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Examples of Iwasawa invariants

by

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0. Let $E = Q(\sqrt{-d})$, $d > 0$, be a quadratic imaginary field of discriminant $-d$ and class number $h = h(-d)$ and let l be an odd rational prime, $(l, d) = 1$. There is a unique $\boldsymbol{Z}_l$-extension of E which is absolutely abelian. Let e_n, $n \geqslant 0$, denote the exact power of l dividing the class number of the nth-layer of the $\boldsymbol{Z}_l$-extension. Under the assumption (A) $l^{n+1} \equiv 1(d)$, the author has given the following formulas for $e_n - e_{n-1}$ ([1]). Let η be a primitive l^n-th root of unity and $\mathfrak{l} = (1-\eta)$ the prime ideal of $Q(\eta)$ lying over l. Let χ be the character of E; χ is a quadratic character of conductor d. Define $a(\tau) = \sum_{i=1}^{\tau-1} \chi(i)$. Let g be a primitive root modulo l^{n+1} and for all $s \geqslant 0$, $g(s) \equiv g^s(l^{n+1})$, $0 < g(s) < l^{n+1}$. For any $s \in \boldsymbol{Z}$, $r \in \boldsymbol{N}$, define s_r by $s_r \equiv s(l^r)$, $0 \leqslant s_r < l^r$. Then

$$e_n - e_{n-1} = \mathrm{ord}_{\mathfrak{l}}(\gamma); \quad \gamma = \sum_{s=0}^{\varphi(l^n)-1} \gamma_s \eta^s;$$

(1)

$$\gamma_s = \sum_{i=0}^{l-2} \Big(a\big(g(s+il^n)\big) - a\Big(g\big(\varphi(l^n)+il^n+s_{n-1}\big)\Big)\Big).$$

Hence the difference $e_n - e_{n-1}$ depends on the $\mathfrak{l}$-order of an algebraic integer in $Q(\eta)$ whose coefficients are certain sums in χ.

For sufficiently large n, $e_n = \mu l^n + \lambda n + c$ for fixed $\mu, \lambda \geqslant 0$, $c \in \boldsymbol{Z}$ ([4], [7]). These λ, μ are the Iwasawa invariants of the given $\boldsymbol{Z}_l$-extension. Our purpose here is to describe some computations of these invariants based on (1).

The contents of this note are as follows: in § 1 we show how to alter (1) in order to dispense with the restrictive assumption (A). In § 2 we show that, in the case $(-d/l) = -1$, a knowledge of e_i for small i often suffices for the determination of μ, λ. Some auxiliary results for $l = 3$ are given in § 3. A short description of the actual computations and the tabulated results are contained in § 4.

I would like to thank John Coates for several useful suggestions, including, in particular, the proof of the lemma of § 3.

1. Fix any $n \geqslant 0$ and let $w = dl^{n+1}$, $l^{n+1} \equiv y(d)$. By [1] we have

(2)
$$e_n - e_{n-1} = \operatorname{ord}_l\Big(\prod_{\substack{0<j<l^n \\ (j,l)=1}} S_j\Big) - (n+1)\varphi(l^n),$$
$$S_j = \sum_{0<\mu<w} \chi\bar{\chi}_1^j(\mu)\mu = l^{n+1}\sum_{i=0}^{l^{n+1}-1} \bar{\chi}_1^j(i) \sum_{\delta=0}^{d-1} \delta\chi(i+\delta l^{n+1}).$$

Substituting y for l^{n+1} and multiplying by $\chi(y)\chi(y^{-1})$ we have

$$S_j = l^{n+1}\chi(y)\sum_i \bar{\chi}_1^j(i)\sum_\delta \delta\chi(iy^{-1}+\delta).$$

If we let

$$w(\mu) = \sum_{\delta=0}^{d-1}\delta\chi(\delta+\mu) \quad \text{and} \quad a(\mu) = \sum_{i=0}^{\mu-1}\chi(i),$$

then it follows easily that $w(\mu) = w(0) + da(\mu)$. Hence

$$S_j = l^{n+1}\chi(y)\sum_i \bar{\chi}_1^j(i)\cdot w(iy^{-1}) = l^{n+1}\chi(y)\sum_i \bar{\chi}_1^j(i)[w(0)+da(iy^{-1})]$$
$$= dl^{n+1}\chi(y)\sum_i a(iy^{-1})\bar{\chi}_1^j(i).$$

