Also, if we use this inequality to make the above proof explicit, we find that $P(\beta^2)$ divides the least common multiple of $P(\varkappa^2)$ and

$$\prod_{p\leqslant\mu} p\cdot\prod_{p^L\leqslant\mu} p^L,$$

where

$$\mu = 4^{10^6\,\mathrm{Norm}(\varkappa^2)}.$$

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Some distribution problems concerning the divisors of integers

by

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Introduction. In this paper we study the distribution (mod1) of $\log d$, where d runs through the divisors of the positive integer n. As usual we denote the number of these divisors by $\tau(n)$.

The sequence $\{\log m, m = 1, 2, 3, \ldots\}$ is not uniformly distributed (mod 1), nevertheless if we set

$$f_n(x) = \frac{1}{\tau(n)}\sum_{\log d\leqslant x\,(\mathrm{mod}\,1)} 1,$$

then on a sequence of integers n of asymptotic density 1, we have that

$$f_n(x)\to x$$

uniformly for

$$0\leqslant x\leqslant 1.$$

Indeed, for each $\lambda < \frac{1}{2}$, there is a sequence of density 1 on which

$$\sup_{0\leqslant\alpha\leqslant\beta\leqslant 1}|f_n(\beta)-f_n(\alpha)-(\beta-\alpha)|\leqslant\frac{1}{(\tau(n))^{\lambda}}.$$

This result was proved in a recent paper of Hall [2].

It follows from this that for each fixed $\alpha\in[0, 1)$, there is a sequence of integers n of density 1 on which

$$\min_{d|n}\|\log d-\alpha\|\to 0,$$

where $\|x\|$ denotes the difference between x and the nearest integer to it, and we consider the following problem. How fast can the left hand side tend to zero on a sequence of density 1, or even on a sequence of positive density? It turns out that this question can be answered very precisely.

In the case $\alpha = 0$, the problem is only interesting if we disregard the divisor $d = 1$ in calculating the minimum above. This suggests that for general α we distinguish two cases, whether we allow

$$\|\log d - \alpha\| = 0 \tag{1}$$

or restrict our attention to the minimum positive value of the expression on the left.

Let M denote the set of those $\alpha \in [0, 1)$ for which there is an integer m satisfying

$$\log m \equiv \alpha \ (\mathrm{mod}\ 1).$$

As e is transcendental there can be at most one such m, and we denote it by $m(\alpha)$. Thus (1) can only hold if $\alpha \in M$ and $d = m(\alpha)$, that is, n must be a multiple of $m(\alpha)$. We take account of this in our results which are as follows.

THEOREM 1. *Let α and c be real numbers, $0 \leqslant \alpha < 1$. The integers n having a divisor d satisfying*

$$0 < \|\log d - \alpha\| < 2^{-\log\log n - c\sqrt{\log\log n}}$$

have asymptotic density

$$\frac{1}{\sqrt{2\pi}} \int_c^\infty e^{-u^2/2}\, du, \tag{2}$$

moreover, if $\alpha \in M$ and we allow equality on the left, the density is increased to

$$\frac{1}{\sqrt{2\pi}} \int_c^\infty e^{-u^2/2}\, du + \frac{1}{m(\alpha)\sqrt{2\pi}} \int_{-\infty}^c e^{-u^2/2}\, du.$$

We can replace c by a function of n tending to $+\infty$ or $-\infty$. We have

THEOREM 2. *Let $f(n) \to \infty$ as $n \to \infty$, and $0 \leqslant \alpha < 1$. Almost all integers n have a divisor d such that*

$$0 < \|\log d - \alpha\| < 2^{-\log\log n + f(n)\sqrt{\log\log n}}.$$

The sequence of integers n having a divisor d satisfying

$$0 < \|\log d - \alpha\| < 2^{-\log\log n - f(n)\sqrt{\log\log n}}$$

has density zero, unless $\alpha \in M$ and we allow equality on the left; in this case the density is $1/m(\alpha)$.

Next, we study the behaviour of

$$\sup_\alpha \min_{d|n} \|\log d - \alpha\|.$$

This is very similar to the case where α is fixed, indeed we give the following result.

