[3] D. H. Lehmer, *An extended theory of Lucas functions*, Ann. of Math. 31 (1930), p. 419–449.
[4] M. Mendès France, *Sur les fractions continues limitées*, Acta Arith. 23 (1973), p. 207–215.
[5] A. Schinzel, *On some problems of the arithmetical theory of continued fractions, I*, Acta Arith. 6 (1961), p. 394–413; *II*, Ibid. 7 (1962), p. 288–298.

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Note on sequences well-spaced and well-distributed among congruence classes

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Let

$$a_1 < a_2 < \ldots \tag{1}$$

be an infinite sequence of positive integers with asymptotic density [1] δ. Then it is said to be a *well-spaced sequence* if there exists a constant $C = C(\delta)$, depending only on δ, so that

$$\sup_{i\geqslant 1}(a_{i+1}-a_i) < C.$$

Suppose $0<\eta<1$. Then the sequence (1) of asymptotic density δ is said to be (η)-*well-distributed* among congruence classes if there exists an absolute constant K so that for all $m \leqslant Kn^{1-\eta}$, we have

$$\Big|\sum_{\substack{a_i\equiv a(m)\\ a_i\leqslant n}} 1-\delta nm^{-1}\Big| \doteq o(\delta nm^{-1}); \qquad a = 0, \ldots, m-1$$

as $n\to\infty$.

Henceforth we shall refer to a sequence which is well-spaced and (η)-well-distributed among congruence classes as an η-*sequence*.

One question that naturally presents itself is whether the function $f_\eta(\delta)$, which denotes [2] $\inf_{\mathscr{A}}\underline{\lim}\, A_2(n)n^{-1}$, where the inf is taken over all η-sequences $\mathscr{A}$ with asymptotic density δ, tends to ∞ as $\delta\to 0$.

In this paper we shall prove the following theorem which shows that $f_{1/2}(\delta)$ is bounded for all $\delta>0$. It is an open question whether there exists $\eta<\frac{1}{2}$ so that $f_\eta(\delta)$ remains bounded for all $\delta>0$.

Theorem. *For every $\delta>0$ there exists a $(\frac{1}{2})$-sequence $\mathscr{A}$ of asymptotic density δ such that*

$$\overline{\lim}\, A_2(n)n^{-1} \leqslant \big(2+o(1)\big)\delta. \tag{2}$$

[1] The asymptotic density of a sequence $\mathscr{A}$, if it exists, is defined to be $\lim_{\eta\to\infty} A(n)n^{-1}$, where $A(n)$ is the counting function of $\mathscr{A}$.

[2] $A_2(n)$ denotes the number of integers $\leqslant n$ of the form a_i+a_j where $a_i, a_j\in\mathscr{A}$.

Proof. Let $\alpha = \frac{1}{2}(5^{1/2}-1)$ and $\mathscr{A}$ consist of the integers a so that

(3) $$a\alpha - [a\alpha] \leqslant \delta.$$

It is clear that $\mathscr{A}$ is well-spaced.

From (3) it follows that if a_i, a_j belong to $\mathscr{A}$ then

$$(a_i+a_j)\alpha - [(a_i+a_j)\alpha] \leqslant 2\delta.$$

To show (2) it suffices to show that the number of fractional parts $(\alpha), (2\alpha), \ldots, (n\alpha)$ falling into any interval (a, b) of $(0, 1)$ is $(b-a)n+o(n)$, as $n\to\infty$. In fact we shall establish this as well as that $\mathscr{A}$ is $(\frac{1}{2})$-well-distributed among congruence classes by showing that there exists an absolute constant $K>0$ so that for every $m \leqslant Kn^{1/2}$ and $0 \leqslant a < m$, the discrepancy of the sequence of fractional parts

(4) $$\big((tm+a)\alpha\big), \quad t = 0, \ldots, [(n-a)m^{-1}]$$

is $o(nm^{-1})$, as $n\to\infty$.

A theorem of Erdös and Turán ([1], p. 55; [2]) asserts that the discrepancy of (4) is

(5) $$< K_1\left(\frac{nm^{-1}}{u} + \sum_{k=1}^{u}\frac{\psi(k)}{k}\right),$$

where K_1 is an absolute constant, and

(6) $$\psi(k) = \Big|\sum_{t=0}^{[(n-a)m^{-1}]} e^{2\pi ik(tm+a)\alpha}\Big|.$$

It is well-known that (3)

(7) $$\|q\alpha\| \geqslant 5^{-1/2}2^{-1}q^{-1}, \quad q \geqslant q_0$$

so that (6) and (7) imply that

$$\psi(k) \leqslant 2\,\|km\alpha\|^{-1} \leqslant 4\cdot 5^{-1/2}mk.$$

Using this in (5) we see that the discrepancy of (4) is

$$< K_2(nm^{-1}u^{-1}+mu),$$

and this is $o(nm^{-1})$ if u is chosen sufficiently large in terms of K_2, and K is chosen sufficiently small in terms of u.

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(3) $\|\theta\|$ is the difference, taken positively, between θ and the nearest integer.

References

[1] P. Erdös, *Problems and results on diophantine approximations*, Asymptotic distribution modulo one (1962).

[2] — and P. Turán, *On a problem in the theory of uniform distribution I, II*, Indagationes Math. 10 (1948), pp. 370–378, 406–413.

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