On substituting in (2), we see that

$$e_n - e_{n-1} = \operatorname{ord}_l\Big(\prod_j \sum_i a(iy^{-1})\bar{\chi}_1^j(i)\Big).$$

Let η be a primitive l^n-th root of unity. Exactly as in our earlier paper, we may write the sum appearing above in terms of an integral basis $1, \eta, \ldots, \eta^{\varphi(l^n)-1}$ and observe that the product is the norm of this sum from $Q(\eta)$ to Q. This gives us, in the notation of § 0,

(3)
$$e_n - e_{n-1} = \operatorname{ord}_l\Big(\sum_{s=0}^{\varphi(l^n)-1}\gamma_s\eta^s\Big),$$
$$\gamma_s = \sum_{i=0}^{l-2}\Big(a\big(y^{-1}\cdot g(s+il^n)\big) - a\big(y^{-1}\cdot g(s_{n-1}+il^n+\varphi(l^n))\big)\Big).$$

2. Let A_n be the l-class group of the nth layer of the $\boldsymbol{Z}_l$-extension of E. Let A be the inductive limit of the A_n under the natural imbedding $A_n \to A_m$, $m \geqslant n$ ([5], [2], [3]). Let $\hat{A} = \operatorname{Hom}(A, Q_l/\boldsymbol{Z}_l)$ and $\Lambda = \boldsymbol{Z}_l[[T]]$. Then there is an exact sequence of Λ-modules:

(4)
$$0 \to \hat{A} \to \bigoplus_{i=1}^{\tau} \Lambda/(f_i^{s_i}) \to D \to 0.$$

Here each f_i is either l or a distinguished irreducible polynomial and D is a Λ-module of finite cardinality, [6].

Let Γ be the Galois group of the $\boldsymbol{Z}_l$-extension over E, so $\Gamma \cong \boldsymbol{Z}$ topologically. Let Γ_n be the unique subgroup of Γ of index l^n and let γ be a topological generator of Γ. Thus γ^{l^n} is a generator of Γ_n. Then Λ is naturally isomorphic to $\varprojlim \boldsymbol{Z}_l[\Gamma/\Gamma_n]$. Under this isomorphism γ corresponds to $1-T$ and, by identifying these two elements, we view Γ as imbedded in Λ. Let $\omega_n = 1-\gamma^{l^n} = 1-(1-T)^{l^n}$.

Multiplication by ω_n is a Λ-homomorphism whose kernel is the submodule fixed by Γ_n. Hence (4) gives rise to the kernel-cokernel sequence:

(5)
$$0 \to \hat{A}^{\Gamma_n} \to \bigoplus_{i=1}^{\tau}\Lambda/(f_i^{s_i})^{\Gamma_n} \to D^{\Gamma_n} \to \frac{\hat{A}}{\omega_n\hat{A}} \to \bigoplus_{i=1}^{\tau}\frac{\Lambda}{(\omega_n, f_i^{s_i})} \to \frac{D}{\omega_n D} \to 0.$$

Observe that $\hat{A}/\omega_n\hat{A} \cong (\widehat{A^{\Gamma_n}})$. If we assume $(-d/l) = -1$, so that there is a unique ramified prime for the $\boldsymbol{Z}_l$-extension, then $A^{\Gamma_n} \cong A_n$ ([4], [5]).

Theorem 1. *If* $\chi(l) = -1$, *then*

$$\#(A_n) = \prod_{i=1}^{\tau}\#\big(\Lambda/(\omega_n, f_i^{s_i})\big):$$

Proof. In view of (5) and the finiteness of D, it suffices to show that each $[\Lambda/(f^s)]^{\Gamma_n}$ is trivial. Assume first that $f = l$. Let $g(T) \in \Lambda$ and assume that $\gamma^{l^n}\cdot g(T) \equiv g(T)(l^s)$, i.e. $\omega_n\cdot g(T) \equiv 0(l^s)$. Let $g(T) = \sum_{i=0}^{\infty} g_i T^i$. If $g(T) \not\equiv 0(l^s)$, choose j such that g_j has minimal l-order among all the coefficients. Say $l^\tau \| g_j$, $\tau < s$. The $g(T) = l^\tau\cdot g'(T)$, where $g'(T)$ has some coefficient prime to l. Now $\omega_n\cdot g(T) \equiv 0(l^s)$ implies $\omega_n\cdot g'(T) \equiv 0(l^{s-\tau})$. Hence $\omega_n\cdot g'(T) \equiv 0(l)$. But $\omega_n \equiv T^{l^n}(l)$ and $g'(T) \not\equiv 0(l)$.