THEOREM 3. *For any real number c, the sequence of integers n for which*

$$\sup_\alpha \min_{d|n} \|\log d - \alpha\| < 2^{-\log\log n - c\sqrt{\log\log n}}$$

has asymptotic density

$$\frac{1}{\sqrt{2\pi}} \int_c^\infty e^{-u^2/2}\, du,$$

and if c is replaced by a function of n tending to $+\infty$ or $-\infty$, the density is respectively zero or 1.

Before embarking on the proofs we would like to make a few remarks. First of all, it is well known that

$$\tau(n) > 2^{\log\log n + c\sqrt{\log\log n}}$$

on a sequence of asymptotic density given by (2), hence the least positive value of

$$\|\log d - \alpha\|, \quad d \mid n$$

behaves roughly like $1/\tau(n)$, corresponding to the simple hypothesis that the fractional parts of $\log d$ are almost equally spaced on the unit interval.

By the way, the present Theorem 2 gives the solution of one of the problems in Hall's paper: Theorem 2 [2] holds if and only if $\mu < \log 2$, not, as the author guessed, if and only if $\mu < 1$.

Proof of Theorem 1. The idea of the proof is that for most integers n, we might expect the minimum value of

$$\|\log d - \alpha\|, \quad d \mid n,$$

to be of the order of magnitude $1/\tau(n)$. Therefore numbers with a sufficiently large number of prime factors should have a divisor d satisfying

$$\|\log d - \alpha\| < 2^{-\log\log n - c\sqrt{\log\log n}},$$

the remaining numbers should not, unless they are multiples of $m(\alpha)$ in the case $\alpha \in M$.

Accordingly we divide the integers $n \leqslant x$ into three main classes. Class 1, which has cardinality

$$\sim \frac{x}{\sqrt{2\pi}} \int_c^\infty e^{-u^2/2} du$$

contains those integers for which

$$\nu(n) \geqslant \log\log x + c(\log\log x)^{1/2} + 3(\log\log x)^{1/3}.$$

The last term on the right does not affect the asymptotic density of the class, being of smaller order than $\sqrt{\log\log x}$, and simply provides some leeway in the analysis; we show that almost all these n have a divisor d satisfying

$$0 < \|\log d - \alpha\| < 2^{-\log\log x - c\sqrt{\log\log x}} \tag{3}$$

the left hand inequality showing that $d \neq m(\alpha)$.

Clearly almost all $n \leqslant x$ exceed $\sqrt{x}$, and there exists an $a = a(c)$ such that for these n,

$$\log\log n + c\sqrt{\log\log n} \geqslant \log\log x + c\sqrt{\log\log x} - a.$$

The second class contains integers $n \leqslant x$ with

$$\nu(n) \leqslant \log\log x + c(\log\log x)^{1/2} - 3(\log\log x)^{1/3}$$

and we prove that the number of integers in this class with a divisor d satisfying

$$0 < \|\log d - \alpha\| < 2^{a - \log\log x - c\sqrt{\log\log x}} \tag{4}$$

is $o(x)$. Evidently the multiples of $m(\alpha)$ in Class 2 have density

$$\frac{1}{m(\alpha)\sqrt{2\pi}} \int_{-\infty}^{c} e^{-u^2/2} du.$$

Class 3 contains the remaining integers $n \leqslant x$ for which $\nu(n)$ satisfies neither of the inequalities above; since the maximum cardinality of a set of integers $n \leqslant x$ with a fixed number of distinct prime factors is

$$\ll \frac{x}{\sqrt{\log\log x}}$$

and the range of values of $\nu(n)$ within Class 3 is at most $6(\log\log x)^{1/3}$, the number of members of the class is

$$\ll \frac{x}{(\log\log x)^{1/6}} = o(x).$$

We remark that throughout the analysis which follows we could replace $o(x)$ wherever it appears by an explicit O-estimate, except at one point. This occurs in the treatment of Class 2, where we use the fact that for $d \neq m(\alpha)$,

$$\|\log d - \alpha\| \neq 0.$$

However, so far as we are aware, no positive lower bound for the left hand side is known, and this limits the precision of our result.

