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The divisors of integers I

by

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Introduction. Let d denote a divisor of the positive integer n and $\tau(n)$ the number of these divisors. For any real number x, let $\{x\} = x-[x]$ denote the fractional part of x. The aim of this note is to study the distribution of the points $\{\log d\}$ in the interval $[0,1]$.

We write

$$f_n(x) = \frac{1}{\tau(n)} \sum_{\{\log d\} \leqslant x} 1$$

and we will show that on a sequence of asymptotic density 1,

$$f_n(x) \to x$$

uniformly for $x \in [0,1]$. For each n, we define the discrepancy

$$\Delta(n) = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |f_n(\beta) - f_n(\alpha) - (\beta - \alpha)|.$$

Notice that some authors do not normalize and would therefore use the term discrepancy for the function $\tau(n)\Delta(n)$ in this case. The main result is as follows.

THEOREM 1. *Provided $\lambda < \frac{1}{2}$, there exists a sequence of asymptotic density* 1 *on which*

$$\Delta(n) \leqslant \frac{1}{(\tau(n))^{\lambda}}.$$

It is clear that for all n,

$$\Delta(n) \geqslant \frac{1}{\tau(n)}$$

and theorems of Aardenne-Ehrenfest [1], Roth [4] and Schmidt [5] give lower bounds for the discrepancy; indeed Schmidt's result shows that for each prime p we have

(1) $$\tau(p^m)\Delta(p^m) \geqslant 10^{-2} \log m$$

for infinitely many m. It is possible that there are infinitely many m for which (1) holds simultaneously for all primes: this seems an interesting problem. We show that there exists a positive c so that for $r>1$,

$$\sum_{n=1}^{\infty}\frac{\tau^2(n)\Delta^2(n)}{n^r}\geqslant c\sum_{n=1}^{\infty}\frac{\tau(n)}{n^r}\tag{2}$$

which suggests that the $\frac{1}{2}$ in Theorem 1 may be best possible: it does not prove this for it is quite possible that there is a thin sequence on which $\Delta(n)$ is large enough to make (2) hold.

There is a rather curious corollary to Theorem 1 which leads to another problem.

THEOREM 2. *Provided $\mu<\frac{1}{2}\log 2$, almost all integers n have a divisor d, not equal to 1, such that for some positive integer m,*

$$d=e^m\left(1+O\left(\frac{1}{\log^\mu n}\right)\right).$$

This follows from the first theorem and the fact that on a sequence of asymptotic density 1,

$$\frac{\log\tau(n)}{\log 2}\sim\log\log n.$$

My guess is that the conclusion holds for $\mu<1$, and this result would be best possible.

In the proof of Theorem 1 we need a result of Erdös and Turán [2]: let $x_1,x_2,\ldots,x_N$ be any real numbers,

$$f(x)=\frac{1}{N}\sum_{\{x_i\}\leqslant x}1,$$

$$S_m=\frac{1}{N}\sum_{i=1}^{N}e^{2i\pi m x_i},\qquad m\in Z^+.$$

Then if

$$\Delta=\sup_{0\leqslant\alpha<\beta\leqslant 1}|f(\beta)-f(\alpha)-(\beta-\alpha)|$$

and T is any positive integer,

$$\Delta\ll\frac{1}{T}+\sum_{m=1}^{T}\frac{|S_m|}{m}\tag{3}$$

where the constant implied by Vinogradov's notation $\ll$ is independent of T and the numbers x_i. Indeed, throughout this note $\ll$ will always imply an absolute constant. By Fourier's theorem, we have that

$$f(\beta)-f(\alpha)-(\beta-\alpha)=\sum_{m=-\infty}^{\infty}{}'\frac{e^{2i\pi m\beta}-e^{2i\pi m\alpha}}{2i\pi m}\bar{S}_m$$

where the sum on the right is over non-zero integers m, so that for all α,

$$\int_0^1|f(\beta)-f(\alpha)-(\beta-\alpha)|^2d\beta=\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}|S_m|^2;$$

this shows that

$$2\pi^2\Delta^2\geqslant\sum_{m=1}^{\infty}\frac{1}{m^2}|S_m|^2.$$

Proof of the theorems. We set

$$\tau(n,\theta)=\sum_{d|n}d^{i\theta}$$

and the Erdös–Turán estimate (3) gives

$$\Delta(n)\ll\frac{1}{T}+\sum_{m=1}^{T}\frac{1}{m}\frac{|\tau(n,2\pi m)|}{\tau(n)}.$$