Now assume that f is a distinguished irreducible polynomial. Then $\Lambda/f^s \cong \boldsymbol{Z}_l[T]/f^s$ as Λ-modules. Let $g(T) \in \boldsymbol{Z}_l[T]$ and assume that $\omega_n\cdot g(T) \equiv 0(f^s)$. By unique factorization in $\boldsymbol{Z}_l[T]$, either $g(T) \equiv 0(f)$ or $\omega_n \equiv 0(f)$. We are done if we exclude the second possibility. If f/ω_n, the $\Lambda/(\omega_n, f^s)$ maps onto $\Lambda/(f)$ and is therefore infinite. But, by the sequence (5) and the remarks immediately following it, this cannot be.

Theorem 2. *Let* ζ_n *be a primitive* n*-th root of unity and* f *a distinguished irreducible polynomial,* $f \nmid \omega_n$ *any* n. *Then*

$$[\Lambda\colon (\omega_n, f^s)] = [\Lambda\colon (\omega_{n-1}, f^s)]\big[\boldsymbol{Z}_l[\zeta_n]\colon \big(f(1-\zeta_n)^s\big)\big] \quad \textit{for} \quad n \geqslant 1.$$

For $n = 0$ *we have*

$$[\Lambda\colon (T, f^s)] = l^{s\cdot\operatorname{ord}_l f(0)} = \big[\boldsymbol{Z}_l\colon \big(f^s(0)\big)\big].$$

Proof. For $n = 0$, $[\Lambda\colon (T, f^s)] = \big[\boldsymbol{Z}_l[T]\colon (T, f^s)\big]$. Mapping T to 0 we see that this equals $\big[\boldsymbol{Z}_l\colon \big(f^s(0)\big)\big]$. In the general case we again have $[\Lambda\colon (\omega_n, f^s)] = \big[\boldsymbol{Z}_l[T]\colon (\omega_n, f^s)\big]$. Note that $\omega_n = \omega_{n-1}\cdot\pi_n$, where π_n is an irreducible polynomial of degree $\varphi(l^n)$. In fact π_n is the minimal polynomial of $1-\zeta_n$ over Q. We will use the exact sequence of Λ-modules:

$$Z_l[T]\xrightarrow{G}\frac{Z_l[T]}{(\omega_{n-1})}\oplus\frac{Z_l[T]}{(\pi_n)}\to\frac{Z_l[T]}{(\omega_{n-1},\pi_n)}\to 0.$$

Clearly, the kernel of G is $(\omega_n)\subseteq Z_l[T]$. Hence

$$[Z_l[T]\colon (\omega_n,f^s)] = [G(Z_l[T])\colon G(\omega_n,f^s)][(\omega_n)\colon(\omega_n)\cap(\omega_n,f^s)] = [G(Z_l[T])\colon G(\omega_n,f^s)].$$

On the other hand, if we let $R_1 = Z_l[T]/(\omega_{n-1})$ and $R_2 = Z_l[T]/(\pi_n)$,

$$[R_1\oplus R_2\colon G(Z_l[T])][G(Z_l[T])\colon G(\omega_n,f^s)] = [R_1\oplus R_2\colon G(\omega_n,f^s)].$$

Using the fact that $G(\omega_n,f^s) = G(f^s)$, we conclude that

$$[Z_l[T]\colon (\omega_n,f^s)] = [R_1\oplus R_2\colon G(f^s)][Z_l[T]\colon(\omega_{n-1},\pi_n)]^{-1}.$$