We begin by considering the first class. Let $I = I(x)$ be the interval

$$\left(\exp\left((\log\log x)^3\right),\ x^{1/(\log\log x)^2}\right)$$

and suppose that n has t prime factors, $p_1, \ldots, p_t$ lying in $I(x)$. Then we may assume that these prime factors are distinct, moreover that if n is in the first class,

$$\log\log x + c(\log\log x)^{1/2} + 2(\log\log x)^{1/3} < t < 2\log\log x.$$

For the number of exceptions to the first assumption is

$$\leqslant x \sum_{p \in I} \frac{1}{p^2} = o(x)$$

while the second follows from the fact that the normal number of distinct prime factors of n outside $I(x)$ is $5\log\log\log x$. Suppose that

$$[r\log p_i] \equiv h_i \pmod{r}, \quad 1 \leqslant i \leqslant t$$

and that we can find a set of ε's, $\varepsilon_i = 0$ or 1 for $1 \leqslant i \leqslant t$ such that

$$\varepsilon_1 h_1 + \varepsilon_2 h_2 + \ldots + \varepsilon_t h_t \equiv h \pmod{r}$$

where

$$h = \max(1, [r\alpha]).$$

Evidently

$$r(\varepsilon_1 \log p_1 + \varepsilon_2 \log p_2 + \ldots + \varepsilon_t \log p_t - \alpha) \equiv b \pmod{r}$$

where

$$h - r\alpha \leqslant b \leqslant h - r\alpha + t$$

and so if

$$d = p_1^{\varepsilon_1} p_2^{\varepsilon_2} \ldots p_t^{\varepsilon_t}$$

certainly

$$d \mid n, \quad \|\log d - \alpha\| \leqslant t/r.$$

Moreover, the choice of h ensures that the ε_i are not all zero, and so

$$d > \exp\left((\log\log x)^3\right).$$

Therefore $d \neq m(a)$ if x is sufficiently large. We let r be the integer part of

$$2^{\log\log x + c(\log\log x)^{1/2} + (\log\log x)^{1/3}}$$

and it follows that for sufficiently large x, d satisfies (3). In order to establish the existence of a suitable set $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ we need the following lemma, adapted from Theorem 2 of Erdös and Rényi [1].

LEMMA 1. *Let G be an Abelian group of order r, and*

$$t\log 2 \geqslant \log r + 2\log\log r.$$

Then for all but possibly $o\left(\binom{r}{t}\right)$ choices of the distinct elements $g_1, g_2, \ldots, g_t$ of G, every element of G may be written in the form

$$\varepsilon_1 g_1 + \varepsilon_2 g_2 + \ldots + \varepsilon_t g_t.$$

This result is uniform in r and t.

We let G be the group of residue classes (mod r) under addition, and note that r and t satisfy the requirement of the lemma. It will therefore be sufficient to show that for almost all the integers n under consideration, the corresponding classes h_i are distinct and unexceptional in the sense of the lemma. For this we need the following result.

LEMMA 2. *There exists an absolute constant $\beta > 0$ such that if E is any sub-interval of $[0, 1)$ and l is the length of E, then*

$$\sum_{\substack{u<p<v \\ \log p \in E \,(\mathrm{mod}\, 1)}} \frac{1}{p} = l\left(\log\left(\frac{\log v}{\log u}\right) + O\left(\frac{1}{\log u}\right)\right) + O\left(e^{-\beta\sqrt{\log u}}\right).$$

This follows easily from the classical result

$$\pi(z) = \int_2^z \frac{dw}{\log w} + O\left(ze^{-2\beta\sqrt{\log z}}\right)$$

and we suppress the details. Now suppose that

$$u = \exp\left((\log\log x)^3\right), \qquad v = x^{1/(\log\log x)^2}$$

so that (u, v) is the interval $I(x)$, and let $P(h)$ denote the set of primes p in $I(x)$ satisfying

$$[r\log p] \equiv h \,(\mathrm{mod}\, r),$$

that is, the fractional part of $\log p$ lies in the interval

$$E(h) = \left[\frac{h}{r}, \frac{h+1}{r}\right)$$

of length $l = 1/r$. We deduce from Lemma 2 that

$$\sum_{p \in P(h)} \frac{1}{p} = \frac{1}{r}\left(1 + O\left(\frac{1}{(\log\log x)^4}\right)\right)\log\left(\frac{\log v}{\log u}\right) \ll \frac{\log\log x}{r}.$$

Evidently the number of integers $n \leqslant x$ for which the corresponding h_i are not all distinct is

$$\ll x \sum_{0 \leqslant h < r}\left(\sum_{p \in P(h)} \frac{1}{p}\right)^2 \ll \frac{x}{r}(\log\log x)^2 = o(x).$$

