2 \neq 0$; in the latter cases we have the lower bound $c_{11}/Q$. Similar estimates hold for the factors of $W_2$. Hence $|a_1a_2| > (c_{12}/Q)$, and it therefore suffices to prove either $\Phi = 0$ or

$$|\Phi| > (kS)^{-\gamma^{2k} Z^{q_{22} Q_{22}}}.\tag{10}$$

We now define $p_i = p_i(r, q)$ ($0 \leq i \leq 2, 1 \leq j \leq 2$). It is plain from (2) that

$$f(s + r, q, \ldots, s + r, q) = a_0 \left( s + \frac{r}{q} \right) - \beta_1 \varepsilon_1 - \beta_2 \varepsilon_2$$

where, in the notation of Lemma 3,

$$\varepsilon_1 = \varepsilon(s q + r, 0, \frac{1}{q}), \quad \varepsilon_2 = \varepsilon(0, s q + r, \frac{1}{q}).$$

We conclude from Lemma 3 that

$$\Psi = a_1^{-m_1} a_2^{-m_2} \varepsilon_1^{-m_1} \varepsilon_2^{-m_2} (\log \gamma)^{-m_3}$$

is contained in the field generated by

$$g_2, g_3, a_1 (0 \leq i \leq 4), \quad \beta_1 (1 \leq j \leq 2), \quad \beta_2 (1 \leq j \leq 2),$$

$$\varepsilon_1, \varepsilon_2, \gamma^{\frac{1}{2}}$$

over the rationals. By Lemma 2, this field has degree at most $c_4 Q_1$, and so, in particular, $\Psi$ has degree at most $c_4 q_4$. Further, it is clear from Lemmas 1 and 3, and the equations $p_i(x) = p_i(x') - q_i(x) - q_i(x')$, that for $\gamma^{2k}$ and (11) are both algebraic integers with the maximum of their absolute values at most $\gamma^{2k}$. Denoting by $g$ the leading coefficient of the minimal integral polynomial of $\gamma$, it is clear from the explicit expressions for $\Psi$ that

$$\gamma^{2k} \gamma^2 \Psi = g^{2k} \gamma^{2k} \gamma^2 (m_1 + m_2 + m_3 + m_4 + \ldots)$$

is an algebraic integer with the maximum of the absolute values of its conjugates at most

$$(L + 1)^4 (L + 4 + 1)^{2k} (m_1 + m_2 + m_3 + m_4 + \ldots) (c_{12} S)^{\frac{1}{2k}} Z^{q_{22} Q_{22}} < (kS)^{2k} Z^{q_{22} Q_{22}}.$$

Since $\Psi$ has degree at most $c_4 Q_2$, the assertion of the lemma follows on noting that either $\Psi = 0$ or the norm of (12) is at least 1.

**Lemma 10.** Let $J$ be an integer satisfying $0 \leq J \leq 110$. Then

$$\varphi_{m_1, m_2, m_3, m_4}(s + \frac{r}{q}, \ldots, s + \frac{r}{q}) = 0$$

for all integers $g, r, s$ with $q$ even, $(r, q) = 1$, $1 \leq q = 2k^{2k}$, $1 \leq s < k^{2k} - 1$, $1 \leq r < q$

and all non-negative integers $m_1, m_2, m_3, m_4$ with $m_1 + m_2 + m_3 + m_4 \leq k/2^j$.

**Proof.** The lemma is valid for $J = 0$ by Lemma 7. We suppose that $J$ is an integer with $0 \leq J < 110$, and we assume that the lemma holds
for \( J = 0, \ldots, I \). We proceed to deduce its validity for \( J = I + 1 \). We define

\[
\varphi_j = 2h^{2J}, \quad S_j = h^{2J+4r}, \quad T_j = [2h^{2J+1}] \quad (J = 0, \ldots, I)
\]

and we assume that there are integers \( q', r', s' \) with \( q' \) even, \( (r', s') = 1 \),

\[
1 \leq q' \leq Q_{I+1}, \quad 1 \leq s' \leq S_{I+1}, \quad 1 \leq r' < q',
\]

and non-negative integers \( m'_1, m'_2, m'_3, m'_4 \) with \( m'_1 + m'_2 + m'_3 + m'_4 \leq T_{I+1} \) satisfying

\[
\varphi_{m'_1 m'_2 m'_3 m'_4} \left( s' + \frac{r'}{q'} \right), \ldots, s' + \frac{r'}{q'} \neq 0
\]

and we shall derive a contradiction. Further, we assume that \( m'_1, m'_2, m'_3, m'_4 \) are chosen minimally so that

\[
\varphi_{\mu_1 \mu_2 \mu_3 \mu_4} \left( s' + \frac{r'}{q'} \right), \ldots, s' + \frac{r'}{q'} = 0
\]

for all non-negative integers \( \mu_1, \ldots, \mu_4 \) with \( \sum_{j=1}^4 \mu_j < \sum_{j=1}^4 m'_j \).