At this point we compare $G(f^s)$ with $\bar f^s R_1\oplus\bar f^s R_2$. Let $(\bar a\bar f^s,\bar b\bar f^s)\in\bar f^s R_1\oplus\bar f^s R_2$. This element is in $G(f^s)$ iff there exists a $c\in Z_l[T]$ such that $af^s\equiv cf^s(\omega_{n-1})$ and $bf^s\equiv cf^s(\pi_n)$. Since we have assumed that f does not divide ω_n, this is equivalent to $a\equiv c(\omega_{n-1})$ and $b\equiv c(\pi_n)$; i.e. $(\bar a,\bar b)\in G(Z_l[T])$. It follows that the injection $R_1\oplus R_2\to\bar f^s R_1\oplus\bar f^s R_2$ given by multiplication by f^s takes $G(Z_l[T])$ onto $G(f^s)$. Hence

$$[R_1\oplus R_2\colon G(Z_l[T])] = [\bar f^s R_1\oplus\bar f^s R_2\colon G(f^s)].$$

Thus we have

$$\begin{aligned}[Z_l[T]\colon (\omega_n,f^s)] &= [R_1\oplus R_2\colon \bar f^s R_1\oplus\bar f^s R_2][\bar f^s R_1\oplus\bar f^s R_2\colon G(f^s)][Z_l[T]\colon(\omega_{n-1},\pi_n)]^{-1}\\ &= [R_1\oplus R_2\colon \bar f^s R_1\oplus\bar f^s R_2]\\ &= [R_1\colon\bar f^s R_1][R_2\colon\bar f^s R_2] = [Z_l[T]\colon(\omega_{n-1},f^s)][Z_l[T]\colon(\pi_n,f^s)].\end{aligned}$$

The first factor is $[\Lambda\colon(\omega_{n-1},f^s)]$. We evaluate the second factor by considering the map $T\to 1-\zeta_n$ of $Z_l[T]$ to $Z_l[\zeta_n]$ with kernel (π_n). This gives the equality

$$[Z_l[T]\colon (f^s,\pi_n)] = [Z_l[\zeta_n]\colon (f^s(1-\zeta_n))].$$

Recall that e_n is the exact power of l dividing $\#(A_n)$.

COROLLARY 1. *If* $\chi(l) = -1$, *then*

$$e_n-e_{n-1} = \sum_{i=1}^{\tau}s_i\cdot\operatorname{ord}_{1-\zeta_n}[f_i(1-\zeta_n)].$$

Also

$$e_0 = \sum_{i=1}^{\tau}s_i\cdot\operatorname{ord}_l f_i(0).$$

Proof. By the theorems $e_n-e_{n-1} = \sum[Z_l[\zeta_n]\colon (f_i(1-\zeta_n))^{s_i}]$. But the index of $(f_i(1-\zeta_n)^{s_i})$ in $Z_l[\zeta_n]$ is equal to the l-part of the global norm $N(f_i(1-\zeta_n))^{s_i}$, from $Q(\zeta_n)$ to Q. And, as usual, $\operatorname{ord}_l N(a) = \operatorname{ord}_{1-\zeta_n}(a)$, for $a\in Q(\zeta_n)$. The second statement follows directly from Theorems 1 and 2.

Note that since f_i is either l or a distinguished polynomial, we must have $\operatorname{ord}_l(f_i(0))\geqslant 1$. Thus $e_0\geqslant\sum_{i=1}^{\tau}s_i$.

It has been shown by Iwasawa [4] that for n sufficiently large $e_n = \mu l^n+\lambda n+c$ where c is an integer constant and μ,λ are determined by

$$\mu = \sum_{f_i=l}s_i,\qquad \lambda = \sum_{f_i\neq l}s_i\cdot\deg(f_i).$$

The few corollaries below enable us to evaluate μ,λ in very many cases, when $\chi(l) = -1$, based only on a knowledge of e_i for small i.

COROLLARY 2. *If* $\chi(l) = -1$ *and* $e_n-e_{n-1}<\varphi(l^n)$ *for some* $n>1$, *then* $\mu = 0$.

Proof. By Corollary 1,

$$e_n-e_{n-1} = \sum_i s_i\operatorname{ord}_{1-\zeta_n}(f_i(1-\zeta_n))\geqslant\sum_{f_i=l}s_i\cdot\varphi(l^n) = \mu\cdot\varphi(l^n).$$

COROLLARY 3. *If* $\chi(l) = -1$ *and* $e_n-e_{n-1}<\varphi(l^n)$ *for some* $n\geqslant 1$, *then* $e_n-e_{n-1} = \lambda$.