Next we estimate the number of integers $n \leqslant x$ corresponding to an exceptional set of residue classes $h_1, h_2, \ldots, h_t$. Let $t < 2\log\log x$ as we may assume any $p_1, p_2, \ldots, p_t$ be any primes in $I(x)$. The number of $n \leqslant x$ with precisely these prime factors in $I(x)$ is equal to the number of integers not exceeding $x/p_1 p_2 \ldots p_t$ with no prime factor in $I(x)$. Notice that if p is a prime in $I(x)$ and $\log\log x > 3$, certainly

$$p < \frac{x}{p_1 p_2 \ldots p_t}$$

so that a result of van Lint and Richert [4] derived by Selberg's upper bound method gives the estimate

$$\ll \frac{x}{p_1 p_2 \ldots p_t}\prod_{p \in I}\left(1 - \frac{1}{p}\right)$$

for the number of such $n \leqslant x$. This estimate may also be deduced from a theorem of Hall [3]. Hence for any $h_1, h_2, \ldots, h_t$ the number of integers $n \leqslant x$ with t prime factors in $I(x)$ satisfying

$$[r\log p_i] \equiv h_i \,(\mathrm{mod}\, r) \quad \text{for} \quad 1 \leqslant i \leqslant t$$

is

$$\ll x\prod_{p \in I}\left(1 - \frac{1}{p}\right)\prod_{i=1}^{t}\left(\sum_{p \in P(h_i)} \frac{1}{p}\right) \ll x\left(\frac{\log u}{\log v}\right)\left(\frac{1}{r}\log\left(\frac{\log v}{\log u}\right)\right)^t.$$

By Lemma 1, the number of exceptional sets $h_1, \ldots, h_t$ is

$$o\left(\binom{r}{t}\right) = o\left(\frac{r^t}{t!}\right)$$

so the number of $n \leqslant x$ corresponding to exceptional sets of residue classes of cardinality t is

$$o\left(x\frac{\log u}{\log v} \cdot \frac{1}{t!}\log^t\left(\frac{\log v}{\log u}\right)\right).$$

Lemma 1 is uniform in t, hence we may sum over t and this is $o(x)$.

Therefore almost all the integers in Class 1 have a divisor d satisfying (3), that is, a divisor other than $m(a)$ with the required property. Thus it is immaterial in Class 1 whether we allow $m(a)$ as a divisor or not.

We now turn our attention to the second class, and in the case $a \in M$, we begin with the remark that the multiples of $m(a)$ in the class have asymptotic density

$$\frac{1}{m(a)\sqrt{2\pi}}\int_{-\infty}^{c} e^{-u^2/2}\,du.$$

For they are numbers of the form $nm(a)$, $n \leqslant x/m(a)$, and since

$$\nu\big(nm(a)\big) = \nu(n) + O(1)$$

they satisfy

$$\nu(n) \leqslant \log\log x + c(\log\log x)^{1/2} - 3(\log\log x)^{1/3} + O(1)$$
$$\leqslant \log\log\left(\frac{x}{m(a)}\right) + \big(c + o(1)\big)\sqrt{\log\log\left(\frac{x}{m(a)}\right)}.$$

This gives the density above.

Next, we show that the number of members of the class with a divisor d satisfying (4) is $o(x)$. As in the treatment of Class 1, we set

$$u = \exp\big((\log\log x)^3\big)$$

and we begin by showing that at most $o(x)$ integers $n \leqslant x$ have a divisor d satisfying (4) whose greatest prime factor does not exceed u. Evidently it will be sufficient to weaken the condition on d to

$$(5) \qquad 0 < \|\log d - a\| < (\log x)^{-1/2}$$

and to show that

$$\sum{}' \frac{1}{d} = o(1)$$

where the dash denotes that every prime factor of d is less than or equal to u, and that d satisfies (5). Let

$$H = \exp\big((\log\log x)^7\big).$$

Then

$$\sum_{d>H}{}' \frac{1}{d} \leqslant \frac{1}{\log H}\sum{}' \frac{\log d}{d} \leqslant \frac{1}{\log H}\prod_{p\leqslant u}\left(1-\frac{1}{p}\right)^{-1}\sum_{p\leqslant u}\frac{\log p}{p-1}$$
$$= O\left(\frac{\log^2 u}{\log H}\right) = o(1).$$