Let \( Z = 10S_{I+1} \), and let \( \varphi(x_1, x_2, x_3, x_4) \) be the function defined in Lemma 8 for this choice of \( Z \). Let \( \psi(z) = \varphi_{m'_1 m'_2 m'_3 m'_4} (x, z, s, z, z) \). Then, by our inductive hypothesis, we see that, for all integers \( q, r, s, \) with \( q' \) even, \( (r, q) = 1 \),

\[
1 \leq q \leq Q_{I+1}, \quad 1 \leq s \leq S_{I+1}, \quad 1 \leq r < q,
\]

and each integer \( m \) satisfying \( 0 \leq m \leq T_{I+1} \), we have

\[
\psi_m \left( s + \frac{r}{q} \right) = 0,
\]

for \( \psi_m \left( s + \frac{r}{q} \right) \) is given by

\[
\sum_{j_1=1}^{m_1} \cdots \sum_{j_4=1}^{m_4} j_1! j_2! j_3! j_4! \varphi_{m_1+j_1, \ldots, m_4+j_4} \left( s \right. \left. + \frac{r}{q} \right), \ldots, s + \frac{r}{q}
\]

and the partial derivatives vanish because \( m_1 + j_1 + \cdots + m_4 + j_4 \leq T_I \).

Now write

\[
F(x) = \prod_{q=1}^{Q_I} \prod_{s=1}^{S_I} \prod_{r=1}^{q' \text{ even}} \prod_{m=q'}^{T_{I+1}} \left( z - s - \frac{r}{q} \right)^{T_{I+1}}.
\]

Then by (13), \( \psi(x) F(x) \) is regular in the disc \( |x| \leq 5S_{I+1} \). Hence, denoting by \( \theta \) and \( \Theta \) the upper bound of \( |\psi(x)| \) and the lower bound of \( |F(x)| \), respectively, on the circle \( |x| = 5S_{I+1} \), we conclude from the maximum modulus principle that

\[
|\psi \left( s' + \frac{r'}{q'} \right) | \leq |F \left( s' + \frac{r'}{q'} \right) | \frac{\theta}{\Theta}.
\]

Now, by Lemma 8, we have

\[
\theta \leq h^{4k (10S_{I+1})^{-2k^{1/3} S_{I+1}}}.
\]

Further, since for any \( e \) with \( |e| = 5S_{I+1} \),

\[
|e - e - \frac{r}{q}| \geq 2 \left| \left( s' + \frac{r'}{q'} \right) - (e + \frac{r}{q}) \right|
\]

and since also the number of sets \( q, r, e \) which occur in the definition of \( F(x) \) is at least

\[
\frac{1}{2} S_I (\varphi(2) + \varphi(4) + \cdots + \varphi(Q_I)) = a_k^L q^2 S_I,
\]

where \( Q_I \) denotes the largest even integer not exceeding \( Q_I \) and \( \varphi \) denotes Euler's function (cf. [1], p. 155), we clearly have

\[
\theta > 2 a_n q^{2} S_{I+1} \left| F \left( s' + \frac{r'}{q'} \right) \right|.
\]

As

\[
Q_I = 4S_{I+1} h^{-6s}, \quad S_I = S_{I+1}, h^{-14}, \quad T_{I+1} > a_n k, \quad L \leq h^{-3},
\]

it follows readily from (14), (15), (16) that

\[
\left| \psi \left( s' + \frac{r'}{q'} \right) \right| < 2^{-a_0 q^{2} S_{I+1}}.
\]

On the other hand, the hypotheses of Lemma 9 are satisfied with

\[
Q = Q_{I+1}, \quad \frac{S}{S_{I+1}} = q = q', \quad r = r', \quad s = s', \quad m_j = m'_j (1 \leq j \leq 4),
\]

and, by virtue of our initial assumption, we have \( \psi(s' + r'/q') \neq 0 \). We conclude from Lemma 9 that

\[
\left| \psi \left( s' + \frac{r'}{q'} \right) \right| > (h S_{I+1})^{-a_0 h L_4 S_{I+1} L_{I+1} ^{2+1}}.
\]

But, as \( L \leq h^{-3}, \ L_4 \leq h^{-15}, \ I < 110 \), we see that the estimates (17) and (18) are contradictory for \( k \) sufficiently large. This completes the proof of the lemma.
IV. Completion of the proof. Let \( P_k(x), 1 \leq k \leq n \) be polynomials of degree \( q_k - 1 \) respectively. Put \( \sigma = \sum_{k=1}^{n} q_k \). Let \( \omega_1, \ldots, \omega_n \) be arbitrary complex numbers, and put \( \Delta = \max_{k} |\omega_k| \).