Proof. By Corollary 2, we have $\mu = 0$. Hence

$$\varphi(l^n)>e_n-e_{n-1} = \sum_i s_i\operatorname{ord}_{1-\zeta_n}f_i(1-\zeta_n).$$

Let $f_i = \sum_{j=0}^{d_i}a_{ij}T^j$. Then

$$\operatorname{ord}_{1-\zeta_n}(f_i(1-\zeta_n))\geqslant\min_{0\leqslant j\leqslant d_i}\left(\operatorname{ord}_{1-\zeta_n}(a_{ij}(1-\zeta_n)^j)\right) = \min_{0\leqslant j<d_i}\left(d_i, j+\varphi(l^n)\cdot\operatorname{ord}_l(a_{ij})\right).$$

Since f_i is distinguished, $\operatorname{ord}_l a_{ij}>0$. It then follows from the inequality

$$\varphi(l^n)>e_n-e_{n-1}\geqslant\sum_i s_i\min_j\left(d_i, j+\varphi(l^n)\operatorname{ord}_l(a_{ij})\right)\tag{6}$$

that in each summand the minimum is achieved uniquely at d_i. Hence $e_n-e_{n-1} = \sum s_i d_i = \lambda$.

COROLLARY 4. *Assuming* $\chi(l) = -1$ *and* $\mu = 0$, *the formula* $e_n = \lambda n+c$ *is valid for all* n *such that* $\lambda<\varphi(l^{n+1})$.

Proof. The formula is valid at n_0 iff $e_n-e_{n-1} = \lambda$ for all $n\geqslant n_0+1$. By (6) we see that this holds whenever $\lambda<\varphi(l^n)$, since $d_i\leqslant\lambda$.

COROLLARY 5. *If* $(-d/l) = -1$, $\mu = 0$ *and* $e_0 = 1$, *then* $e_1-e_0>l-1$ *implies* $\lambda = l-1$.

Proof. Since $\mu = 0$ and $e_0 = 1$, we have $\tau = 1$, $s_1 = 1$, and $f_1 = f$ is an irreducible polynomial of degree λ. Then, as above, we have for $n = 1$,

$$e_1-e_0\geqslant\min_{0<j<\lambda}\left(\lambda, j+(l-1)\operatorname{ord}_l(a_j), (l-1)\operatorname{ord}_l f(0)\right).$$

In this case, however, $\operatorname{ord}_l f(0) = e_0 = 1$ and the rightmost term in brackets equals $l-1$. Since, by hypothesis, $e_1 - e_0 > l-1$, there must be another term equal to $l-1$ in the brackets. But λ is the only possibility.

By a more careful attention to detail, one can conclude in this situation that $f(0) = (1-\zeta_1)^{l-1}$ exactly.

3. Remarks on the case $l = 3$. In this case one can proceed a bit further with the formulas of § 1. Let $M(\tau) = \sum_{i=0}^{\tau} \chi(i) = a(\tau+1)$. Then it can be shown that $\mu = 0$ for the Z_3-extension of $Q(\sqrt{-d})$, $(d, 3) = 1$, if $M(3^{-1} \bmod d) \not\equiv 0(3)$ ([1]). The following lemma was suggested by the results of the computation described in § 4. I am indebted to J. Coates for the proof.

LEMMA 1. *Let $Q(\sqrt{-d})$ be an imaginary quadratic field of discriminant d, class number h, and character χ. Then*

$$(3-\chi(3))h = 2M([d/3]), \qquad [\ ] \text{ denotes greatest integer.}$$

Proof. First assume $d \equiv 1(3)$ and $d-1 = 3\tau$. We start with the well known $-dh = \sum_{i=1}^{d-1} \chi(i)\cdot i$. Observing that $\chi(3) = -1$, we have