Notice that for these large d's we have not used (5). In the remaining case $d \leqslant H$, we drop the condition on the prime factors of d. The sum of the reciprocals of the d's satisfying

$$(6) \qquad m - (\log x)^{-1/2} < \log d - a < m + (\log x)^{-1/2}$$

is

$$(7) \qquad \ll (\log x)^{-1/2} + e^{-m},$$

and since $d \leqslant H$ we have

$$m \leqslant (\log\log x)^7 + O(1).$$

Next, since e is transcendental, except in the special case $a \in M$, $d = m(a)$, we have

$$\|\log d - a\| \neq 0.$$

Let $d(x)$ be the smallest positive integer such that

$$0 < \|\log d - a\| < (\log x)^{-1/2}.$$

Then

$$d(x) \to \infty \quad \text{as } x \to \infty.$$

As we remarked earlier, we do not know how fast $d(x) \to \infty$ and this limits the precision of our result. Let the integer nearest to $\log d(x)$ be $m_0(x)$. Then the ranges (6) with $m < m_0(x)$ are empty, except the range corresponding to $m(a)$: we may assume $m(a) < m_0(x)$. However, this range contains only the one d, $m(a)$ itself, which does not satisfy (5). Therefore we sum (7) for $m \geqslant m_0(x)$ and obtain

$$\sum_{d\leqslant H}{}' \frac{1}{d} \ll \frac{(\log\log x)^7}{(\log x)^{1/2}} + \frac{1}{d(x)} = o(1).$$

It remains to consider those integers in the second class with a divisor d satisfying (4), but no such divisor all of whose prime factors are less than or equal to u. We refer to these integers as belonging to the fourth class, and we have to show that their number is $o(x)$.

We begin by excluding from the class numbers with no prime factor exceeding

$$w = x^{1/\log\log x}.$$

It follows from the results of van Lint and Richert [4] and Hall [3] quoted above that the cardinality of the excluded set is

$$\ll x\prod_{w<p\leqslant x}\left(1-\frac{1}{p}\right) = o(x).$$

Thus if n is in Class 4 we write

$$n = mp, \quad p > w.$$

The number of such n for which m itself belongs to Class 4 is

$$(8) \qquad \ll \sum_{m\leqslant x/w}' \pi\left(\frac{x}{m}\right) \ll \frac{x\log\log x}{\log x} \sum_{m\leqslant x}' \frac{1}{m}$$

where the dash denotes that m belongs to the fourth class. In order to estimate the sum on the right, we write

$$m = qp_1p_2\ldots p_t$$

where $p_1, p_2, \ldots, p_t$ are those prime factors of m which exceed u, written in increasing order. We restrict our attention to those m for which they are distinct, that is

$$u < p_1 < p_2 < \ldots < p_t$$

the contribution of the exceptional m's to (8) being

$$\ll \frac{x\log\log x}{\log x} \sum_{p>u} \frac{1}{p^2} \sum_{r\leqslant x} \frac{1}{r} \ll \frac{x}{u}\log\log x = o(x).$$

Since m belongs to Class 4, it has a divisor d which may be written

$$d = fp_1^{\varepsilon_1}p_2^{\varepsilon_2}p_3^{\varepsilon_3}\ldots p_t^{\varepsilon_t}, \quad f|q, \quad \varepsilon_i = 0 \text{ or } 1 \text{ for } 1\leqslant i\leqslant t$$

satisfying (4). By hypothesis, d must have a prime factor greater than u, therefore there exists a j, $1\leqslant j\leqslant t$ such that $\varepsilon_j = 1$. But now the fractional part of $\log p_j$ is determined to lie within the union of $2^{t-1}\tau(q)$ sub-intervals of $[0, 1)$ each of length

$$(9) \qquad l = 2^{a+1-(\log\log x)-c(\log\log x)^{1/2}}$$

according to the possible choices of f, ε_i ($i\neq j$, $1\leqslant i\leqslant t$). We refer to Lemma 2, with $v = x$, and find that if $\sum^*$ is the sum over possible p_j's, then

$$\sum^* \frac{1}{p_j} \ll 2^{t+a-\log\log x-c(\log\log x)^{1/2}}\tau(q)\log\left(\frac{\log x}{\log u}\right).$$