**Lemma 11.** The number of zeroes of the function

\[
F(z) = \sum_{s=1}^{n} P_k(z) e^{\omega_k z}
\]

in an arbitrary circular disk of radius \( R \) in the complex plane is at most

\[
3(\sigma - 1) + 4R\Delta.
\]

Proof see [4], p. 58.

Now Lemma 10 implies

\[
G_{m_1, \ldots, m_k}(s; \ldots) = 0
\]

for all integers \( s \) with \( 1 \leq s \leq L+1 \), and all integers \( m_1, \ldots, m_k \) with \( 0 \leq m_1, \ldots, m_k \leq L \). Putting

\[
G(z) = \sum_{s=1}^{n} G_{m_1, \ldots, m_k}(s; \ldots)
\]

and noting that \( L + \frac{1}{12} \leq 2^{1/12}L \), it is clear from Lemma 10 that

\[
G(z) = 0 \quad (1 \leq s \leq L+1, \ 0 \leq m \leq L^{1/6}).
\]

The left side of (23) can be written in the form

\[
\sum_{s=1}^{L} \sum_{s=1}^{L} \sum_{s=1}^{L} \cdots \sum_{s=1}^{L} \left[ \prod_{k=1}^{n} r(\lambda_k, 4, \mu, s) q(\lambda_k, \lambda_3, m_1 - \mu_3, \ldots, m_n - \mu_n) \right]
\]

where \( r(\lambda_1, 4, \mu, s) \) denotes the function

\[
\left\{ \begin{array}{c}
\prod_{j=1}^{n} \frac{\partial^{m_j - \mu_j - \mu_j} f(x_1, \ldots, x_n)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \gamma^{\lambda_j x_j}
\end{array} \right\}
\]

evaluated at the point \( x_1 = \ldots = x_n = s + \frac{1}{12} \) and

\[
q(\lambda_k, \lambda_4, \mu, \nu, \tau) = \sum_{s=1}^{L} \sum_{s=1}^{L} q(\lambda_1, \ldots, \lambda_k) p(\lambda_1, \nu, \frac{\omega_k}{4}) p(\lambda_2, \tau, \frac{\omega_k}{4}) (2\pi i \lambda_k)^{\nu}(\tau)^{\mu}
\]

In particular, for \( m_1 = \ldots = m_n = 0 \) we see that

\[
\sum_{s=1}^{L} \sum_{s=1}^{L} \sum_{s=1}^{L} \cdots \sum_{s=1}^{L} f(s + \frac{1}{12}, \ldots, s + \frac{1}{12}) q(\lambda_k, \lambda_4, 0, 0, 0) = 0
\]

for \( 1 \leq s \leq L+1 \). Furthermore, (22) implies that each of these zeroes has multiplicity at least \( \frac{1}{2}k^2 \). On the other hand, (23) can be put in the form of (19) with

\[
P_2(z) = \sum_{s=0}^{L} f(s + \frac{1}{2}, \ldots, s + \frac{1}{2}) g(\lambda_k, \lambda_4, 0, 0, 0) \gamma^{s/4},
\]

and \( \omega_k = \lambda_k \log y \). By Lemma 11, (23) should have no more than

\[
3(\lambda_4 + 1)(L + 1) + 4(\lambda_4 + 1) \lambda_4 \log y
\]

zeroes in the circle \( |s| \leq L + \frac{3}{4} \). But for sufficiently large \( k \), the number (24) is clearly less than \( Lk^{1/3} \), so (23) is identically zero, and we conclude that

\[
q(\lambda_3, \lambda_4, \nu, \tau, \nu) = 0 \quad (0 \leq \lambda_3 \leq L, \ 0 \leq \lambda_4 \leq L).
\]

Now let \( \nu_1, \nu_2, \nu_3 \) be any three integers with \( 0 \leq \nu_1, \nu_2, \nu_3 \leq L \), and suppose that \( q(\lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \) for all integers \( \lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3 \) with

\[
0 \leq \lambda_4 \leq L, \ 0 \leq \nu_1, \nu_2, \nu_3 \leq L, \ \sum_{j=1}^{3} \nu_j \leq \sum_{j=1}^{3} \nu_j.
\]