$$3dh = -3\chi(3)dh = \sum_{i=1}^{d-1} \chi(3i)\cdot 3i = \sum_{i=1}^{\tau} + \sum_{i=\tau+1}^{2\tau} + \sum_{i=2\tau+1}^{3\tau}$$
$$= \sum_{i=1}^{\tau} \chi(3i)\cdot 3i + \sum_{i=1}^{\tau} \chi(3i-1)(3i-1+d) + \sum_{i=1}^{\tau} \chi(3i-2)(3i-2+2d)$$
$$= \sum_{i=1}^{d-1} \chi(i)i + d\sum_{i=1}^{\tau} \chi(3i-1) + 2d\sum_{i=1}^{\tau} \chi(3i-2)$$
$$= -dh + d\Big(\sum_{i=1}^{\tau} \chi(3i-1) + \sum_{i=1}^{\tau} \chi(3i-2) + \sum_{i=1}^{\tau} \chi(3i-2)\Big)$$
$$= -dh + d\Big(-\sum_{i=1}^{\tau} \chi(3i) + \sum_{i=1}^{\tau} \chi(3i-2)\Big).$$

Now observe that by change of variable,

$$\sum_{i=1}^{\tau} \chi(3i-2) = \sum_{i=0}^{\tau-1} \chi(3\tau-3i-2) = \sum_{i=0}^{\tau-1} \chi(-3i-3) = \sum_{i=0}^{\tau-1} \chi(i+1) = M(\tau).$$

Hence we have

$$3dh = -dh + d(2M(\tau)),$$

which is the desired result.

The case $d \equiv 2(3)$ follows similarly.

For $d \equiv 0(3)$, $d = 3\tau$, we proceed as follows:

$$-dh = \sum_{i=1}^{d} \chi(i)\cdot i = \sum_{i=1}^{\tau} + \sum_{i=\tau+1}^{2\tau} + \sum_{i=2\tau+1}^{3\tau}$$
$$= \sum_{i=1}^{\tau} \big(\chi(i)\cdot i + \chi(i+\tau)(i+\tau) + \chi(i+2\tau)(i+2\tau)\big)$$
$$= \sum_{i=1}^{\tau} \big(\chi(i) + \chi(i+\tau) + \chi(i+2\tau)\big)i + \tau\sum_{i=1}^{\tau} \chi(i+\tau) + 2\tau\sum_{i=1}^{\tau} \chi(i+2\tau).$$

The first summand here is zero, since we can write χ as the product of a character of conductor 3 and a character of conductor τ. So

$$-dh/\tau = \sum_{i=1}^{\tau} \chi(i+\tau) + 2\sum_{i=1}^{\tau} \chi(i+2\tau) = \sum_{i=\tau+1}^{3\tau} \chi(i) + \sum_{i=2\tau+1}^{3\tau} \chi(i)$$
$$= M(3\tau) - M(\tau) + M(3\tau) - M(2\tau) = -\big(M(\tau) + M(2\tau)\big).$$

Using the general relation $M(\mu) = M(d-\mu-1)$ and the fact that $\chi(\tau) = 0$, we arrive at

$$dh/\tau = 2M(\tau) \quad \text{or} \quad 3h = 2M([d/3]).$$

COROLLARY. *For $d \equiv \pm 1(3)$,*

$$(3-\chi(3))h = 2\big(M(3^{-1} \bmod d) - \chi(3)\big).$$

Proof. It suffices to show that $M([d/3]) = M(3^{-1}) - \chi(3)$. We will treat the case $d \equiv 1(3)$, $d-1 = 3\tau$. Now

$$M([d/3]) = M\left(\frac{d-1}{3}\right) = M\left(d - \frac{d-1}{3} - 1\right) = M\left(\frac{2d-2}{3}\right)$$
$$= M\left(\frac{2d+1}{3}\right) - \chi\left(\frac{2d+1}{3}\right) = M(3^{-1} \bmod d) - \chi(3).$$

COROLLARY. *If $Q(\sqrt{-d})$, $(d, 3) = 1$, has class number divisible by 3, then the invariant μ for the Z_3-extension of $Q(\sqrt{-d})$ is zero.*

Proof. A necessary condition that μ be nonzero is $M(3^{-1}) \equiv 0(3)$. By the above corollary, this would imply $h \equiv 2(3)$. So, in fact, if $h \equiv 0$, $1(3)$, then $\mu = 0$.