Since $a = O(1)$, and

$$t \leqslant \log\log x + c(\log\log x)^{1/2} - 3(\log\log x)^{1/3}$$

this is

$$\ll \tau(q)2^{-3(\log\log x)^{1/3}}\log\left(\frac{\log x}{\log u}\right).$$

Hence

$$\sum' \frac{1}{m} \ll \sum \frac{1}{q} \sum_{t\leqslant t_0} \sum_{j=1}^{t} \sum \frac{1}{p_1\ldots p_{j-1}p_{j+1}\ldots p_t} \sum^* \frac{1}{p_j}$$

$$\ll \sum_q \frac{\tau(q)}{q} \sum_{t\leqslant t_0} \frac{t\cdot 2^{-3(\log\log x)^{1/3}}}{(t-1)!}\log^t\left(\frac{\log x}{\log u}\right)$$

$$\ll t_0^2 2^{-3(\log\log x)^{1/3}} \prod_{p\leqslant u}\left(1-\frac{1}{p}\right)^{-2} \sum_t \frac{1}{t!}\log^t\left(\frac{\log x}{\log u}\right)$$

$$\ll t_0^2 2^{-3(\log\log x)^{1/3}}(\log x)(\log u)$$

$$\ll 2^{-3(\log\log x)^{1/3}}(\log x)(\log\log x)^5.$$

Therefore

$$\frac{x\log\log x}{\log x}\sum' \frac{1}{m} = o(x).$$

The remaining integers in Class 4 are of the form

$$n = mp, \quad p > w$$

where m itself does not belong to the fourth class. Hence n has a divisor d, of the form fp, where $f|m$, satisfying (4), and so the fractional part of $\log p$ lies in the union of $\tau(m)$ sub-intervals of $[0, 1)$, each of length l, given by (9). We require

Lemma 3. *There exist absolute positive constants A and β such that if E is any subinterval of $[0, 1)$ and l is the length of E, then*

$$\sum_{\substack{w<p\leqslant y\\ \log p\in E \,(\mathrm{mod}\,1)}} 1 \leqslant \frac{Aly}{\log y} + O(ye^{-\beta\sqrt{\log w}}).$$

The proof is as indicated in Lemma 2, β being the same. We have the following

Corollary. *Setting $y = x/m$, where $m\leqslant x/w$, and choosing w as above, we have*

$$\sum_{\substack{w<p\leqslant x/m\\ \log p\in E \,(\mathrm{mod}\,1)}} 1 \ll \frac{lx}{m\log w}.$$

We are now ready to estimate the cardinality of the set of integers n specified above. Notice that

$$(10) \qquad \nu(n) \leqslant \log\log x + c(\log\log x)^{1/2} - 3(\log\log x)^{1/3},$$

and we restrict our attention to those n for which

$$\tau(n) \leqslant 2^{\log\log x + c(\log\log x)^{1/2} - 2(\log\log x)^{1/3}}. \tag{11}$$

To estimate the number of exceptional n note that

$$\tau(n) \leqslant 2^{\omega(n)}$$

where $\omega(n)$ denotes the number of prime factors of n counted according to multiplicity. Hence if n satisfies (10) but not (11), then

$$\omega(n) - \nu(n) > (\log\log x)^{1/3}.$$

But

$$\sum_{n\leqslant x} \big(\omega(n) - \nu(n)\big) = O(x)$$

so we have discounted at most

$$O\left(\frac{x}{(\log\log x)^{1/3}}\right) = o(x)$$

numbers. The remaining set of integers n in Class 4 has cardinality

$$\leqslant \sum_{m\leqslant x/w}{}' \sum_{w<p\leqslant x/m}^{(m)} 1$$

the inner sum being over p's for which the fractional part of $\log p$ lies in the union of intervals corresponding to m, the dash denoting that

$$\tau(m) \leqslant 2^{\log\log x + c(\log\log x)^{1/2} - 2(\log\log x)^{1/3}}.$$

By the corollary to Lemma 3, this is

$$\ll \frac{xl}{\log w} \sum{}' \frac{\tau(m)}{m} \ll x \cdot 4^{-(\log\log x)^{1/3}} \log\log x = o(x).$$

This completes our treatment of Class 4, and so of Class 2. We have shown that the asymptotic density of integers in the class with a divisor d satisfying

$$0 < \|\log d - a\| < 2^{-\log\log n - c\sqrt{\log\log n}}$$

is zero; but if $a \in M$ and we allow equality on the left, the density is increased to

$$\frac{1}{m(a)\sqrt{2\pi}} \int_{-\infty}^{c} e^{-u^2/2} du.$$

This completes the proof of Theorem 1, and we indicate the changes needed to prove the first part of Theorem 3. It is plain that Theorem 2 and the second part of Theorem 3, where c is replaced by a function of n, are simple corollaries.