The Schwarz inequality implies

$$\Delta^2(n)\tau^2(n)\ll\frac{\tau^2(n)}{T^2}+(\log T)\sum_{m=1}^{T}\frac{1}{m}|\tau(n,2\pi m)|^2.\tag{4}$$

Also

$$2\pi^2\tau^2(n)\Delta^2(n)\geqslant\sum_{m=1}^{\infty}\frac{1}{m^2}|\tau(n,2\pi m)|^2,$$

and we will deduce (2) from this straightaway. Indeed, a well known result of Ramanujan [3] gives

$$\sum_{n=1}^{\infty}\frac{|\tau(n,2\pi m)|^2}{n^r}=\frac{\zeta^2(r)\zeta(r+2i\pi m)\zeta(r-2i\pi m)}{\zeta(2r)}$$

for $r>1$. Therefore in this range

$$2\pi^2\sum_{n=1}^{\infty}\frac{\tau^2(n)\Delta^2(n)}{n^r}\geqslant F(r)\sum_{n=1}^{\infty}\frac{\tau(n)}{n^r},$$

where

$$F(r)=\frac{1}{\zeta(2r)}\sum_{m=1}^{\infty}\frac{1}{m^2}|\zeta(r+2i\pi m)|^2.$$

Since

$$\zeta(r+2i\pi m)=O(\log 2m)$$

uniformly for $r\geqslant 1$ and $m\in Z^+$, the infinite series is uniformly convergent, and $F(r)$ is continuous, for $r\geqslant 1$. Hence on the interval $1\leqslant r\leqslant 2$, $F(r)$ attains its lower bound, which is not zero as $\zeta(s)$ is non-zero for $\operatorname{Re}s\geqslant 1$.

Moreover, for $r \geqslant 2$,

$$F(r) \geqslant 540/\pi^6,$$

and so

$$F(r) \geqslant c > 0 \quad \text{for} \quad r \geqslant 1,$$

completing the proof of (2).

Next, we prove Theorem 1. Let $0 < y \leqslant 1$, and $\omega(n)$ denote the number of prime factors of n counted according to multiplicity. From (4) we deduce that

$$\sum_{n \leqslant x} \Delta^2(n)\tau^2(n)y^{\omega(n)} \ll \frac{1}{T^2} x\log^3 x + (\log T)\sum_{m=1}^{T}\frac{1}{m}\sum_{n\leqslant x} y^{\omega(n)}|\tau(n, 2\pi m)|^2.$$

We select $T = [\log^2 x]$ and $y = \frac{1}{2}$, so that the first term on the right is $o(x)$. Rather than find an asymptotic formula for the sum on the right we show that there exists an A so that

$$\sum_{m \leqslant \log^2 x}\frac{1}{m}\sum_{n\leqslant x}\frac{1}{2^{\omega(n)}}|\tau(n, 2\pi m)|^2 \ll x(\log\log x)^A$$

as this is sufficient for the application. For $\operatorname{Re} s > 1$ we have

$$\sum_{n=1}^{\infty}\frac{|\tau(n, 2\pi m)|^2}{2^{\omega(n)}n^s} = \frac{\zeta^2(s, \frac{1}{2})\zeta(s+2i\pi m, \frac{1}{2})\zeta(s-2i\pi m, \frac{1}{2})}{\zeta(2s, \frac{1}{4})}$$

where

$$\zeta(s, y) = \prod_p \left(1-\frac{y}{p^s}\right)^{-1}.$$

Hence for $c > 1$,

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)\frac{|\tau(n, 2\pi m)|^2}{2^{\omega(n)}} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^s\zeta^2(s, \frac{1}{2})\zeta(s+2i\pi m, \frac{1}{2})\zeta(s-2i\pi m, \frac{1}{2})}{s(s+1)\zeta(2s, \frac{1}{4})}ds.$$

Replacing x by $2x$, and observing that for $n \leqslant x$, $1-n/2x \geqslant \frac{1}{2}$, we have that

$$\sum_{n\leqslant x}\frac{|\tau(n, 2\pi m)|^2}{2^{\omega(n)}} \ll \frac{(2x)^c}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(c+it, \frac{1}{2})^2\zeta(c+it+2i\pi m, \frac{1}{2})\zeta(c+it-2i\pi m, \frac{1}{2})|}{|(c+it)(c+1+it)\zeta(2c+2it, \frac{1}{4})|}dt.$$