It is a simple consequence of this corollary that $\mu_3 = 0$ whenever $d \equiv 1(3)$.

4. A fortran program was written for IBM 360/70 to compute $e_1 - e_0$ by use of equation (2). For $l = 3, 5, 7$ we have treated all d up to 3000 with $(-d/l) = -1$, $l \mid h(-d)$ (Table 1, 2, 3). Some cases of $(-d/l) = +1$ were also computed for testing and comparison (Table 4). For $l = 3$, $M(3^{-1} \bmod d)$ was computed; this resulted in the formulation of lemma, § 3. Table 5 summarizes the consequences of applying the corollaries of § 2 to the contents of Tables 1, 2, 3. For $l = 3$, the computation of $e_1 - e_0$

was not always sufficient to determine λ. Hence the program was enlarged to compute e_2-e_1. The additional results are included in Table 1 where necessary and in Table 4 where available. The program input was selected by hand from Gauss' tables. Hence the set of tabulated discriminants may not be complete. Note that if $(-d/l) = 1$ and $l \nmid h(-d)$, then $\lambda = \mu = e_n = 0$.

Table 1. $\boldsymbol{Z}_l$-extension of $Q(\sqrt{-d})$

$l = 3,\ (-d/3) = -1$

d	h	e_1-e_0	e_2-e_1	d	h	e_1-e_0	e_2-e_1
31	3	1		1432	6	1	
				1480	12	2	2
139	3	1		1588	6	1	
				1720	12	1	
199	9	1		1732	12	1	
211	3	2	2	1972	12	1	
244	6	1		2047	18	4	
247	6	1		2068	12	1	
283	3	1		2071	30	1	
307	3	1		2104	12	1	
331	3	1		2155	12	2	3
367	9	1		2167	18	1	
379	3	3		2191	30	1	
424	6	1		2227	6	1	
436	6	1		2260	12	2	
439	15	1	1	2344	18	1	
451	6	1	1	2440	12	1	
472	6	1	1	2443	6	2	4
499	3	1	1	2479	24	2	2
547	3	1	1	2488	12	1	
628	6	1		2491	12	1	
643	3	1		2503	21	2	3
655	12	1		2515	6	2	2
679	18	1		2563	6	2	2
751	15	2	2	2599	30	2	3
808	6	1		2644	18	1	
823	9	1		2647	15	1	
835	6	1		2680	12	4	
856	6	2	4				
883	3	1		2728	12	1	
907	3	2	3	2740	12	1	
964	12	1		2767	21	1	
1048	6	1		2791	39	1	
1096	12	2	4	2824	24	1	
1108	6	1		2872	12	2	2
1144	12	2	3	2911	42	2	3
1192	6	1		2920	12	2	3
1336	12	3		2923	6	2	2

Table 2. $\boldsymbol{Z}_l$-extension of $Q(\sqrt{-d})$

$l = 5,\ (-d/5) = -1$

d	h	e_1-e_0	d	h	e_1-e_0
47	5	1	1748	20	1
103	5	1	1823	45	1
127	5	2	1867	5	1
143	10	1	1887	20	1
303	10	1	1928	20	2
347	5	1	2063	45	1
443	5	1	2087	35	1
488	10	1	2152	10	2
523	5	2	2203	5	1
683	5	1	2243	15	3
787	5	1	2247	20	1
788	10	1	2347	5	1
803	10	2	2363	10	1
872	10	1	2407	20	1
923	10	1	2452	10	1
947	5	1	2483	20	2
1007	30	1	2487	20	1
1123	5	1	2532	20	1
1223	35	1	2543	35	1
1253	20	1	2552	20	1
1268	10	1	2603	20	1
1327	15	1	2643	10	1
1427	15	1	2647	15	1
1492	10	1	2683	5	2
1567	15	1	2708	30	1
1592	20	1	2712	20	1
1643	10	1	2743	20	1
1652	20	1	2843	15	1
1688	10	2	2887	25	1
1707	10	1	2948	20	1
1723	5	2	2983	20	1
1747	5	1	2987	20	1