Then (21) with \( m_1 = \nu_1, m_2 = \nu_2, m_3 = \nu_3, m_4 = 0, m_5 = 0 \) gives

\[
\sum_{s=0}^{L} \sum_{s=0}^{L} \cdots \sum_{s=0}^{L} r(\lambda_3, \lambda_4, 0, 0, 0, s) q(\lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (1 \leq s \leq L+1),
\]

and, as above, we conclude that \( q(\lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \) for \( 0 \leq \lambda_3, \lambda_4 \leq L \). It follows by induction that

\[
q(\lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (0 \leq \lambda_3, \lambda_4 \leq L, \ 0 \leq \nu_1, \nu_2, \nu_3 \leq L).
\]

Now choose \( \lambda_3, \lambda_4 \) so that \( q(\lambda_3, \ldots, \lambda_j) \neq 0 \) for some \( \lambda_1, \lambda_2, \lambda_3 \). We conclude that the determinant \( \Delta \) of coefficients of the \( (L+1)^3 \) equations

\[
q(\lambda_3, \lambda_4, \nu_1, \nu_2, \nu_3) = 0 \quad (0 \leq \nu_1, \nu_2, \nu_3 \leq L)
\]

must vanish. But it is well known (1) that

\[
\Delta = (\Delta_1\Delta_2\Delta_3)^{L+1},
\]

where

\[
\Delta_j = 2! \ldots L! p^j(1/2)_{(L+1)} \quad (j = 1, 2), \quad \Delta_3 = 2! \ldots L!(-2)^{L+1}.
\]

Since \( p^j(1/2) \neq 0 \ (j = 1, 2) \), it follows that \( \Delta \neq 0 \), which is a contradiction. Thus (2) cannot be valid, and the proof of the theorem is complete.

(1) Cf. Lemmas 6 and 7 of [1].
Scharfe untere Abschätzung für die Anzahlfunktion

der B-Zwillinge

von

KARL-HEINZ INDEKOFER (Frankfurt am Main)

1. Einleitung. Es sei $\mathcal{S}$ die Menge aller natürlichen Zahlen, die sich
als Summe zweier Quadrate von ganzen Zahlen darstellen lassen. Die
Elemente von $\mathcal{S}$ heißen B-Zahlen. Das Paar $(n, n+1)$ nennen wir B-Zwilling,
wenn sowohl $n \in \mathcal{S}$ als auch $n+1 \in \mathcal{S}$ ist. Nach G. Rieger [6] gilt für

die Anzahl $(\#) B_2(x) = |\{n \leq x: n \in \mathcal{S}, n+1 \in \mathcal{S}\}$ der B-Zwillinge unterhalb

$x$ die obere Abschätzung

\begin{equation}
B_2(x) \leq \omega(\log x)^{-1}.
\end{equation}

Bezüglich der Abschätzung von $B_2(x)$ nach unten ist bisher nur (vgl.

[2], [7])

\begin{equation}
B_2(x) \geq o(x)(\log x)^{-(1+\varepsilon)}
\end{equation}

bekannt, wobei die Konstante $o(x)$ nur von $\varepsilon$ abhängt. In dieser Note
soll mit Hilfe des Selbergschen Siebes eine Abschätzung von $B_2(x)$ nach
unten gegeben werden, welche (1.2) verbessert und die richtige Größen-
ordnung von $B_2(x)$ angibt. Wir zeigen

Satz. Für die Anzahl der B-Zwillinge unterhalb $x$ gilt

\begin{equation}
B_2(x) \geq x(\log x)^{-1}.
\end{equation}

Der Beweis beruht auf zwei Lemmata, welche Abschätzungen von
Spezialfällen des linearen Siebes nach unten und des großen Siebes nach
oben enthalten.

$\varepsilon_1, \varepsilon_2, \ldots$ bezeichnen positive Konstanten.

2. Hilfssätze. Sei $b(\cdot)$ die charakteristische Funktion von $\mathcal{S}, B_1\colon = \{m \in \mathbb{N} : p|m, p \text{ prim } \Rightarrow p = 1 \ (4)\}$ und

$B_2\colon = \{m \in \mathbb{N} : p|m, p \text{ prim } \Rightarrow p = 3 \ (4)\}$.

(*) $\#(n: \ldots)$ bezeichnet die Anzahl der $n$ mit den Eigenschaften...