We need an estimate for $|\zeta(s, \frac{1}{2})|$ when $\operatorname{Re} s > 1$. In this half-plane,

$$\log\zeta(s, y) = \sum_{r=1}^{\infty}\sum_p \frac{y^r}{rp^{rs}} = y\log\zeta(s) + \sum_{r=2}^{\infty}\sum_p\frac{y^r-y}{rp^{rs}}$$

and it follows that for $\operatorname{Re} s > 1$,

$$|\zeta(s, \tfrac{1}{2})| \leqslant 2|\zeta(s)|^{1/2}$$

moreover

$$\frac{1}{|\zeta(2s, \frac{1}{4})|} \leqslant \prod_p\left(1+\frac{1}{p^2}\right) < 2.$$

In order to estimate the integral above we consider the ranges $|t| \leqslant 1$, $1 < |t| < 2\pi m - 1$, $|t \pm 2\pi m| \leqslant 1$, $|t| > 2\pi m + 1$ separately, and we use the fact that for $|u| \geqslant 1$,

$$\zeta(c+iu) = O(\log 2|u|).$$

Thus the integrals over the ranges mentioned are respectively:

$$\ll (\log 2m)\int_{-1}^{1}\frac{dt}{|c-1+it|};$$

$$\ll \log^2 2m;$$

$$\ll m^{-2}(\log^{3/2}2m)\int_{-1}^{1}\frac{dt}{|c-1+it|^{1/2}};$$

$$\ll \frac{1}{m}\log^2 2m;$$

plainly the first two estimates are the largest, and so

$$\sum_{n\leqslant x}\frac{|\tau(n, 2\pi m)|^2}{2^{\omega(n)}} \ll x^c\left(\log^2 2m + (\log 2m)\log\frac{1}{c-1}\right).$$

We select $c = 1+1/\log x$, and deduce that

$$\sum_{m\leqslant\log^2 x}\frac{1}{m}\sum_{n\leqslant x}\frac{1}{2^{\omega(n)}}|\tau(n, 2\pi m)|^2 \ll x(\log\log x)^3$$

as required, with $A = 3$. Therefore

$$\sum_{n\leqslant x}\Delta^2(n)\tau^2(n)2^{-\omega(n)} \ll x(\log\log x)^4.$$

The integers $n \leqslant x$ for which $\Delta(n) > (\tau(n))^{-\lambda}$ may be divided into two classes: if both

$$\omega(n) \leqslant \log\log x + u\sqrt{\log\log x}$$

and

$$\tau(n) \geqslant 2^{\log\log x - u\sqrt{\log\log x}}$$

then n belongs to the first class; otherwise it belongs to the second class. If $u \to \infty$ with x, all but $o(x)$ integers $n \leqslant x$ satisfy both the conditions above, and so the second class contains only $o(x)$ numbers. Now let $\sum'$ run over elements of the first class. Then

$$\sum_{n\leqslant x}{}' \tau^{2-2\lambda}(n) 2^{-\omega(n)} \ll x(\log\log x)^4$$

and so

$$\sum_{n\leqslant x}{}' 1 \ll x(\log\log x)^4 2^{(2\lambda-1)\log\log x + (3-2\lambda)u\sqrt{\log\log x}} = o(x)$$

if $\lambda < \frac{1}{2}$, and u increases more slowly than $\sqrt{\log\log x}$, say

$$u = u(x) = (\log\log x)^{1/4}.$$

This completes the proof of Theorem 1.

References

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On the average length of finite continued fractions

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Let a and n be positive integers, $1 \leqslant a < n$, $(a, n) = 1$, and let

$$\frac{a}{n} = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_{l(a,n)}}}},$$

where $c_1, c_2, \ldots, c_{l(a,n)}$ are positive integers, $c_{l(a,n)} > 1$. Put

$$L(n) = \sum_{\substack{1\leqslant a<n \\ (a,n)=1}} l(a, n).$$

Denote by $r(n)$ the number of solutions of the equation

$$n = xx' + yy'$$

in positive integers x, x', y, y', for which $x > y$, $x' > y'$, $(x, y) = 1$, $(x', y') = 1$.

Recently H. Heilbronn [3] proved that if $n > 2$, then

$$L(n) = \tfrac{3}{2}\varphi(n) + 2r(n) \tag{1}$$

and

$$r(n) = \frac{6\ln 2}{\pi^2}\varphi(n)\ln n + O\left(n\left(\sum_{d|n}\frac{1}{d}\right)^3\right),$$

where φ is Euler's function.

For the numbers a and n we compute positive integers q_i and r_i such that

$$r_0 = n,\ r_1 = a;\quad r_{i-1} = q_i r_i + r_{i+1}\ (i = 1, 2, \ldots, m)$$

and

$$r_0 > r_1 > \ldots > r_m > r_{m+1} = 0.$$