Table 3. Z_l-extension of $Q(\sqrt{-d})$
$l = 7$, $(-d/7) = -1$

d	h	$e_1 - e_0$
151	7	3
71	7	1
431	21	1
463	7	1
487	7	1
536	14	1
596	14	1
743	21	1
807	14	1
827	7	1
863	21	1
935	28	1
1031	35	1
1163	7	1
1171	7	1
1311	28	1
1479	28	2
1523	7	1
1527	14	1
1703	28	2
2011	7	1
2024	28	1
2055	28	1
2083	7	1
2087	35	1
2111	49	1
2123	14	1
2179	7	1
2251	7	1
2279	56	1
2335	14	1
2431	28	1
2503	21	1
2507	14	1
2543	35	1
2564	28	1
2571	14	1
2612	14	1
2703	28	1
2767	21	1

Table 4. Z_l-extension of $Q(\sqrt{-d})$
$(-d/l) = +1$

	d	$h(-d)$	$e_1 - e_0$	$e_2 - e_1$
$l = 3$	11	1	1	1
	20	2	1	1
	23	3	1	1
	35	2	2	2
	56	4	2	2
	68	4	1	1
	104	6	1	1
	116	6	1	1
	152	6	1	1
	3299	27	2	2
	3896	36	2	2
$l = 5$	19	1	1	1
	31	3	1	1
	136	4	2	2
	139	3	1	1
	199	9	1	1
	211	3	1	1
	244	6	1	1
	1311	28	3	3
$l = 7$	19	1	1	1
	31	3	1	1
	52	2	1	1
	136	4	2	2
	139	3	1	1
	199	9	1	1
	244	6	1	1

Table 5. Relation of first layers
to invariants when $(-d/l) = -1$

	e_0	$e_1 - e_0$	$e_2 - e_1$	λ	e_n	n_0
$l = 3$	1	1		1	$n+1$	0
	1	2	2	2	$2n+1$	0
	1	2	3	3	$3n$	1
	1	2	4	4	$4n-1$	1
	1	3		3	$3n+1$	0
	1	4		4	$4n+1$	0
	2	1		1	$n+2$	0
	2	4	3	3	$3n+3$	1
$l = 5, 7$	1	1		1	$n+1$	0
	1	2		2	$2n+1$	0
	1	3		3	$3n+1$	0
	2	1		1	$n+2$	0

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Некоторые свойства дзета-функции Римана на критической прямой

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Цель этой заметки: дать некоторые дополнения к предшествующей заметке [5], и попробовать применить дзета-функцию Римана в релятивистской космологии.

Пусть $0 < \gamma' < \gamma''$ — ординаты соседних нулей дзета-функции Римана $\varrho' = \frac{1}{2}+i\gamma'$, $\varrho'' = \frac{1}{2}+i\gamma''$, и, $\{t_0\}$ — последовательность значении $t_0 > 0$ таких, что

(а) $$\gamma' < t_0 < \gamma'',$$

(б) $$Z'(t_0) = 0,$$

(b) $$t_0 \to +\infty.$$

Пусть $\{\tilde{t}_0\}$ — подпоследовательность последовательности $\{t_0\}$ такого рода, что

$$|\zeta(\tfrac{1}{2}+i\tilde{t}_0)| > \frac{1}{\tilde{t}_0^{\alpha}}, \quad 0 < \alpha \leqslant 1.$$

Пусть $\tilde{\gamma}', \tilde{\gamma}''$ — ординаты таких соседних нулей функции $\zeta(s)$, что $\tilde{\gamma}' < \tilde{t}_0 < \tilde{\gamma}''$. Символ $\{\tilde{\gamma}', \tilde{\gamma}''\}$ обозначает последовательность таких соседних ординат.

Численные эксперименты с функцей $Z(t)$ показывают, что точки t_0 распределены с небольшим разбросом в окрестностях точек $(\gamma'+\gamma'')/2$.

Обозначим
$$\Delta(t_0) = \min\{t_0-\gamma', \gamma''-t_0\}.$$

Теоретически неисключено, что даже в случае

$$\gamma''-\gamma' > \frac{1}{\gamma'^{\alpha}},$$

$\Delta(t_0)$ — сколь угодно мало, т.е., точка t_0 находится на сколь угодно малом расстоянии или от точки γ' или от точки γ''. Точнее: возникает вопрос об оценке снизу величины $\Delta(t_0)$. В этом направлении имеет место