We divide the integers $n \leqslant x$ into three classes in the same way as before. The integers in Class 3 have zero density, moreover, those in Class 2 with

$$\sup_a \min_{d|n} \|\log d - a\| < 2^{a - \log\log x - c\sqrt{\log\log x}}$$

have zero density; it is sufficient to select a particular a, say $a_1 \notin M$ and notice from the proof of Theorem 1 that the integers in Class 2 with a divisor d satisfying

$$\|\log d - a_1\| < 2^{a - \log\log x - c\sqrt{\log\log x}}$$

have zero density. Hence to complete the proof of the first statement of Theorem 3 we need to show that for almost all members of Class 1,

$$\sup_a \min_{d|n} \|\log d - a\| < 2^{-\log\log x - c\sqrt{\log\log x}}.$$

As in the proof of Theorem 1, we suppose that n has the prime factors $p_1, p_2, \ldots, p_t$ in $I(x)$ and that

$$[r \log p_i] \equiv h_i \pmod r, \quad 1 \leqslant i \leqslant t.$$

An examination of Lemma 1 and the argument preceding it shows that unless the set of residue classes h_i are exceptional, corresponding to a subset of integers of Class 1 of zero density, every residue class $h \pmod r$ is representable in the form

$$\varepsilon_1 h_1 + \varepsilon_2 h_2 + \ldots + \varepsilon_t h_t \equiv h \pmod r, \quad \varepsilon_i = 0 \text{ or } 1.$$

Thus for every a, there is a divisor d satisfying

$$\|\log d - a\| \leqslant t/r,$$

that is,

$$\sup_a \min_{d|n} \|\log d - a\| \leqslant t/r,$$

and with the values of t and r given, this gives all that we require if x is sufficiently large.

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Об одном классе бинарных биквадратичных форм

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Пусть F_n — множество бинарных форм степени $n \geqslant 3$ с целыми коэффициентами.

В теории представлений чисел бинарными формами важное значение имеет следующая теорема, позволяющая эффективным методом исследовать соответствующие уравнения.

Теорема 1 (см. [3], стр. 304). *Положим*

$$f = f(x, y) = aNm(x - \alpha y),$$

где a — целое, $f \in F_n$. Назовем форму f и алгебраическое число α „исключительным", если существует такая нумерация сопряженных α_1, $\alpha_2, \ldots, \alpha_n$, что

$$\frac{\alpha_1 - \alpha_i}{\alpha_2 - \alpha_i} \cdot \frac{\alpha_2 - \alpha_j}{\alpha_1 - \alpha_j} = \frac{1 - \xi_i}{1 - \xi_j}$$

для любых i, j ($i \neq j$, $3 \leqslant i, j \leqslant n$), где $\xi_i \neq 1$ ($i = 3, 4, \ldots, n$) — некоторые корни из 1.

Все решения диофантова уравнения

$$f(x, y) = Ap_1^{z_1} \ldots p_s^{z_s}, \quad (x, y) = 1,$$

в целых $x, y, z_1 \geqslant 0, \ldots, z_s \geqslant 0$ удовлетворяют неравенству

$$\max(|x|, |y|) < C_1 \exp |A|^{\varkappa},$$

где $\varkappa = 2 + \varepsilon$, $\varepsilon > 0$ — любое число, C_1 — вычислимая величина, не зависящая от A, при условии, что f не есть „исключительная" форма.

Класс E_n ($E_n \subset F_n$) „исключительных" форм введен еще в работах [1] и [2]; в [2], в частности показано отсутствие таких форм при $n \geqslant 5$.

В случае $n = 4$ вопрос о существовании „исключительных" форм оставался открытым, известен был лишь единственный пример $f(x, y) = x^4 - 2x^2y^2 - y^4$, указанный в [2].